PRESCRIBING THE SPECTRA OF LOCALLY UNIFORM GEOMETRIES.

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LECTURE 1

THE BASS NOTE AND SPECTRAL RIGIDITY

CHERN LECTURES

BERKELEY 2023.

THE GENERAL PROBLEM

- S A NONCOMPACT SYMMETRIC SPACE (THAT IS A RIEMANNIAN SPACE FOR WHICH GEODESIC INVERSION AT EVERY POINT EXTENDS TO A GLOBAL ISOMETRY) OR A p-ADIC ANALOGUE.

- $\underline{P}$ A SELFADJOINT INVARIANT LINEAR DIFFERENTIAL OPERATOR ACTING ON FUNCTIONS OR FORMS — USUALLY LAPLACIAN.

- $\mathcal{F}$ A FAMILY OF COMPACT OR FINITE VOLUME QUOTIENTS OF S "LOCALLY UNIFORM GEOMETRIES"

- EXAMINE THE SPECTRA OF $\underline{P}$ AS X VARIES OVER $\mathcal{F}$.

IN ALL CASES WE FOCUS ON EXPLICIT CASES — THE SIMPLEST ARE THE RICHEST!

BASS NOTE SPECTRUM

FOR X, P LET

$\mu_P(X)$ = SMALLEST IN ABSOLUTE VALUE NON-ZERO EIGENVALUE (NON ZERO MODE) OF P ON $L^2(X)$.
THE "BASS-NOTE"

$$\mathrm{BASS}_P(\mathcal{Y}) := \overline{\{ \mu_P(X) : X \in \mathcal{Y} \}} \subset [0,\infty) .$$

IT HAS A DISCRETE AND A LIMIT POINT PART

$\cancel{\mathrm{AB}^D_P(\mathcal{Y})}$; $\mathrm{BASS}^D_P(\mathcal{Y})$, $\mathrm{BASS}^L_P(\mathcal{Y})$.

EUCLIDEAN SPACE / GEOMETRY OF NUMBERS

$S = \mathbb{R}^n$ ($n \geq 2$)

$\mathcal{Y}_n$: THE SPACE OF FLAT TORI $X_L = \mathbb{R}^n / L$, L A LATTICE ; SCALE TO HAVE VOLUME $= 1$.

$$\mathcal{Y}_n \cong PGL_n(\mathbb{Z}) \backslash PGL_n(\mathbb{R}) / K .$$

- P A HOMOGENEOUS POLYNOMIAL IN $D_1, D_2, \ldots, D_n$, $D_j = \frac{\partial}{\partial x_j}$ (OR $i \frac{\partial}{\partial x_j}$)

IN THIS RARE CASE ONE CAN COMPUTE THE SPECTRUM OF P ON $L^2(X_L)$:

$$\{ p(m) : m \in L^* \}$$ L^* THE DUAL LATTICE.

$$\mu_P(X_L) = \inf_{m \neq 0} \{ |p(m)| : m \in L^* \} .$$

BASS NOTE SPECTRUM $(n \geq 2)$:

- P-DEGREE 1 : $\mu_P(X) = 0$, $BASS_P(\mathcal{Y}_n) = \{0\}$ NO GAP (PIGEON HOLE)

- P-QUADRATIC :

OVER $\mathbb{R}$, P IS EQUIVALENT TO

$$D_1^2 + D_2^2 + \cdots D_r^2 - D_{r+1}^2 - \cdots - D_{s+r}^2$$, $r+s=n$

(r,s) SIGNATURE.

DEFINITE CASE: $P = \Delta =$ LAPLACIAN :

$$BASS_\Delta(\mathcal{Y}_n) = [0, \gamma_n^2] \cong 0 \longmapsto \gamma_n^2$$

γ_n IS HERMITE'S CONSTANT, KNOWN FOR $n = 2,3,4,5,6,7,8$, 24

E_8 AND LEECH ARE EXREMALS FOR $n=8, n=24$ ($n=24$ COHN/KUMAR)

P – QUADRATIC (INDEFINITE):

• $n = 2$; $P = W = D_1^2 - D_2^2 \cong D_1 D_2$ "WAVE OPERATOR" SIGN(1,1)

$\mathrm{BASS}_W(\mathcal{Y}_2) =$

HALL-FREIMAN RAY

TRANSITION FRACTAL LIKE

DISCRETE POINTS $m/\sqrt{9m^2-4}$

m A MARKOFF NUMBER; COORD OF $x_1^2+x_2^2+x_3^2 = 3x_1x_2x_3$

$\equiv$ MARKOFF SPECTRUM.

