Note on Quantization

Pierre Deligne

The present notes do not pretend to any originality. We have tried to present different aspects of what quantization can mean, including the case of odd variables. Complex polarizations are not considered.

§1.

Let (M, ω) be a symplectic manifold of dimension $2d$. We will use the vague words "big" and "small". For this, we need some notion of size ~ 1 on M. For instance M could carry a Riemannian metric, with curvature of size at most ~ 1 and injectivity radius of size at least ~ 1. The symplectic form should be of size ~ 1, and $\hbar$ is a small number. The 2-form which really matters is $\omega/\hbar$.

Our sign convention for the Poisson bracket corresponding to ω is the following: the Hamiltonian vector field $X(f)$ defined by a function f is given by $X(f)g = \{f, g\}$, and $df = -i_{X(f)}\omega$. If $M = \mathbb{R}^2$ with coordinates p, q and if $\omega = dp \wedge dq$, then $\{p, q\} = 1$.

A quantization of M consists of a complex Hilbert space $\mathcal{H}$ and of a rule to attach to functions f on M operators $f^\wedge$ acting on $\mathcal{H}$. The rule $f \mapsto f^\wedge$ should be $\mathbb{C}$-linear; for f real, $f^\wedge$ should be Hermitian and, in a sense I will not try to make precise, $f \mapsto f^\wedge$ should almost be a homomorphism: $1^\wedge$ should be the identity and for slowly varying functions f and g,

$$(fg)^\wedge \sim f^\wedge g^\wedge \qquad \text{mod } O(\hbar). \tag{1.1}$$

If (1.1) did hold exactly, the $f^\wedge$ would mutually commute. The symplectic structure controls the failure of commutativity

$$[f^\wedge, g^\wedge] \sim -i\hbar\{f, g\}^\wedge \qquad \text{mod } O(\hbar^2). \tag{1.2}$$

Formula (1.2) can be rewritten

$$(X(f)g)^\wedge \sim (i/\hbar)[f^\wedge, g^\wedge] \qquad \text{mod } O(\hbar). \tag{1.2$'$}$$

The symplectic vector field $X(f)$, if it is well behaved at infinity, exponentiates to an automorphism $\exp(X(f))$ of (M, ω), with Taylor's formula reading

$$\exp(X(f))^* g = \exp(\text{derivation } X(f))(g).$$

Formula (1.2)′ should hold in its integrated form

$$(1.3) \qquad (\exp(X(f))^*g)^\wedge \sim \exp((i/\hbar)f^\wedge)g^\wedge \exp(-(i/\hbar)f^\wedge) \qquad \bmod O(\hbar).$$

We don't care about the exact rule $f \mapsto f^\wedge$. It matters mainly mod $O(\hbar)$.

From the deformation quantization point of view, (1.1) is the first term in an expansion (*$*$-product*)

$$(1.1)' \qquad f^\wedge g^\wedge = (fg)^\wedge + c_1(f,g)^\wedge\hbar + c_2(f,g)^\wedge\hbar^2 + \cdots,$$

with $c_i(f,g)$ given by a $\mathbb{C}$-bilinear differential operator in f and g. The formula (1.2) becomes $c_1(f,g) - c_1(g,f) = -i\{f,g\}$. If one considers quantizations $\mathcal{H}(\hbar)$,with $\hbar$ tending to 0, $(1,1)'$ can be an asymptotic expansion in $\hbar$. If one redefines $f \mapsto f^\wedge$ to be $f^\wedge + d_1(f)^\wedge\hbar + \cdots$ with $d_i(f)$ a linear differential operator in f, $(1.1)'$ changes to an *equivalent* $*$-product.

Assumption (1.1) means that $\mathcal{H}$ almost localizes on M: an element h of $\mathcal{H}$ can be decomposed as a sum $\sum h_i$ of elements localized in small regions U_i of M: if $x \in U_i$, $f^\wedge h_i$ is approximately $f(x) \cdot h_i$. Because of the noncommutativity (1.2), localization does not make sense for regions smaller than expressed by the uncertainty principle $\Delta p \Delta q \sim 2\pi\hbar$.

