Prescribing the Spectra of Locally Uniform Geometries

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Lecture 1

The Bass Note and Spectral Rigidity

Chern Lectures

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THE GENERAL PROBLEM

- $S$, a noncompact symmetric space (that is a Riemannian space for which geodesic inversion at every point extends to a global isometry) or a $p$-adic analogue.

- $P$, a self-adjoint invariant linear differential operator acting on functions or forms — usually Laplacian.

- $Y$, a family of compact or finite volume quotients of $S$ "locally uniform geometries."

- Examine the spectra of $P$ as $X$ varies over $Y$.

In all cases we focus on explicit cases — the simplest are the richest!
BASS NOTE SPECTRUM

For \( x, P \) let

\[ \mu_P(x) = \text{smallest in absolute value non-zero eigenvalue (non-zero mode) of } P \text{ on } L^2(x). \]

The "bass-note"

\[ \text{BASS}_P(\mathcal{Y}) := \{ \mu_P(x) : x \in \mathcal{Y} \} \subset [0, \infty). \]

It has a discrete and a limit point part

\[ \text{BASS}^D_P(\mathcal{Y}); \text{ BASS}^D_P(\mathcal{Y}), \text{ BASS}_P^L(\mathcal{Y}). \]

EUCLIDEAN SPACE / GEOMETRY OF NUMBERS

\[ S = \mathbb{R}^n \quad (n \geq 2). \]

\[ \mathcal{N}_n : \text{The space of flat tori } \quad \mathcal{X}_L = \mathbb{R}^n/L \quad \text{, } L \text{ a lattice; scale to have volume } = 1. \]

\[ \mathcal{N}_n \cong \text{PGL}_n(\mathbb{Z})/\text{PGL}_n(\mathbb{R}) \quad /K. \]
A homogeneous polynomial in
\[ D_1, D_2, \ldots, D_n, \quad D_j = \frac{\partial}{\partial x_j} \text{ (or } -\frac{\partial}{\partial x_j} \text{)} \]

In this rare case one can compute the spectrum of \( P \) on \( L^2(\mathbb{R}^n) \):

\[
\{ p(m) : m \in \mathbb{L}^* \} \quad \mathbb{L}^* \text{ the dual lattice.}
\]

\[
\lambda_2 (\mathbb{R}^n) = \inf \sum_{m \rightarrow 0} | p(m) | : m \in \mathbb{L}^* \}
\]

**Bass note spectrum \((n \geq 2)\):**

- **P - degree 1**: \( \mu_p (x) = 0 \), \( \text{Bass}_p (\mathbb{R}) = \{ 0 \} \) no gap.
  (Pigeon hole)

- **P - quadratic**: over \( \mathbb{R} \), \( P \) is equivalent to
  \[
  D_1^2 + D_2^2 + \ldots + D_n^2 - D_{n+1}^2 - \ldots - D_{n+t}^2, \quad t + s = n
  \]

Definite case: \( P = \Delta = \text{Laplacian} \):

\[
\text{Bass}_\Delta (\mathbb{R}^n) = \begin{bmatrix} 0, & \chi_n^2 \end{bmatrix}
\]

\( \chi_n \) is Hermite's constant, known for \( n = 2, 3, 4, 5, 6, 7, 8 \), 24 to 24, \( \text{Cohn/Kumar} \).
P - QUADRATIC (INDEFINITE):

\[ n = 2 \quad ; \quad P = W = D_1^2 - D_2^2 = D_1 D_2 \quad \text{"WAVE OPERATOR" SIGN}(1,1) \]

\[ \text{BASS}_n(\frac{1}{2}) = \]

- HALL-FREIMAN RAY
- TRANSITION
- FRACTAL LIKE
- DISCRETE POINTS

\[ m = \sqrt{9m^2 - 4} \]

\[ m \text{ A MARKOFF NUMBER}; \quad \text{COORD OF} \quad 2x_1^2 + 2x_2^2 + x_3^2 = 3x_1x_2x_3 \]

\[ \text{MARKOFF SPECTRUM} . \]

\[ n \geq 3 \quad : \quad \text{RESOLVED BY MARGULIS'; KEY IS UNIPOTENT RIGIDITY (RATNER) IN PGL_n(\mathbb{Z}) | PGL_n(\mathbb{R})} \]