• $n \geq 3$: RESOLVED BY MARGULIS; KEY IS UNIPOTENT RIGIDITY (RATNER) IN $PGL_n(\mathbb{Z})\backslash PGL_n(\mathbb{R})$

$n = 3, 4$ THE BASS NOTE SPECTRUM IS <u>RIGID</u> THAT IS IT IS INFINITE DISCRETE WITH UNIQUE LIMIT POINT $\{0\}$.

SIGN(2,1)

SIGN(2,2)

SIGN(3,1)

THE INFINITE DISCRETE POINTS CORRESPOND TO THE ANISOTROPIC RATIONAL QUADRATIC FORMS.

$n \geq 5$: $\mathrm{BASS}_P(\mathcal{Y}_n) = \{0\}$, NO GAPS (NO ANISOTROPIC FORMS!)

- FOR POLYNOMIALS P OF DEGREE $d \geq 3$ LESS IS KNOWN; CORRESPOND TO "STAR BODIES" OF MAHLER.

$n=2$: (•) (G. KOTSOVOLIS (2023)): FOR P NONSINGULAR HOMOGENEOUS OF DEGREE $d \geq 3$

$\mathrm{BASS}_P(\mathcal{Y}_2)$ IS AN INTERVAL $[0, m_P]$, $m_P > 0$.

FOR $d=3$; $m_P = \sqrt[4]{\frac{-D_P}{23}}$ IF $D_P < 0$, $m_P = \sqrt[4]{\frac{D_P}{49}}$ IF $D_P > 0$, $D_P = \mathrm{disc}(P)$.

$n=3$: THE HOLY GRAIL CONJECTURE IS (CASSELS; SWINNERTON-DYER, OPPENHEIM)

$\mathrm{BASS}_{D_1 D_2 D_3}(\mathcal{Y}_3)$ IS RIGID. (AND THE SAME FOR $D_1 D_2 \cdots D_n$ ON $\mathcal{Y}_n$, $n \geq 3$).

THE BODY $|x_1 x_2 x_3| < 1$ IS AN AUTOMORPHIC STAR BODY OF MAHLER SO THAT THE SPECTRUM HAS A HOMOGENEOUS DYNAMICS INTERPRETATION. THE HIGHER RANK RIGIDITY CONJECTURES FOR DIAGONAL ACTIONS (SEE EINSIEDLER-KATOK-LINDENSTRAUSS) IMPLY THE ABOVE BASE NOTE RIGIDITY. THEIR WORK SHOWS THAT $\mathrm{BASS}_{D_1 D_2 D_3}(\mathcal{Y}_3)$ HAS ZERO HAUSDORFF DIMENSION.

IT FOLLOWS FROM DAVENPORT AND RODGERS [D-R] THAT (•)

$$\mathrm{BASS}_{D_1(D_2^2+D_3^2)}(\mathcal{Y}_3) = \left[0, \frac{2}{\sqrt{23}}\right].$$

— SEE EXTRA NOTES ON CHERN LECTURE 1.

WE FIXATE ON THREE LOCALLY UNIFORM GEOMETRIES :

$\mathbb{H}^2 = \{z = x+iy \,,\, y>0\}$ HYPERBOLIC PLANE $ds^2 = \dfrac{dx^2+dy^2}{y^2}$

$G = SL(2,\mathbb{R})$ ITS GROUP OF MOTIONS $z \xmapsto{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \dfrac{az+b}{cz+d}$

Γ A DISCRETE SUBGROUP OF G WITH $\mathrm{VOL}(\Gamma \backslash G) < \infty$; $X = \Gamma \backslash \mathbb{H}^2$ IS A HYPERBOLIC 2-ORBIFOLD.