If we were only to assume (1.1) and (1.2), a multiple $\mathcal{H}^n$ of a quantization $\mathcal{H}$ of M would again be a quantization. A "finite multiplicity" assumption is that if f is a C^∞ function with compact support on M, the operator $f^\wedge$ is of trace class. If this holds, I would expect that for some integer m (more precisely, one for each connected component of M), one has

$$\operatorname{Tr}(f^\wedge) \sim m \int f(x)dx$$

where, in Darboux local coordinates, dx is the product of the $dp_i\, dq_i/2\pi\hbar$ (Liouville measure). We want $m = 1$:

$$(1.4) \qquad \operatorname{Tr}(f^\wedge) \sim \int f(x)dx$$

§2. Example: Cotangent Bundles

The basic example of almost localization, in the sense meant above, is when M is a cotangent bundle T^*V and $\mathcal{H}$ the space of half-densities on V. If f is peaked at x, say in a Gaussian fashion, and if g is a real function, with dg slowly varying, then $fe^{ig/\hbar}$ is localized around (x, dg_x) in M. If f is slowly varying, then $f \cdot e^{ig/\hbar}$ is localized around the section $x \mapsto dg_x$ of $M \to V$, laying over the support of f.

The operator $f^\wedge$ has a natural definition when f is affine linear on the fibers of T^*V over V. If f is the pull-back of a function on V, $f^\wedge$ is multiplication by f. If f is linear on each cotangent space, hence identified with a vector field F on V, then $f^\wedge = -i\hbar\mathcal{L}_F$. For those f, (1.2) holds exactly. To define $f^\wedge$ for more complicated functions requires auxiliary choices.

On T^*V, we have a canonical 1-form α: in local coordinates q^i on V, giving local coordinates (p_i, q^i) on T^*V, α is $\sum p_i dq^i$. The trivial unitary line bundle $\mathcal{L}$, with the connection $-i\alpha/\hbar$: $\nabla f = df - i(\alpha/\hbar)f$, is a prequantization line bundle:

its curvature is $-i\omega/\hbar$, for ω the symplectic form $d\alpha$. A real function g on V defines the Lagrangian section $L(g): x \mapsto (dg)_x$ of T^*V. The pull-back to $L(g)$ of $e^{ig/\hbar}$ is a horizontal section of $\mathcal{L}$.

For any symplectic manifold M, let Λ_M or simply Λ be the fiber space over M whose fiber Λ_m at $m \in M$ is the space of Lagrangian (= maximal isotropic) subspaces of the tangent space T_m of M at m. To take into account the Maslov index story, it is best to take prequantization line bundles as living on Λ. They should satisfy (a) (b) below.

(a) The curvature 2-form R is the pull-back from M of $-i\omega/\hbar$:

$$\nabla^2 = -i\omega/\hbar.$$

In particular, on each fiber Λ_m of $\Lambda \to M$, $(\mathcal{L}, \nabla)$ is flat. One has $\pi_1(\Lambda_m) = \mathbb{Z}$ and

(b) the monodromy of $(\mathcal{L}, \nabla)$ on Λ_m is multiplication by i.

Remarks: (i) In the same way that a Hamiltonian vector field $X(f)$ acts on a prequantization line bundle, with the action depending on f, and not only on $X(f)$, it also acts on $(\mathcal{L}, \nabla)$ as above: $X(f)$ is symplectic, hence lifts to Λ, the question is local on Λ, and locally on Λ $(\mathcal{L}, \nabla)$ is the inverse image of a prequantization line bundle on M. If we exponentiate, we see that $\exp(X(f))$ acts by an automorphism of $(M, \omega, \mathcal{L}, \nabla)$. The infinitesimal action on sections of $\mathcal{L}$ is given by

$$\partial_{X(f)} = \nabla_{X(f)} - if/\hbar \tag{2.1}$$

(ii) To make sense of (b), we should make precise our sign convention identifying $\pi_1(\Lambda_m)$ with $\mathbb{Z}$. For the symplectic vector space $\mathbb{R}^2$, with the form $dp \wedge dq$, the generator is given by the path

$$\theta \longmapsto \text{ line spanned by } (q,p) = (\cos\theta, \sin\theta) \qquad (0 \leq \theta \leq \pi).$$

