

Dear Fulton,

I enjoyed very much your notes on connectivity ---.

A few remarks:

- The proof of cor 3 p 24 can be expressed somewhat more algebraically, looking at the diagramm:

$$\begin{array}{ccc} \pi_1(X_0 \times X_0 \cap V) & \longrightarrow & \pi_1(X_0 \times X_0) \\ \downarrow & & \downarrow \\ \pi_1(X \times X \cap V) & \longrightarrow & \pi_1(X \times X) \\ \uparrow s & & \\ \pi_1(X) & & \end{array}$$

: the image of $\pi_1(X)$ in $\pi_1(X)^2$ covers $\text{Im}(\pi_1(X_0) \rightarrow \pi_1(X))^2$

hence this $\pi_1(X_0) \rightarrow \pi_1(X)$ is trivial. The rest as in your paper

- In the same vein, at the places where you assume " X' " irreducible, one can drop the assumption by weakening the conclusion to: "... has an image containing $\text{Im}(\pi_1(\text{normalization of } X) \rightarrow \pi_1(X)) = \text{Im}(\pi_1(\text{smooth locus of } X) \rightarrow \pi_1(X))$ ". Of course, this $\text{Im}(\pi_1 \rightarrow \pi_1)$ is $\pi_1(X)$ if and only if X' is irreducible.

- misprints: p 36, cor 3 (B): assume $C' \neq \emptyset$.

p 36, cor 2, in char: V is "locally" simply connected.

- p 28 cor (B): one could give a proof closer in spirit to [G-L] by proving a stronger theorem: if $X \rightarrow \mathbb{P}^n - H$ is an \'etale covering, possibly with an ∞^d of sheets, they are points of H above which $\geq e$ sheets come together, if $e \leq d, n+1$. Meaning: if one restrict the covering to a small ball around a suitable $h \in H$, they are connected components of it that have $\geq e$ sheets.

Proof a) one can assume by induction that $d \geq n+1$, that over each hyperplane, it happens that n sheets come together, and, by contradiction, that more never do. This enables to extend $X \rightarrow \mathbb{P}^N - H$ to a ramified covering $\bar{X} \rightarrow \mathbb{P}^N$: above small balls, \bar{X} is \amalg (finite coverings). Let $R \subset \bar{X}$ be the locus where n sheets come together: $R \xrightarrow{f} \mathbb{P}^N$ is unramified, above, above small balls, \amalg (finite maps), each $f(R) = Y$ is algebraic, and, R is above Y° zariski open in Y , R is étale. One has to show that $\Delta \subset R \times_{\mathbb{P}^N} X$ is not open. Its openness would contradict

$$\pi_1((\mathbb{P}^N - H) \times Y^\circ \cap V(\Delta)) \rightarrow \pi_1((\mathbb{P}^N - H) \times Y^\circ).$$

Did Lazarsfeld prove the local analogue (= for a small punctured neighbourhood of 0 in affine space) of your theorem with names?

Suggestion: take $Z \xrightarrow{f} \text{Ball} - \{0\}$, with f "algebraic looking".

$Z \xrightarrow[\substack{\text{complex} \\ \text{of } T \subset \bar{Z} \\ \text{analogic}}]{\text{proper}} \bar{Z} \xrightarrow{f} \text{Ball}$. Then, assuming Z irreducible, L linear subspace

- ($0 \in L$) with $n = \dim Z - \text{codim } L \geq 2$, one has, after a shrinking of the ball (depending on L)
- (a) for L general: $\pi_i(f^{-1}L) \cong \pi_i Z$ for $i < n-1$, \Rightarrow for $i = n-1$
 - (b) for any L : $\pi_i(f^{-1}(V(L-0))) \cong \pi_i Z$ for $i = n-1$ (same).

If this suggestion did hold, it would reprove the theorem.

Indeed, write $\mathbb{P}^n = \mathbb{P}(V)$ (set of directions in V), hence $(V - \{0\}) \rightarrow \mathbb{P}(V)$.

Starting from $Z \rightarrow \mathbb{P}(V) \times \mathbb{P}(V)$, it pulls back to
 $\begin{array}{ccc} Z & \xrightarrow{\quad} & \mathbb{P}(V) \times \mathbb{P}(V) \\ \uparrow & & \uparrow \\ Z_1 & \longrightarrow & (V - \{0\}) \times (V - \{0\}) \hookrightarrow V - \{0\} \end{array}$

while $\Delta \subset P(V) \times P(V)$ has over it $\Delta_V \subset V \times V$, the graph of the identity map of $V - \{0\}$ and a linear subspace.

The suggestion, applied to Z , and Δ_V , gives Fulton-Hansen.

This leads to suggestions about the higher π_i . First a reproducibility of the above. Starting from

$$Z \rightarrow P(V) \times P(V),$$

one introduces $Z_2 \subset P(V \times V) : (Z_1 \text{ above } / \text{homotopic})$:

~~P(V) × P(V)~~ $P(V \times V) = \text{join}(P(V) \text{ and } P(V)) \Rightarrow G_m\text{-bundle over } P(V \times \mathbb{R}^n)$
 $(G_m\text{-bundle} = P(V \times V) = P(V \times \mathbb{P}) = P(\mathcal{O} \times V))$, and take pull back of Z .

Then, Bertini applied to $Z_2 \subset P(V \times V)$ and to $P(\text{diagonal } V \subset V \times V)$ gives Fulton-Hansen for $Z \rightarrow P(V) \times P(V)$. Bertini for higher π_i .

The ~~suggestion~~ suggest:

Conjecture
Suggestion. Take $Z \xrightarrow{f} P(V) \times P(V)$, ~~smoothable~~,
 f proper, Z ^{smooth} irreducible. Then

$$0 \rightarrow \pi_2 f^* \Delta \rightarrow \pi_2 Z \rightarrow Z \rightarrow \pi_1 f^* \Delta \rightarrow \pi_1 Z \rightarrow 0$$

$$\pi_i f^* \Delta \xrightarrow{\sim} \pi_i Z \quad (i > 2)$$

up to $i = \dim f(Z) - \text{codim } \Delta$, the injectivity part being lost for this last i . [†]

Of course, I want a variant with $f^* V(\Delta)$ ~~for~~ for non proper Z . The map $\pi_2 Z \rightarrow \mathbb{Z}$ is as follows: one Z , we have the line bundles $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$. Look at the homotopy sequences of the corresponding G_m -principal bundle: $\pi_2(Z) \xrightarrow{\delta} \pi_1(\mathbb{C}^*) = \mathbb{Z}$, and take the difference of the maps to \mathbb{Z} for $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$

[†] I went overboard. This is for f finite. In general, a correction like in the conjecture in my Bombay talk.

It vanishes on the image of $\pi_2(f^{-1}\Delta)$, because $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$ become isomorphic on $f^{-1}(\Delta)$

— I guess that Lazanfeld observed that the proof of Cor p 28 + his local theorem gives that for

$$X \longrightarrow Y \hookrightarrow \mathbb{P}^N \quad : \begin{array}{l} Y \text{ closed, smooth} \\ X/Y \text{ 2-sheeted covering,} \end{array}$$

if $2 \operatorname{codim} Y + d \leq N$, then X is simply connected.

What does one know about higher π_i or H_i — possibly assuming X smooth?

I am going to Moscow in a few days, for 2 weeks, and will take your ms with me, to leave it there. Could you send me a new one?

Yours truly,

P. Deligne