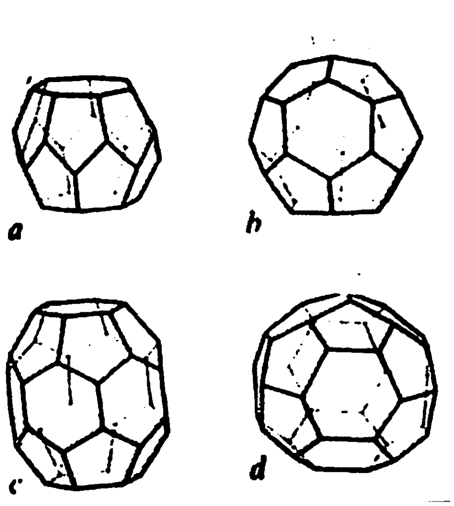
$\mathbb{H}^3 = \{w = (y,z) \,;\, y>0, z \in \mathbb{C}\}$ HYPERBOLIC 3-SPACE $ds^2 = \dfrac{dy^2+dz^2}{y^2}$

$G = SL_2(\mathbb{C})$ GROUP OF MOTIONS

Γ A DISCRETE SUBGROUP OF G WITH $\mathrm{VOL}(\Gamma \backslash G) < \infty$

$X = \Gamma \backslash \mathbb{H}^3$ IS A HYPERBOLIC 3-ORBIFOLD.

3-REGULAR FINITE CONNECTED GRAPH (NO LOOPS)

$X =$ 

CAN BE REALIZED AS

$\Gamma \backslash PGL_2(\mathbb{Q}_2) / PGL(\mathbb{Z}_2)$

- THE SET OF FINITE CONNECTED 3-REGULAR GRAPHS IS DENOTED

$$\boxed{\text{CUBIC}}$$

- THE SET OF FINITE VOLUME HYPERBOLIC d-DIMENSIONAL ORBIFOLDS IS DENOTED

$$\boxed{\text{HYP}_d} \qquad d = 2,3$$

- FOR $X \in \text{CUBIC}$, $|V(X)| = n$ IS EVEN

THE LAPLACIAN Δ IS

$$\Delta f(v) = d_v f(v) - \sum_{w \sim v} f(w) = 3f(v) - \sum_{w \sim v} f(w)$$

FOR $f : V(X) \to \mathbb{C}$

$$\sigma_\Delta(X) = \{ 0 = \lambda_0(X) < \lambda_1(X) \leq \cdots \leq \lambda_{n-1}(X) \} \subset [0,6].$$

- FOR $X \in \text{HYP}_d$, THE $L^2(X)$ SPECTRUM OF THE LAPLACIAN Δ ON FUNCTIONS ON X IS

$$\sigma_\Delta(X) = \{ 0 = \lambda_0(X) < \lambda_1(X) \leq \cdots \} \subset [0,\infty).$$

FOR $\mathcal{Y}$ A SUBSET OF CUBIC OR HYP_d

$$\mathrm{BASS}_\Delta(\mathcal{Y}) = \overline{\{ \lambda_1(X) : X \in \mathcal{Y} \}} .$$

- THE UNIVERSAL COVER OF $X \in$ CUBIC IS T_3 THE THREE REGULAR TREE AND ITS L^2 SPECTRUM IS

$$\sigma_\Delta(T_3) = [\alpha, 6-\alpha] , \quad \alpha = 3 - 2\sqrt{2} = 0.1715\ldots$$

(KESTEN ; SALLY-SHALIKA 2-ADIC PLANCHEREL MEASURE).

- THE UNIVERSAL COVER OF $X \in HYP_d$ IS $\mathbb{H}^d$, AND ITS L^2-SPECTRUM IS

$$\sigma_\Delta(\mathbb{H}^d) = \left[\left(\frac{d-1}{2}\right)^2, \infty \right) .$$

(HARISH CHANDRA IN GENERAL)

"LINEAR PROGRAM" FOR CUBIC SATURATES:

$$\text{TRACE}\,(\Delta(X)) = 3n = \lambda_1 + \lambda_2 \cdots + \lambda_{n-1}$$

$$\text{SO} \qquad \mu_\Delta(X) = \min_{j\geq 1} \lambda_j \leq \frac{\lambda_1 + \cdots + \lambda_{n-1}}{n-1} \leq \frac{3n}{n-1} \leq 4 \qquad (n \geq 4)$$

FOR $n=4$ THIS IS ACHIEVED AND IS UNIQUE!

