11/30/2011

Dear Vasiu,

Let $\mathcal{C}$ be a tannakian category over an algebraically closed field $k$. Your question amounts to: does $\mathcal{C}$ always have a fiber functor over $k$. I believe it is true. The following argument is presumably too pedestrian, hiding which "compacity arguments" are really relevant.

Let $A$ be the set of strictly full subcategories of $\mathcal{C}$, stable by $\otimes$, subquotients and duals. Let $A_f \subset A$ be the set of those generated by finitely many objects, using the same operations. For $\alpha \in A$, I note $\mathcal{C}_\alpha$ the corresponding subcategory (as defined: $\mathcal{C}_\alpha = \alpha$). The set $A$ is ordered by inclusion. I assume we already know the existence and unicity ~~of~~ up to isomorphism of fiber functors for the $\mathcal{C}_\alpha$, $\alpha \in A$.

If $\mathcal{C}_\alpha \subset \mathcal{C}_\beta$, it makes sense to say that a fiber functor $\omega_\beta$ on $\mathcal{C}_\beta$ extends (on the nose) a fiber functor $\omega_\alpha$ on $\mathcal{C}_\alpha$: If an extension up to isomorphism exists, an actual extension exists too.

Let us order the set of $(\alpha, \omega_\alpha)$: $\alpha \in A$, $\omega_\alpha$ fiber functor on $\mathcal{C}_\alpha$, by "$\omega_\beta$ extends $\omega_\alpha$". To avoid set theoretical difficulties, one could consider only the fiber functors $\omega_\alpha$ such that the $\omega_\alpha(X)$ take value in the set of vector spaces $k^n$, $n \in \mathbb{N}$.

The ordered set of $(\alpha, \omega_\alpha)$ is inductive (Bourbaki Ens Ch 3 §2, n°4) : If $I$ is a totally ordered subset, the "union" of the $(\alpha, \omega_\alpha)$ is a $\mathcal{T}_0$ in $A$ $(= \bigcup_{(\alpha,\omega_\alpha)\in I} \mathcal{T}_\alpha)$ with ~~the~~ fiber functor $\omega_0$ characterized by $\omega_0 | \mathcal{T}_\alpha = \omega_\alpha$. This "union" majorizes $I$. By Bourb loc cit Th 2, the ordered set of the $(\alpha, \omega_\alpha)$ has a maximal element : $(\mathcal{T}_1, \omega_1)$. To prove that $\mathcal{T}_1 = \mathcal{T}$, it suffices to prove the following

<u>Lemma 1</u> Let $\mathcal{T}'$ be in $A$ and $\mathcal{T}''$ be in $A_f$. Let $\langle \mathcal{T}', \mathcal{T}'' \rangle$ be generated by $\mathcal{T}'$ and $\mathcal{T}''$ : it is in $A$. Then, any fiber functor $\omega'$ on $\mathcal{T}'$ can be extended (on the nose, or up to isomorphism, this amounts to the same) to $\langle \mathcal{T}', \mathcal{T}'' \rangle$

I first prove

<u>Lemma 2</u> Suppose $\mathcal{T}'$ is in $A_f$ too. "Restriction" is then a equivalence of categories

(fiber functors on $\langle \mathcal{T}', \mathcal{T}'' \rangle$) $\longrightarrow$ (triples of a fiber functor $\omega'$ on $\mathcal{T}'$, $\omega''$ on $\mathcal{T}''$ and an isomorphism $\tau$ of the restrictions of $\omega'$ and $\omega''$ to $\mathcal{T}' \cap \mathcal{T}''$ ($\mathcal{T}' \cap \mathcal{T}''$ is also in $A_f$)).

We may assume that $\langle \mathcal{T}', \mathcal{T}'' \rangle$ is the category of representations of a group $G$, and that for invariant subgroups $A$ and $B$, $\mathcal{T}'$ (resp $\mathcal{T}''$) is the subcategory of representations where $A$ (resp $B$) acts trivially. That they generate $\langle \mathcal{T}', \mathcal{T}'' \rangle$ means that $A \cap B = \{e\}$. The intersection $\mathcal{T}' \cap \mathcal{T}''$ is the category of representation on which $AB$ acts trivially