(iii) Let V be a symplectic vector space, and Λ_0 be the space of Lagrangian linear subspaces of V. If S_1 and S_2 in Λ_0 intersect transversally, one defines as follows a preferred homotopy class of paths from S_1 to S_2. Identify S_2 to $S_1^\vee$ by $s_2 \mapsto \omega(s_2, s_1)$. Choose a basis e_i of S_1 and let e_1' be the dual basis of S_2. The path is

$$\theta \longmapsto \text{ subspace spanned by } (\cos\theta)e_i + (\sin\theta)e_i' \qquad (0 \leq \theta \leq \pi/2).$$

It depends only on the quadratic form for which the basis (e_i) is orthonormal. The choice of this quadratic form running over a contractible set, the homotopy class of the path does not depend on the choice. If V is of dimension $2d$, the preferred path from S_1 to S_2, followed by the preferred path from S_2 to S_1, is d times the preferred generator of $\pi_1(\Lambda_0)$.

We now come back to the case where M is a cotangent bundle T^*V. Let $\Lambda^0 \subset \Lambda$ be the subbundle with $\Lambda_m^0 \subset \Lambda_m$ consisting of those Lagrangian subspaces L of T_m intersecting transversally the vertical subspace of T_m^* (tangent to the fibers of $T^*V \to V$). If we pull back the prequantization line bundle on T^*V to Λ^0, it extends uniquely to $\mathcal{L}_\Lambda$ on Λ satisfying the conditions (a), (b).

Let L be a Lagrangian subvariety of M. Let s be the section of Λ over L: $x \mapsto$ tangent space of L at x. The section s has values in Λ^0 if and only if L is locally of the form $L(g)$. As L is Lagrangian, $s^*\mathcal{L}_\Lambda$ is flat and it makes sense to speak of a slowly varying section of $s^*\mathcal{L}$ on L. Generalizing the $f \cdot e^{ig/\hbar}$ considered previously, to a slowly varying half-density u on L with values in $s^*\mathcal{L}$ corresponds $[u]$ in $\mathcal{H}$, with

$$\|[u]\|^2 \sim \int_S \langle u, u\rangle \,. \tag{2.2}$$

It is localized near $\mathrm{Supp}(u) \subset S \subset M = T^*V$.

A function f defines an automorphism $\exp(X(f))$ of $(M, \mathcal{L}, \nabla)$ (Remark (ii) above), and an automorphism $\exp(-(i/\hbar)f^\wedge)$ of $\mathcal{H}$. Those automorphisms should (almost) preserve the construction $u \mapsto [u]$:

$$[\exp(X(f))^*u] \sim \exp((i/\hbar)f^\wedge)[u]. \tag{2.3}$$

Inner products are given as follows. Fix (S_i, u_i) $(i = 1, 2)$ as above, and assume that S_1 and S_2 intersect transversally. Let x be an intersection point. The Lagrangian subspaces $s_1(x)$ and $s_2(x)$ of T_xM intersect transversally, and the preferred class of paths γ from $s_1(x)$ to $s_2(x)$ (Remark (iii) above) gives an isomorphism $\gamma^*\colon \mathcal{L}_{s_2(x)} \to \mathcal{L}_{s_1(x)}$. The symplectic structure puts $s_1(x)$ and $s_2(x)$ in duality, identifying the line of half-densities of S_1 at x with the dual of the similar line for S_2. The inner product $\langle u_1(x), \gamma^*u_2(x)\rangle$ is hence just a number. The stationary phase approximation gives

$$\langle [u_1], [u_2]\rangle \sim \sum_{x \in S_1 \cap S_2} (2\pi i\hbar)^{d/2}\, \langle u_1(x), \gamma^*u_2(x)\rangle \tag{2.4}$$

for M of dimension $2d$.

Our convention for inner products $\langle a, b\rangle$ is: antilinear in a, linear in b.

§3.

The flat case is when M is an affine space, with a translation invariant symplectic form. In the flat case, one can define a quantization as the data of $f \mapsto f^\wedge$, just for f a linear function, with (1.2) holding exactly. The purely imaginary linear functions form a Lie algebra for the bracket $-i\hbar\{f, g\}$, and we want a unitary representation $f \mapsto f^\wedge$ of this Lie algebra, with $i^\wedge = i \cdot \mathrm{Id}$. It is more convenient to ask for a unitary representation of the corresponding Lie group. Formula (1.4) becomes a request for irreducibility. Weyl's quantization $f \mapsto f^\wedge$, for a general f, is given by

$$\exp(\ell)^\wedge = \exp(\ell^\wedge) \tag{3.1}$$

for ℓ a purely imaginary linear function, a general $f^\wedge$ being deduced from (3.1) by considering f as a superposition of $\exp(\ell)$, by Fourier transform. For Weyl quantization, formula (1.4) holds exactly.