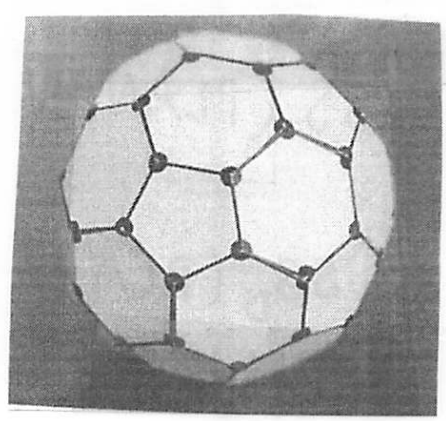
$X = K_4$  ; $\mu = 4$

$X = K_{3,3}$  ; $\mu = 3$

⋮

$X =$ PETERSON  ; $\mu = 2$

⋮

$X =$ FOOTBALL  ; $\mu = 0.234..$

$$\alpha = 3 - 2\sqrt{2} = 0.1715..$$

THEOREM (- - - -) BASS NOTE SPECTRUM FOR CUBIC

$\mathrm{BASS}_\Delta(\mathrm{CUBIC}) =$ 

$\mathrm{BASS}^D_\Delta(\mathrm{CUBIC}) = \{4, 3, 2, 2, 2, 2, \ldots\} \subset (\alpha, 4]$

DISCRETE AND INFINITE !

$\mathrm{BASS}^L_\Delta(\mathrm{CUBIC}) = [0, \alpha]$

IF PLANAR CONSISTS OF THE MEMBERS OF CUBIC THAT ARE PLANAR ; THEN THE BASS NOTE SPECTRUM RIGIDIFIES

$\mathrm{BASS}_\Delta(\mathrm{PLANAR}) =$ 

$\mathrm{BASS}^D_\Delta(\mathrm{PLANAR}) = \{4, 2, 2, \ldots\} \subset [0, 4]$

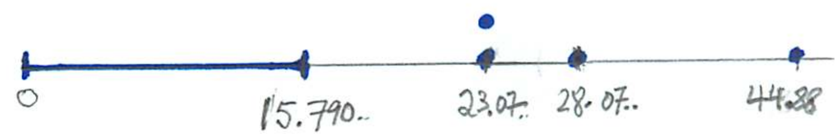
$\mathrm{BASS}^L_\Delta(\mathrm{PLANAR}) = \{0\}.$

- DISCUSS FURTHER IN LECTURE 2.

THEOREM: BASS NOTE DUAL OF HYP_2

(KRAVCHUK, MAZAC, PAL 2021)

$BASS(HYP_2) =$



$BASS_D(HYP_2)$:

X	$[2,3,7]$	$[2,4,5]$	$[2,3,8]$	$[3,3,4]$
$\lambda_1(X)$	44.888...	28.079	23.078..	23.078...

$[a,b,c]$ IS THE TRIANGLE GROUP WITH ORDERS a, b, c.

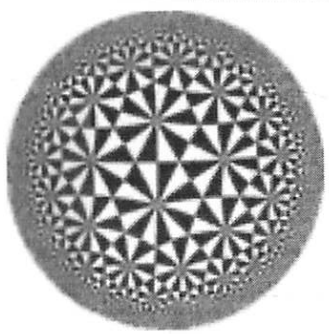
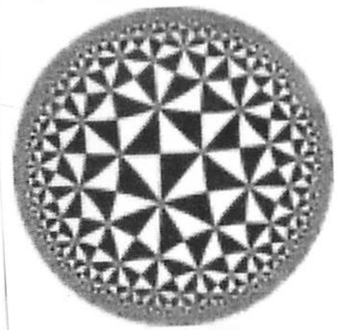
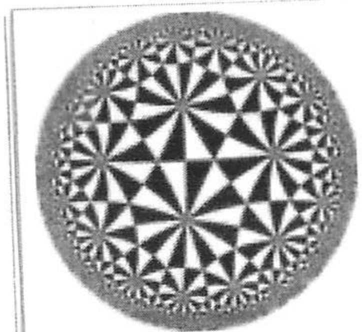
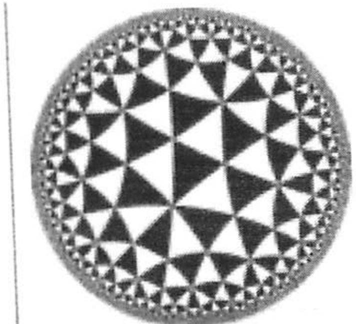
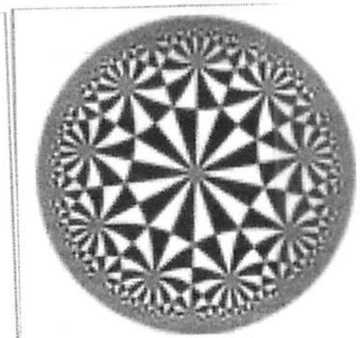
(ALL ARITHMETIC AND CONGRUENCE!)