\[ n = 3, 4 \quad \text{THE BASS NOTE SPECTRUM IS RIGID} \]

\[ \text{THAT IS IT IS INFINITE DISCRETE WITH UNIQUE LIMIT POINT} \] \[ \mathbf{\Sigma}_0^2 \]

\[ \text{THE INFINITE DISCRETE POINTS CORRESPOND TO THE ANISOTROPIC RATIONAL QUADRATIC FORMS} \]

\[ n \geq 5 \quad \text{BASS}_n(\frac{1}{2}) = \mathbf{\Sigma}_0^2 \quad \text{NO GAPS} \quad \text{(NO ANISOTROPIC FORMS!)} \]
For polynomials $P$ of degree $d \geq 3$, less is known; correspond to "star bodies" of Mahler.

$n=2$: Kotsovoulos (2023): For $P$ nonsingular homogeneous of degree $d \geq 3$

\[ \text{Bass}_P(\mathbb{F}_2) \text{ is an interval } [0, m_P], m_P > 0. \]

For $d=3$:
\[ m_P = \sqrt[4]{\frac{-D_p}{a_3}} \quad \text{if } D_p < 0, \quad m_P = \sqrt[4]{\frac{D_p}{49}} \quad \text{if } D_p > 0, \]

\[ D_p = d_{15C}(\mathbb{F}_2). \]

$n=3$: The Holy Grail Conjecture is (Cassels; Swinnerton-Dyer, Oppenheim)

\[ \text{Bass}_{D_1D_2D_3}(\mathbb{F}_3) \text{ is rigid. (And the same for } D_1D_2D_n \text{ on } \mathbb{F}_n \text{ for } n \geq 3).} \]

The body \[ (x_1x_2x_3) < 1 \text{ is an automorphic star body of Mahler so that the spectrum has a homogeneous dynamics interpretation. The higher rank rigidity conjectures for diagonal actions (see Einsiedler-Katok-Lindenstrauss) imply the above base note rigidity. Their work shows that } \text{Bass}_{D_1D_2D_3}(\mathbb{F}_3) \text{ has zero Hausdorff dimension.} \]

It follows from Davenport and Rodgers [D-R] that

\[ \text{Bass}_{D_1(D_2^2 + D_3^2)}(\mathbb{F}_3) = [0, \frac{2}{\sqrt{23}}]. \]

See extra notes on Chern Lecture 1.
We fixate on three locally uniform geometries:

\[ H^2 = \{ z = x + iy, y > 0 \} \] hyperbolic plane \[ ds^2 = \frac{dx^2 + dy^2}{y^2} \]

\[ G = \text{SL}(2, \mathbb{R}) \text{ its group of motions} \]

\[ \Gamma \text{ a discrete subgroup of } G \text{ with } \text{Vol}(\Gamma \backslash G) < \infty; \]

\[ X = \Gamma \backslash H^2 \text{ is a hyperbolic 2-orbifold.} \]

\[ H^3 = \{ w = (y, z), y > 0, z \in \mathbb{C} \} \] hyperbolic 3-space \[ ds^2 = \frac{dy^2 + dz^2}{y^2} \]

\[ G = \text{SL}_2(\mathbb{C}) \text{ group of motions} \]

\[ \Gamma \text{ a discrete subgroup of } G \text{ with } \text{Vol}(\Gamma \backslash G) < \infty \]

\[ X = \Gamma \backslash H^3 \text{ is a hyperbolic 3-orbifold.} \]

\[ \Gamma \text{ regular finite connected graph (no loops)} \]

\[ \text{can be realized as} \quad \Gamma \backslash \text{PGL}_2(\mathbb{Q}_2) / \text{PGL}(\mathbb{Z}_2) \]
• The set of finite connected 3-regular graphs is denoted \textbf{Cubic}.

• The set of finite volume hyperbolic $d$-dimensional orbifolds is denoted \textbf{Hyp}_d, \quad d = 2, 3.