Understood in those terms, quantization in the flat case is unique, up to isomorphism. If M is the cotangent bundle of an affine space V, §2 gives its Schrödinger model as $L^2(V)$: if V_0 is the vector space of translations of V, one has $M = V_0^\vee \times V$; for f (the pull-back of) a function on V, $f^\wedge$ is multiplication by f. For ℓ a linear form on $V_0^\vee$, identified with a vector $v \in V_0$, $\ell^\wedge$ is $-i\hbar\partial_v$, and $\exp(i\ell)^\wedge$ is $\psi(x) \mapsto \psi(x + \hbar v)$.

This uniqueness assertion depends on the assumption that the flat symplectic variety M is of finite dimension.

Remark. In the flat case, the data on Λ_M of a prequantization line bundle defines a quantization $\mathcal{H}$, as defined above, up to unique isomorphism. This is made plausible by the fact that the group of automorphisms of

$$(\text{affine symplectic space } M, \mathcal{L}_\Lambda, \nabla)$$

and of

$$(\text{affine symplectic space } M, \mathcal{H}, f \mapsto f^\wedge)$$

are the same extension of the affine symplectic group (Sp $\ltimes$ translations) by U(1).

The relation between $\mathcal{L}_\Lambda$ and the quantization $\mathcal{H}$ can be fixed as follows. Let $\mathcal{H}_\infty$ be the C^∞-vectors in $\mathcal{H}$ for the action of the Heisenberg group. In the Schrödinger model these are the Schwartz functions on V. Let $\mathcal{H}_{-\infty} \supset \mathcal{H}$ be the dual of $\mathcal{H}_\infty$. A linear Lagrangian subspace L of M defines a section s_L of Λ_M. A $s_L^*\mathcal{L}_\Lambda$-valued half-density u on L, of the form (translation invariant half-density)$\cdot$(flat section $s^*\mathcal{L}_\Lambda$), defines $[u]$ in $\mathcal{H}_{-\infty}$, and the inner product formula (2.4) holds exactly, to give the inner product of continuous superpositions of $[u]$. Formula (2.3) holds exactly, for f quadratic and u as above. In particular, if a linear form ℓ vanishes on L, one has $\ell^\wedge[u] = 0$.

§4.

We now go over to the superworld. A super Hilbert space is a mod 2 graded complex vector space H with an even sesquilinear form (u,v), antilinear in u and linear in v, for which

$$(v,u) = (-1)^{p(u)p(v)}(u,v)^-,$$

with a positivity and a completeness condition. As positivity condition, we take

$$(4.1)\qquad \begin{aligned} (u,u) > 0 &\quad \text{for} \quad u \text{ even}, \quad u \neq 0 \\ -i(u,u) > 0 &\quad \text{for} \quad u \text{ odd}, \quad u \neq 0. \end{aligned}$$

This positivity condition is stable under tensor product if (,) for $H' \otimes H''$ is defined by

$$(u' \otimes u'', v' \otimes v'') = (-1)^{p(u'')p(v')}(u',v')(u'',v''),$$

in accordance with the sign rule. The adjoint $T^\dagger$ of an operator T is defined by

$$(4.2)\qquad (Tu, v) = (-1)^{p(T)p(u)}(u, T^\dagger v).$$

The physicists don't like (4.1), and prefer to work with the ordinary mod 2 graded Hilbert space with the inner product $\langle\ ,\ \rangle$ defined by

$$(4.3)\qquad \begin{aligned} (u,v) &= \langle u,v\rangle \quad \text{for} \quad u,v \text{ even} \\ (u,v) &= i\langle u,v\rangle \quad \text{for} \quad u,v \text{ odd}. \end{aligned}$$

The adjoint $T^\dagger$ is related to the corresponding ordinary adjoint T^* by

$$(4.4)\qquad \begin{aligned} T^\dagger &= T^* \quad \text{for } T \text{ even} \\ T^\dagger &= iT^* \quad \text{for } T \text{ odd}. \end{aligned}$$

The definition (4.2) has the unsettling effect that the eigenvalues of an odd self-adjoint operator are in $i^{1/2}\mathbb{R}$. However, the eigenspaces being neither even nor odd, there is not much wish to consider them.