$$BASS_L(HYP_2) = \lambda_1(T) = [0, \lambda_{MAX}(T)]$$

WHERE T IS THE TEICHMULLER SPACE OF SURFACES OF SIGNATURE $g=0$, $[2,2,2,3]$

$$\lambda_{MAX}(T) = 15.79023\ldots$$

IS ACHIEVED AT A UNIQUE $X \in T$ WHICH IS COMMESURABLE WITH $X' = [2,3,9]$ AND HAS THE SAME λ_1.

X	 (2 3 7)	 (2 4 5)	 (2 3 8)	 (3 3 4)	 (2 3 9)
$\lambda_1(x)$	44.888..	28.079...	23.078..	23.078...	15.7902....
AREA(X)	0.1496..	0.9424..	0.2618...	0.5235..	0.3490..

COMMENTS ABOUT THE PROOF

- IT IS BASED ON WELL DEVELOPED BOOTSTRAP METHODS THAT CONSTRAIN SPECTRA OF TWO DIMENSIONAL CONFORMAL FIELD THEORIES USING CONSISTENCY CONDITIONS AND POSITIVITY OF CORRELATION FUNCTIONS TO SET UP LINEAR PROGRAMS.

FOR THE SIMPLER CASE AT HAND ONE DECOMPOSES $L^2(\Gamma \backslash SL_2(\mathbb{R}))$ INTO IRREDUCIBLES; AND FORMS FOUR POINT CORRALATION FUNCTIONS INVOLVING HOLOMORPHIC AND ANTI-HOLOMORPHIC FORMS. THE LINEAR PROGRAM GIVES UPPER BOUNDS FOR $\lambda_1(\Gamma \backslash \mathbb{H})$ THAT DEPEND ONLY ON THE LEAST k FOR WHICH $\Gamma \backslash \mathbb{H}$ HAS HOLOMORPHIC FORMS OF WEIGHT k.

- FOR $\Gamma = [2,3,7]$ THE UPPER BOUND APPARENTLY SATURATES AT THE TRUE VALUE!

- THE SATURATION IS SIMILAR TO THE $\boxed{\text{COHN-ELKIES}}$ LINEAR PROGRAM WHICH SATURATES FOR THE 8 AND 24 DIMENSIONAL SPHERE PACKING PROBLEM.

$\boxed{\text{HARTMAN-MAZAC-RASTELLI}}$ GIVE A DIRECT RELATION BETWEEN CERTAIN CONFORMAL BOOT STRAP LINEAR PROGRAMS AND COHN-ELKIES AS WELL AS THE EXTREMAL "MAGIC" FUNCTIONS OF VIAZOVSKA AND VIAZOVSKA ETAL WHICH ESTABLISH THIS SATURATION.

BASS NOTE SPECTRUM FOR ARITHMETIC SURFACES

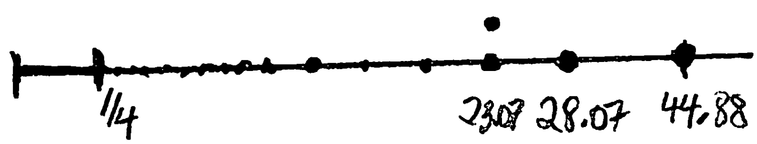
THE BASIC ARITHMETIC GROUP IS THE MODULAR GROUP

$$SL_2(\mathbb{Z}) \quad \text{IN} \quad SL_2(\mathbb{R})$$

GROUPS Γ WHICH ARE COMMENSURABLE TO $SL_2(\mathbb{Z})$ OR GROUPS LIKE IT ARE CALLED <u>ARITHMETIC</u>,

<u>CONJECTURE:</u>

$\mathrm{BASS}_\Delta(\mathrm{ARITH}_2) =$



$$\mathrm{BASS}^D_\Delta(\mathrm{ARITH}_2) = \underbrace{\{44.88,\ 28.07,\ \ldots\ldots\}}_{\text{INFINITE}} \subset \left(\tfrac{1}{4}, \infty\right)$$

$$\mathrm{BASS}^L_\Delta(\mathrm{ARITH}_2) = \left[0, \tfrac{1}{4}\right]$$

QUITE A LOT IS KNOWN TOWARDS THIS:

- THAT $\text{BASS}_\Delta(\text{ARITH}_2)$ IS DISCRETE IN $(\frac{1}{4},\infty)$ FOLLOWS FROM THE FACT THAT THE NUMBER OF $X \in \text{ARITH}_2$ WITH $\text{VOL}(X) \leq C$ IS FINITE (BOREL-PRASAD) AND USING THE TRACE FORMULA

$$\varlimsup_{\text{VOL}(X)\to\infty} \lambda_1(X) \leq \frac{1}{4} \qquad \text{(HUBER)}.$$

- A RECENT BREAKTHROUGH BY HIDE-MAGEE GIVES $\text{BASS}_\Delta^L(\text{ARITH}_2)$ IS INFINITE AND CONTAINS $\{0,\frac{1}{4}\}$
- EVEN MORE RECENTLY MAGEE SHOWS THAT IT CONTAINS $[0,\frac{1}{4}]$.

- SO THE CONJECTURE IS RESOLVED EXCEPT FOR THE PART THAT $\text{BASS}_\Delta^D(\text{ARITH}_2)$ IS INFINITE.

- WHAT THEY (HIDE-MAGEE) SHOW IS THAT A RANDOM n-SHEETED COVER X_n OF A FIXED Y HAS

$$\lim_{n\to\infty} \lambda_1(X_n) = \min\left(\lambda_1(Y), \frac{1}{4}\right).$$

- THE PROOF MAKES USE OF RESULTS FROM $\mathbb{C}^*$-ALGEBRAS; STRONG CONVERGENCE OF SPECTRA IN FREE PROBABILITY (HAAGERUP-THORBJORSEN; BORDENAVE-COLLINS)

BASS NOTE SPECTRUM FOR CONGRUENCE SURFACES

SUBGROUPS OF $SL_2(\mathbb{Z})$ WHICH CONTAIN

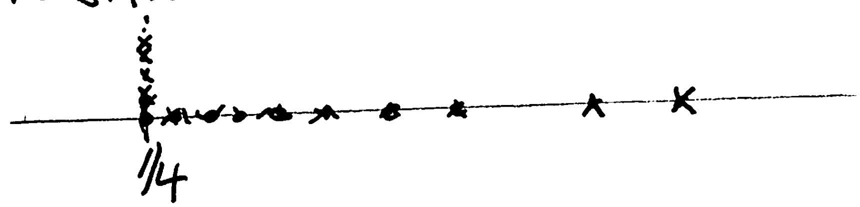
$$\Gamma(N) = \{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv I\,(N) \} \quad \text{SOME } N \in \mathbb{N}$$

AND GROUPS LIKE THESE ARE CALLED CONGRUENCE.

THE HOLY GRAIL IS THE RIGIDIFICATION OF THE BASS NOTE SPECTRUM AT $1/4$:

CONJECTURE (RAMANUJAN-SELBERG + EPSILON).

$$\mathrm{BASS}_\Delta(\mathrm{CONG}_2) =$$



$\mathrm{BASS}^D_\Delta(\mathrm{CONG}_2) \subset (\tfrac{1}{4}, \infty)$ IS INFINITE (+EPS) WITH ONLY LIMIT POINT $1/4$.

$$\mathrm{BASS}^L_\Delta(\mathrm{CONG}_2) = \{\tfrac{1}{4}\}.$$

KNOWN:

$$\mathrm{BASS}_\Delta(\mathrm{CONG}_2) \subset [0.238037\,,\ 44.88..]$$

(KIM-S, BLOMER-BRUMLEY).

BASS NOTE ON SPINORS:

GIVEN $X = \Gamma \backslash \mathbb{H}$ A HYPERBOLIC ORBIFOLD AND A SPIN STRUCTURE DEFINE THE BASS NOTE A THE BOTTOM OF THE SPECTRUM OF THE DIRAC LAPLACIAN ~~OPERATOR~~ D ON SPINORS (NO FORCED ZERO MODE!).