• For $x \in \text{Cubic}$, $|V(x)| = n$ is even, the Laplacian $\Delta$ is

\[
\Delta f(u) = d_e f(u) - \sum_{w \sim u} f(w) = 3 f(u) - \sum_{w \sim u} f(w)
\]

For $f : V(x) \to \mathbb{C}$

\[
\sigma_{\Delta} (x) = \{ 0 = \lambda_0 (x) < \lambda_1 (x) \leq \cdots \leq \lambda_{n-1} (x) \} \subset [0, \infty). \]

• For $x \in \text{Hyp}_d$, the $L^2(x)$ spectrum of the Laplacian $\Delta$ on functions on $X$ is

\[
\sigma_{\Delta} (x) = \{ 0 = \lambda_0 (x) < \lambda_1 (x) \leq \cdots \} \subset [0, \infty). \]
For $\tilde{\gamma}$ a subset of cubic or hyperbolic

$$\text{Bass}_\Delta(\tilde{\gamma}) = \sum \lambda_i(x), \; x \in \tilde{\gamma}$$

- The universal cover of $X \in \text{cubic}$ is $T_3$, the three regular tree and its $L^2$-spectrum is

$$\sigma_\Delta(T_3) = [\alpha, 6-\alpha], \; \alpha = 3 - 2\sqrt{2} = 0.1715...$$

(KESTEN; SALLY-SHALIKA $2$-ADIC PLANCHEREL MEASURE)

- The universal cover of $X \in \text{hyperbolic}$ is $\mathbb{H}^d$, and its $L^2$-spectrum is

$$\sigma_\Delta(\mathbb{H}^d) = \left[\left(\frac{d-1}{2}\right)^2, \infty\right)$$

(HARISH CHANDRA IN GENERAL)
"Linear Program" for cubic saturates:

\[
\text{Trace } (\Delta(X)) = 3n = \lambda_1 + \lambda_2 + \ldots + \lambda_{n-1}
\]

so \[
\mu_\Delta(X) = \min_{j \neq 1} \lambda_j \leq \frac{\lambda_1 + \ldots + \lambda_{n-1}}{n-1} \leq \frac{3n}{n-1} \leq 4 \quad (n \geq 4)
\]

For \(n = 4\) this is achieved and is unique!

\[
X = K_4 \quad ; \quad \mu = 4
\]

\[
X = K_{3,3} \quad ; \quad \mu = 3
\]

\[
X = \text{Peterson} \quad ; \quad \mu = 2
\]

\[
X = \text{Football} \quad ; \quad \mu = 0.234
\]

\[
\kappa = 3 - 2\sqrt{2} = 0.1715
\]
Theorem: Bass Note Spectrum for Cubic

\[ \text{BASS}_\Delta (\text{CUBIC}) = [0, \alpha] \]

\[ \text{BASS}_D^\Delta (\text{CUBIC}) = \{ 4, 3, 2, 2, 2, ... \} \subset (\alpha, 4] \]

Discrete and Infinite!

If planar consists of the members of cubic that are planar, then the bass note spectrum rigidifies

\[ \text{BASS}_\Delta (\text{PLANAR}) = [0, \alpha] \]

\[ \text{BASS}_D^\Delta (\text{PLANAR}) = \{ 4, 2, 2, ... \} \subset (0, 4] \]

\[ \text{BASS}_L^\Delta (\text{PLANAR}) = \{ 0 \} \]

Discuss further in Lecture 2.
Theorem: Bass Note Dual of $\text{Hyp}_2$

(Kravchuk, Mazac, Pal 2021)

$Bass(\text{Hyp}_2) = \ldots$

$Bass_d(\text{Hyp}_2):$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$[2,3,7]$</th>
<th>$[2,4,5]$</th>
<th>$[2,3,8]$</th>
<th>$[3,3,4]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda(x)$</td>
<td>44.888\ldots</td>
<td>28.079</td>
<td>23.078\ldots</td>
<td>23.078\ldots</td>
</tr>
</tbody>
</table>

$[a,b,c]$ is the triangle group with orders $a, b, c$.

(all arithmetic and congruence!)