In a quantization of a supermanifold M, to functions f on M should correspond operators $f^\wedge$, acting on a super Hilbert space $\mathcal{H}$, with $f^\wedge$ of the same parity as f. If f is real, one should have $f^\wedge = f^{\wedge\dagger}$. Because of (3.4), if M is a supermanifold, the physicists will declare to be "real" the real even f, and the $i^{-1/2}\eta$ for η real. With this rule, if α_1 and α_2 are odd and "real", $i\alpha_1\alpha_2$ is "real" (as well as real, being even). Example: on $\mathbb{R}^{1,1}$, with coordinates (x, η), if we put $\theta = i^{-1/2}\eta$, the vector field $\partial_\theta + i\theta\partial_x$ is a multiple of a real vector field:

$$\partial_\theta + i\theta\partial_x = i^{1/2}(\partial_\eta + \eta\partial_x).$$

§5.

Let now M be a super affine space, with a translation-invariant symplectic form. In imitation of §3, we define a quantization as the data of a super Hilbert space $\mathcal{H}$, and of $\ell \mapsto \ell^\wedge$ associating to a linear function on M an operator on $\mathcal{H}$, with ℓ and $\ell^\wedge$ of the same parity. The quantization $\ell \mapsto \ell^\wedge$ should be linear, real: $\ell^\wedge = (\ell^\wedge)^\dagger$ for ℓ real, map 1 to Id, and obey

$$[\ell^\wedge, m^\wedge] = -i\hbar\{f, g\}^\wedge. \tag{5.1}$$

One should also require some irreducibility.

The condition (5.1) requires that for real linear odd functions ℓ, the Poisson bracket $\{\ell, \ell\}$ (a constant) be negative:

$$\{\ell, \ell\} \leq 0. \tag{5.2}$$

Indeed, $[\ell^\wedge, \ell^\wedge] = 2\ell^{\wedge 2} = 2\ell^{\wedge\dagger}\ell^\wedge = 2i\ell^{\wedge *}\ell^\wedge$ must be $-i\hbar\{\ell, \ell\}$. The sign in (5.2) is due to the choices of sign in (4.1) (leading to (4.4)) and (5.1), the latter repeating (1.2).

As in §3, one extends $\ell \mapsto \ell^\wedge$ to polynomial functions by requiring that for a product of even or odd linear functions, one has

$$(\ell_1 \dots \ell_n)^\wedge = (1/n!) \sum_\sigma \pm\, \ell^\wedge_{\sigma(1)} \dots \ell^\wedge_{\sigma(n)} \qquad (\sigma \in S_n)$$

with $\pm$ given by the sign rule. In particular, $\pm$ is $+$ if the ℓ_i are all even, and $\varepsilon(\sigma)$ if they are all odd.

§6.

Let W be a real vector space with a positive definite symmetric bilinear form B. Let M be the affine space defined by ΠW: it is of dimension $(0, \dim W)$, and linear forms on W are odd functions on M. Define a Poisson bracket by

$$\{\ell, m\} = -B^{-1}(\ell, m) \tag{6.1}$$

for $\ell, m \in W^\vee$.

If e_i is an orthonormal basis of W, with dual basis e^i, and if e^i is viewed as an odd function on M, the Poisson bracket (6.1) is given by the super symplectic form

$$\omega = \tfrac{1}{2}\sum de^i de^i. \tag{6.2}$$

On W^*, let Q be the quadratic form $\frac{1}{2}B^{-1}(\ell,\ell)$. If $\mathcal{H}$ is a mod 2 graded module over the (mod 2 graded) Clifford algebra $C(W^*, Q)$, with the module structure written $\mathrm{cl}(\ell)$ for $\ell \in W^*$, (5.1) holds for

$$\ell^\wedge = (i\hbar)^{1/2}\mathrm{cl}(\ell). \tag{6.3}$$

For $\dim W = 1$, one can take $\mathcal{H} = (\mathbb{C}\text{ even})\oplus(\mathbb{C}\text{ odd})$, with $\mathrm{cl}(e^1) = \begin{pmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}$. For the obvious Hilbert space structure of $\mathcal{H}$, and the corresponding super Hilbert space structure, one has then $\mathrm{cl}(e^1)^* = \mathrm{cl}(e^1)$ and for any ℓ in V^*,