$\boxed{\text{• ON THE UNIVERSAL COVER } \mathbb{H} \text{ THE } L^2\text{-SPECTRUM OF } D \text{ IS } [0,\infty).}$

• GESTEAU-PAL-SIMMONS-DUFFIN-XU HAVE RUN THE CONFORMAL BOOTSTRAP WHICH YIELDS A WEALTH OF INFORMATION ON $BASS_D(HYP_2)$ AND ESPECIALLY ITS RELATION TO $BASS_\Delta(HYP_2)$.

THE BASS NOTE SPECTRUM OF D ON $ARITH_2$ IS MORE MYSTEROUS.

THEOREM (A. ADVE, V. GIRI 2023) $BASS_D^L(ARITH)$ HAS INFINITELY MANY POINTS, IN PARTICULAR IS NOT RIGID.

• THE PROOF USES CAREFULLY ENGINEERED LARGE ABELIAN COVERS OF AN ARITHMETIC SURFACE OF GENUS 2.

BASS NOTE SPECTRUM FOR HYP_3:

CONJECTURE:

$\mathrm{BASS}_\Delta(HYP_3) =$ 

$\mathrm{BASS}^D_\Delta(HYP_3) \subset (1,\infty)$ AND IS DISCRETE AND INFINITE.

$\mathrm{BASS}^L_\Delta(HYP_3) = [0,1]$.

COMMENTS

- JORGENSEN AND THURSTON SHOWED THAT HYP_3 IS WELL ORDERED BY VOLUME. THE LIMIT POINTS CORRESPOND TO VOLUMES OF CUSPED Y's WITH $Y_j \to Y$ SUITABLY. THEIR SET IS THE "VOLUME SPECTRUM OF HYP_3".

- COLBOIS/COURTOIS COUPLED WITH CHAVEL/DODZIUK SHOW THAT

$$\lambda_1(Y_j) \to \lambda_1(Y) \quad \text{AS} \quad j \to \infty$$

SINCE $\lambda_1(Y) \leq 1$ AND

$$\varlimsup_{VOL(X)\to\infty} \lambda_1(X) \leq 1 \qquad \text{(HUBER)}$$

IT FOLLOWS THAT

$$\mathrm{BASS}_\Delta(HYP_3) \text{ IS DISCRETE IN } (1,\infty).$$

IT IS KNOWN THAT $\lambda_1(Y) \geq 1$ FOR CERTAIN EXPLICIT CUSPED Y'S (GRUNEWALD-ELSTRODT-MENNICKE)

$$\Longrightarrow \qquad \mathrm{BASS}^{L}_{\Delta}(\mathrm{HYP}_3) \supset \{0,1\}.$$

- BONIFACIO-MAZAC AND PAL HAVE A WEALTH OF DATA ON $\mathrm{BASS}_{\Delta}(\mathrm{HYP}_3)$ AS WELL AS BOOTSTRAP BOUNDS RELATING $\mathrm{BASS}_{\Delta}(\mathrm{HYP}_3)$ AND $\mathrm{BASS}_{\Delta_1}(\mathrm{HYP}_3)$ (SEE BELOW)

INTERESTINGLY THE $X \in \mathrm{HYP}_3$ WITH LARGEST BASS NOTE THAT THEY FIND IS NOT THE MARSHALL-MARTIN ORBIFOLD OF SMALLEST VOLUME ($= 0.0391..$) BUT RATHER ONE OF VOLUME 0.0527. THE MARSHALL-MARTIN ORBIFOLD DOES APPEAR TO MAXIMIZE THE BASE NOTE OF THE LAPLACIAN ON OTHER TENSORS.

FOR $\mathcal{Y} = \mathrm{ARITH}_3$ AS WELL AS FOR $\mathcal{Y} = \mathrm{CONG}_3$ THE BASS NOTE SPECTRA OF Δ HAVE SIMILAR CONJECTURED FORMULATIONS AND STRUCTURE TO CONG_2 AND ARITH_2.