$Bass_L(\text{Hyp}_2) = \lambda_1(T) = [0, \lambda_{\text{max}}(T)]$

Where $T$ is the Teichmüller space of surfaces of signature $g=0$, $[2,2,2,3]$

$\lambda_{\text{max}}(T) = 15.79023\ldots$

is achieved at a unique $x \in T$ which is commensurable with $x' = [2,3,9]$ and has the same $\lambda_1$. 
<table>
<thead>
<tr>
<th>$X$</th>
<th>$\lambda(x)$</th>
<th>AREA($x$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(237)</td>
<td>44.888...</td>
<td>0.1496...</td>
</tr>
<tr>
<td>(245)</td>
<td>28.079...</td>
<td>0.9424...</td>
</tr>
<tr>
<td>(238)</td>
<td>23.078...</td>
<td>0.2618...</td>
</tr>
<tr>
<td>(334)</td>
<td>23.078...</td>
<td>0.5235...</td>
</tr>
<tr>
<td>(239)</td>
<td>15.7902...</td>
<td>0.3490...]</td>
</tr>
</tbody>
</table>
• It is based on well developed bootstrap methods that constrain spectra of two dimensional conformal field theories using consistency conditions and positivity of correlation functions to set up linear programs.

For the simpler case at hand one decomposes $L^2(\Gamma \backslash \text{SL}_2(\mathbb{R}))$ into irreducibles; and forms four point correlation functions involving holomorphic and anti-holomorphic forms. The linear program gives upper bounds for $\mathcal{A}_4(\Gamma \backslash \mathbb{H})$ that depend only on the least $k$ for which $\Gamma \backslash \mathbb{H}$ has holomorphic forms of weight $k$.

• For $\Gamma = [2, 3, 7]$ the upper bound apparently saturates at the true value!

• The saturation is similar to the Cohn-Elkies linear program which saturates for the 8 and 24 dimensional sphere packing problem.

Hartman-Mazac-Rastelli give a direct relation between certain conformal bootstrap linear programs and Cohn-Elkies as well as the extremal "magic" functions of Viazovska and Viazovska et al which establish this saturation.
The basic arithmetic group is the modular group \( SL_2(\mathbb{Z}) \) in \( SL_2(\mathbb{R}) \). Groups \( \Gamma \) which are commensurable to \( SL_2(\mathbb{Z}) \) or groups like it are called arithmetic.

**Conjecture:**

\[
\begin{align*}
\text{BASS}_\Delta(\text{ARITH}_2) &= \{ 0, \frac{1}{4} \} \\
\text{BASS}_\Delta^0(\text{ARITH}_2) &= \{ 44.88, 28.07, \ldots \} \subset (\frac{1}{20}, \infty) \\
\text{BASS}_\Delta^1(\text{ARITH}_2) &= [0, \frac{1}{4}]
\end{align*}
\]
Quite a lot is known towards this:

- That $\text{Bass}_\Delta (\text{Arith}_2)$ is discrete in $(\frac{1}{4}, \infty)$ follows from the fact that the number of $x \in \text{Arith}_2$ with $\text{Vol}(x) < c$ is finite (Borel-Prasad) and using the trace formula
  $$\lim_{\text{Vol}(x) \to \infty} \lambda_1(x) \leq \frac{1}{4} \quad (\text{Huber})$$

- A recent breakthrough by Hide-Magee gives $\text{Bass}_\Delta^L (\text{Arith}_2)$ is infinite and contains $[0, \frac{1}{4}]$.

- Even more recently Magee shows that it contains $[0, \frac{1}{4}]$.

So the conjecture is resolved except for the part that $\text{Bass}_\Delta^D (\text{Arith}_2)$ is infinite.

What they show is that a random $n$-sheeted cover $X_n$ of a fixed $Y$ has
$$\lim_{n \to \infty} \lambda_1(X_n) = \min(\lambda_1(Y), \frac{1}{4}).$$

The proof makes use of results from $C^*$ algebras; strong convergence of spectra in free probability (Haagerup-Thorbjørnsen, Bordewijk-Collins)
Subgroups of $SL_2(\mathbb{Z})$ which contain $\Gamma(n) = \{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv I(n) \}$ some new and groups like these are called congruence.