$$\ell^\wedge = (\ell^\wedge)^\dagger. \tag{6.4}$$

A general W can be written as an orthogonal direct sum of lines, and the corresponding tensor product of super Hilbert spaces is a Clifford module for which (6.4) holds, if $\ell^\wedge$ is defined by (6.3). If $\mathcal{H}_1$ is a graded submodule with the induced Hilbert space structure, (6.4) continues to hold for $\mathcal{H}_1$. This proves the existence of a super Hilbert space $\mathcal{H}$ with a Clifford module structure, irreducible as a graded module, for which (6.4) holds when $\ell^\wedge$ is defined by (6.3).

We now assume W of even dimension. There are the two isomorphism classes of $\mathcal{H}$ as above, exchanged by the parity change $\mathcal{H} \mapsto \mathcal{H}\otimes L$, for L a $(0,1)$-dimensional super Hilbert space. Let $(e^1, \ldots, e^n)$ be an orthonormal basis (for B^{-1}), and $[e^1, \ldots, e^n]$ be the corresponding density on M. Another orthonormal basis would give the same, or the opposite density. The choice of the density $[e^1, \ldots, e^n]$ picks out one of the two isomorphism classes of $\mathcal{H}$: the one for which the following analogue of (1.4) holds. For any f on M, the supertrace $\mathrm{Tr}(f^\wedge)$ is given by

$$\mathrm{Tr}(f^\wedge) = \hbar^{n/2}\int [e^1, \ldots, e_n] f \tag{6.5}$$

Justification: writing $\mathcal{H}$ as a tensor product, one reduces to the case $n = 2$. For $n = 2$, $\mathrm{cl}(e^1)\mathrm{cl}(e^2)$ has square $-1/4$ and eigenvalues $\pm i/2$. It follows that $(e_1e_2)^\wedge = \frac{1}{2}(e_1^\wedge e_2^\wedge - e_2^\wedge e_1^\wedge) = e_1^\wedge e_2^\wedge = i\hbar\mathrm{cl}(e_1)\mathrm{cl}(e_2)$ has eigenvalues $\mp\hbar/2$, hence supertrace $\pm\hbar$.

§7.

A super symplectic flat space M as in §5 can be decomposed as $M^+ \times M^-$, with M^+ even and M^- odd, as in §6. The tensor product of quantizations of M^+ and M^- gives one for M. If M^+ is the cotangent bundle of an affine space V, and if M^- is obtained as in §6 from an even-dimensional quadratic vector space W, this tensor product can be realized as the space of L^2-densities on V with values in a Clifford module for $W^\vee$, and an analogue of (1.4), (6.5) holds.

§8.

Let V be a manifold and W be an orthogonal vector bundle on V, with a connection ∇ respecting the structural symmetric bilinear form B. We take M to be the fiber product over V of T^*V and ΠW. For any supermanifold S, a S-point of M (Supersymmetry 2.8), that is a morphism from S to M, can be identified with the data of an S-point $f\colon S \to V$ of V, of an even section of $f^*T^*_V$, and of an odd section of f^*W. A local coordinate system (q^i) on V and a local basis (e_α) of W give a local coordinate system (q^i, p_i, ψ^α) on M.

We assume that W is even dimensional, oriented and Spin. It gives rise to a bundle of super Hilbert spaces $\mathcal{H}_V$ on V, reproducing point by point what we got in §6, and the connection ∇ on W gives one on $\mathcal{H}_V$.

Let $\mathcal{H}$ be the super Hilbert space of half-densities on V with values in $\mathcal{H}_V$. To any function f on M, of degree ≤ 1 on the fibers of M/V, when p_i is viewed as of degree 1 and ψ^α as of degree 1/2, one associates an operator $f^\wedge$ as follows.

(a) For (the pull-back of) a function f on V, $f^\wedge$ is multiplication by f.

(b) For the odd function defined by a section e of $W^\vee$,

$$e^\wedge = (i\hbar)^{1/2}\mathrm{cl}(e)$$

as in (6.3). This is extended as in §5 to define $f^\wedge$ for any function f on ΠW.