The triples $(\omega', \omega'', \tau)$ are all isomorphic: as all $\omega'$ (resp all $\omega''$) are isomorphic, it suffices to see that $(\omega', \omega'', \tau_1)$ and $(\omega', \omega'', \tau_2)$ are isomorphic. Indeed $\tau_1$ and $\tau_2$ differ by an automorphism of $\omega' | \mathcal{T}' \cap \mathcal{T}''$, and such an automorphism lifts to an automorphism of $\omega'$:

$$G/A(k) \longrightarrow G/AB(k) \quad \text{is onto.}$$

We hence have here categories with just one isomorphism class of objects, and the question is to compare automorphisms groups. We need to check

$$G \xrightarrow{\sim} \{(g', g'') \in G/A, G/B \mid g' \text{ and } g'' \text{ have same image in } G/AB\}$$

$$\begin{array}{ccc} G & \longrightarrow & G/A \\ \downarrow & \square & \downarrow \\ G/B & \longrightarrow & G/AB \end{array}$$

Injectivity of this morphism of groups amounts to $A \cap B = \{e\}$.
Surjectivity: if ~~$g'' = g'ab$, $\{(g', g'')\}$~~ $\tilde{g}'$, $\tilde{g}''$ left $g'$, $g''$, $\tilde{g}'' = \tilde{g}'.ab$ and $\tilde{g}''b = \tilde{g}'a \in G$ maps to $(g', g'')$.

<u>Lemma 3</u>. Same as lemma 2, but $\mathcal{T}'$ only assumed to be in $A$

Let $B$ be the set of $\mathcal{T}_\alpha$ $(\alpha \in A_f)$ contained in $\mathcal{T}'$. One has $\langle \mathcal{T}', \mathcal{T}'' \rangle = \bigcup_{\beta \in B} \langle \mathcal{T}_\beta, \mathcal{T}'' \rangle$. One has equivalences

(fiber functors on $\mathcal{T}$) $\xrightarrow{\sim}$ (fiber functors $\omega_\beta$ on the $\mathcal{T}_\beta$'s, plus a compatible system of isomorphisms $\omega_\beta | \mathcal{T}_\gamma \xrightarrow{\sim} \omega_\gamma$ for $\mathcal{T}_\gamma \subset \mathcal{T}_\beta$; compatible: condition for $\mathcal{T}_\delta \subset \mathcal{T}_\gamma \subset \mathcal{T}_\beta$)

Same for fiber functors on the $\langle \mathcal{T}', \mathcal{T}'' \rangle = \bigcup \langle \mathcal{T}_\beta, \mathcal{T}'' \rangle$.

The $\mathcal{T}_\beta \cap \mathcal{T}''$ have a largest ~~smallest~~ element : ~~they~~ if $\mathcal{T}'' = \mathrm{Rep}(G'')$, they correspond to invariant subgroups of $G''$, subgroups are closed subschemes, and one uses the noetherian property. If $\beta_0$ is such that $\mathcal{T}_{\beta_0} \cap \mathcal{T}''$ is the largest $\mathcal{T}_\beta \cap \mathcal{T}''$, for any $\beta > \beta_0$, extending $\omega_\beta = \omega | \mathcal{T}_\beta$ to $\langle \mathcal{T}_\beta, \mathcal{T}'' \rangle$ amounts to extending $\omega_\beta | \mathcal{T}_\beta \cap \mathcal{T}''$ to $\mathcal{T}''$ (lemma 2), that is to extend $\omega_{\beta_0}$ from $\mathcal{T}_{\beta_0} \cap \mathcal{T}''$ to $\mathcal{T}''$. If we choose one such extension, we get up to unique isomorphism a system of extensions of the $\omega_\beta$ to $\langle \mathcal{T}_\beta, \mathcal{T}'' \rangle$, and by gluing them an extension of $\omega$ to $\langle \mathcal{T}', \mathcal{T}'' \rangle$.

This is all we need to conclude that $\mathcal{T}' = \mathcal{T}$.

This proof should be cleaned up. After all, we are proving that some projective system of gerbs (of the fiber functors on the $\mathcal{T}_\alpha$, $\alpha \in A_f$), where "projective system" is taken in a 2-categorical sense, has a non empty projective limit (again limit in a 2-categorical sense). May be such a translation would make the "compacity arguments" used clearer. They were two of them: "fiber functor is a property of finite type (cf Bourb. Ens Ch 3 §4.5)", and the noetherian property for subgroups.

The same arguments give unicity up to isomorphism of $\omega$, over $k$: we get a maximal $\mathcal{C}_\alpha$ ($\alpha \in A$) over which we have an isomorphism, and extend further if $\mathcal{C}_\alpha \neq \mathcal{C}$.

Best

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