• WHAT APPEARS TO BE QUITE DIFFERENT IS THE BASS NOTE SPECTRUM OF THE HODGE LAPLACIAN Δ_1 ON 1-FORMS. BY THE HODGE THEOREM THE BASS NOTE (MEANING λ_0 IN THIS CASE) OF Δ_1 IS 0 IFF X IS A RATIONAL HOMOLOGY 3-SPHERE.

• THE L^2 SPECTRUM OF Δ_1 ON $\mathbb{H}^3$ IS $[0,\infty)$

RIGIDITY QUESTION: IS $\mathrm{BASS}_{\Delta_1}(\mathrm{HYP}_3)$ RIGID? THAT IS DOES $\lambda_0(\Delta_1(X_j)) \to 0$ ALONG EVERY INFINITE SEQUENCE OF X_j'S ?

STICKING TO HYPERBOLIC RATIONAL HOMOLOGY 3-SPHERES, THE SPECTRUM OF Δ_1 CONSISTS OF TWO PARTS CORRESPONDING TO THE EXACT AND COEXACT FORMS. THE EXACT PART IS THE NON-ZERO EIGENVALUES OF THE LAPLACIAN Δ_0 ON FUNCTIONS AND DENOTE BY $\lambda^*_{\mathrm{COEXACT}}(X)$ THE SMALLEST EIGENVALUE OF Δ_1 ON CO-EXACT FORMS. LIN AND LIPNOWSKI GIVE APPLICATIONS OF THE $\lambda^*_{\mathrm{COEXACT}}$ GAP TO THE SEIBERG-WITTEN EQUATIONS ON X.

THIS LEADS TO THE $\mathrm{BASS}_{\mathrm{COEXACT}}(\mathrm{HYP}_3)$ QUESTION.

LET IHS DENOTE THE SUBSET OF HYP_3 WHICH ARE INTEGRAL HOMOLOGY SPHERES. FOR N AN INTEGER LET $P(N)$ DENOTE THE MEMBERS X OF HYP_3 FOR WHICH $\pi_1(X)$ IS GENERATED BY FEWER THAN N ELEMENTS.

IN THEIR STUDY OF $\lambda^*_{\mathrm{COEXACT}}$, ABDURRAHMAN, ADVE, GIRI, LOWE AND ZUNG SHOW THAT

- $\mathrm{BASS}_{\mathrm{COEXACT}}(\mathrm{IHS})$ IS NOT RIGID, IN FACT HAS INFINITELY MANY LIMIT POINTS.
- FOR ANY N;
  $\mathrm{BASS}_{\mathrm{COEXACT}}(\mathrm{IHS} \cap P(N))$ IS RIGID.

RIGIDITY OF THE SPECTRUM:

DEFINITION: A SEQUENCE X_j OF QUOTIENTS $\Gamma\backslash S$ BENYAMINI-SCHRAAM CONVERGES TO S IF ARBITRARY LARGE BALLS ABOUT THE RANDOM POINT IN X_j ARE ISOMETRIC WITH SUCH A BALL IN S, AS $j \to \infty$.

- M. FRACZYK HAS SHOWN THAT IF $VOL(X) \to \infty$ WITH $X \in CONG_d$, $d=2,3$ THEN X B-S CONVERGES TO $\mathbb{H}^d$.

IT FOLLOWS THAT BOTH

$$BASS_{\not D}(CONG_2) \qquad \text{SPINORS}$$

AND

$$BASS_{\Delta_1}(CONG_3) \,. \qquad \text{1-FORMS}$$

ARE RIGID; THEY ARE INFINITE DISCRETE IN $(0,\infty)$ WITH 0 AS THE ONLY LIMIT POINT.

HIGHER RANK S:

VARIOUS RIGIDITY SETS IN · LEADING TO SPECTRAL RIGIDITY.

(1) THE FAMILY $\mathcal{F}$ OF QUOTIENTS ARE ALL ARITHMETIC (MARGULIS); AND USUALLY EVEN CONGRUENCE SO THE SPECTRA OF ANY P BECOME PART OF THE GENERAL RAMANUJAN CONJECTURES (DISCUSS IN LECTURE 4).

(2) ANY SEQUECE OF QUOTIENTS X_j OF S B-S CONVERGE TO S AS $\mathrm{VOL}(X_j) \to \infty$

ABERT-BERGERON-BRINGER-GELANDER-NIKOLOV-RAMBAULT-SEMET.

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