(c) If f is a function on T^*V, linear on each fiber of T^*V/V and identified with a vector field F,

$$f^\wedge = -i\hbar\nabla_F.$$

In (c), if x in $\mathcal{H}$ is the product of a section h of $\mathcal{H}_V$ by a half-density v, $\nabla_F(x) := \nabla_F(h)\cdot v + h\cdot \mathcal{L}_F(v)$.

The super vector space of operators $f^\wedge$, for f of degree ≤ 1 on the fibers of M/V, is stable under bracket. By (1.2), the bracket of operators corresponds to a Poisson bracket, which we now compute.

For functions on ΠW, it is as in §6: functions on V commute with functions on ΠW and for ℓ, m odd linear functions, identified with sections of $W^\vee$, one has

$$\{\ell, m\} = -B^{-1}(\ell, m).$$

For f a function on T^*V, linear on the fibers and corresponding to a vector field F on V, $\{f, \ \}$ is ∇_F on functions on ΠW.

It remains to compute $[f^\wedge, g^\wedge]$ as being of the form $-i\hbar\{f,g\}^\wedge$ when f and g are linear functions on T^*V, corresponding to vector fields F and G. We have

$$\begin{aligned}[f^\wedge, g^\wedge] &= [-i\hbar\nabla_F, -i\hbar\nabla_G] = (-i\hbar)^2\left(\nabla_{[F,G]} + R(F,G)\right)\\ &= -i\hbar\left(\{f,g\}^\wedge_{T^*V} - i\hbar R(F,G)\right)\end{aligned}$$

for $\{\ ,\ \}_{T^*V}$ the Poisson bracket on T^*V and F the curvature 2-form of W, with values in $\mathrm{SO}(W)$ (which acts on $\mathcal{H}_V$).

Let S^β_α be the matrix of $S \in \mathrm{SO}(W)$, relative to a local basis e_α of W: S is $(x^\alpha) \mapsto (S^\alpha{}_\beta x^\beta)$. Let e^α be the dual basis, and move the indices up and down using B. The action of S on $\mathcal{H}_V$ is then the Clifford multiplication by $S_{\alpha\beta}e^\alpha e^\beta$ (sum on α and β) in the Clifford algebra. This follows from

$$\begin{aligned}[S_{\alpha\beta}e^\alpha e^\beta, e^\gamma] &= S_{\alpha\beta}(e^\alpha[e^\beta, e^\gamma] - [e^\alpha, e^\gamma]e^\beta) \qquad \text{(superbrackets)} \\ &= \tfrac{1}{2}S_{\alpha\beta}(e^\alpha b^{\beta,\gamma} - b^{\alpha,\gamma}e^\beta) = -S_{\alpha\beta}b^{\alpha\gamma}e^\beta = -S^\gamma{}_\beta e^\beta = S(e^\gamma)\end{aligned}$$

for the Lie algbra action.

If we apply this to the curvature, and if ψ^α are the odd functions on M corresponding to the e^α, this gives $[f^\wedge, g^\wedge] = -i\hbar\{f,g\}^\wedge$ for

$$\{f,g\} = \{f,g\}_{T^*V} - R(F,G)_{\alpha,\beta}\psi^\alpha\psi^\beta$$

This Poisson bracket corresponds to the following symplectic 2-form Ω. Let α_{T^*V} be the canonical 1-form pdq on T^*V, as well as its pull-back to M. Let $\alpha_{\Pi W}$ be the 1-form on ΠW whose inverse image by a section ψ is $\frac{1}{2}(\psi, \nabla\psi)$. If (e_α) is a local basis of W, and if the connection is given by the endomorphism-valued 1-form $\gamma^\alpha{}_\beta$:

$$\nabla(x^\alpha e_\alpha) = (dx^\alpha + \gamma^\alpha{}_\beta x^\beta)e_\alpha,$$

one has

$$\alpha_{\Pi W} = \tfrac{1}{2}\psi_\alpha(d\psi^\alpha + \gamma^\alpha{}_\beta\psi^\beta).$$

With these notations,

$$\Omega = d\alpha_{T^*V} + d\alpha_{\Pi V}.$$

If (e_α) is an orthonormal basis, one has

$$d\alpha_{\Pi V} = \tfrac{1}{2}\sum_\alpha(d\psi^\alpha + \gamma^\alpha{}_\beta\psi^\beta)^2 + \sum_{\alpha,\beta}R^\alpha{}_\beta\psi^\alpha\psi^\beta.$$