The Holy Grail is the rigidification of the Bass note spectrum at $\frac{1}{4}$:

**Conjecture (Ramanujan–Selberg + Epsilon):**

$$Bass_\Delta(\text{cong}_2) = \frac{1}{4}$$

$$Bass_D^\Delta(\text{cong}_2) \subset (\frac{1}{4}, \infty) \text{ is infinite (for)}$$

$$Bass_{\Delta}^L(\text{cong}_2) = \frac{1}{4} + \frac{3}{4} \mathbb{J}$$

Known:

$$Bass_\Delta(\text{cong}_2) \subset [0.238037, 44.88]$$

(Kim–S, Blomer–Brumley).
BASS NOTE ON SPINORS:

\[
\text{Given } X = \Gamma \backslash H \text{ a hyperbolic orbifold and a spin structure define the Bass note at the bottom of the spectrum of the Dirac Laplacian } D \text{ on spinors (no forced zero mode!).}
\]

- On the universal cover \( \hat{\Gamma} \) the \( L^2 \)-spectrum of \( D \) is \( [0, \infty) \).
- Gesteau-Pal-Simmons-Duffin-\text{-Xu} have run the conformal bootstrap which yields a wealth of information on \( \text{Bass}_D^\Delta(\text{Hyp}_2) \) and especially its relation to \( \text{Bass}_\Delta(\text{Hyp}_2) \).

The Bass note spectrum of \( D \) on \( \text{Arith}_2 \) is more mysterious.

**Theorem** (A. Adve, V. Giri 2023) \( \text{Bass}_D^L(\text{Arith}) \) has infinitely many points, in particular is not rigid.

- The proof uses carefully engineered large abelian covers of an arithmetic surface of genus 2.
BASS NOTE SPECTRUM FOR HYP$_3$:

CONJECTURE:

\[\text{BASS}_{\Delta}^{\alpha}(\text{HYP}_3) = \overline{0,1}\]

\[\text{BASS}_{\Delta}^{\beta}(\text{HYP}_3) \subset (1,\infty) \text{ AND IS DISCRETE AND INFINITE.}\]

\[\text{BASS}_{\Delta}^{\lambda}(\text{HYP}_3) = [0,1].\]

COMMENTS

- JORGENSEN AND THURSTON SHOWED THAT HYP$_3$ IS WELL ORDERED BY VOLUME. THE LIMIT POINTS CORRESPOND TO VOLUMES OF CUSPED $y_i$'S WITH $y_j \to y$ SUITABLY. THEIR SET IS THE "VOLUME SPECTRUM OF HYP$_3$".

- COLBOIS/COURTOIS COUPLED WITH CHAVAL/DOHJIIUK SHOW THAT

\[\lambda_i(y_j) \to \lambda_i(y) \text{ AS } j \to \infty\]

SINCE $\lambda_i(y) \leq 1$ AND

\[\lim_{\text{Vol}(x) \to \infty} \lambda_i(x) \leq 1 \quad \text{(HUBER)}\]

IT FOLLOWS THAT

\[\text{BASS}_{\Delta}(\text{HYP}_3) \text{ IS DISCRETE IN } (1,\infty).\]
It is known that $\lambda_1(\gamma) \geq 1$ for certain explicit cusped $\gamma$'s (Grunewald-Elstrodt-Mennicke).

$\Rightarrow \quad \text{Bass}_{\Delta}^{-1}(\text{hyp}_3) \subset \{0, 1\}^2.$

Bonifacio-Mazac and Pal have a wealth of data on $\text{Bass}_{\Delta}(\text{hyp}_3)$ as well as bootstrap bounds relating $\text{Bass}_{\Delta}(\text{hyp}_3)$ and $\text{Bass}_{\Delta_1}(\text{hyp}_3)$ (see below).

Interestingly, the $x \in \text{hyp}_3$ with largest Bass note that they find is not the Marshall-Martin orbifold of smallest volume ($= 0.0391...$) but rather one of volume 0.0527. The Marshall-Martin orbifold does appear to maximize the base note of the Laplacian on other tensors.

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For $\gamma = \text{Arith}_3$ as well as for $\gamma = \text{Cong}_3$, the Bass notes of spectra of $\Delta$ have similar conjectured formulations and structure to $\text{Cong}_2$ and $\text{Arith}_2$. 
What appears to be quite different is the Bass note spectrum of the Hodge Laplacian \( \Delta_1 \) on 1-forms. By the Hodge theorem, the Bass note (meaning \( \lambda_0 \) in this case) of \( \Delta_1 \) is 0 iff \( X \) is a rational homology 3-sphere.

The \( L^2 \) spectrum of \( \Delta_1 \) on \( H^3 \) is \( [0, \infty) \).

Rigidity question: Is Bass (\( \Delta_1 \)) rigid? That is does \( \lambda_0(\Delta_1(X_j)) \rightarrow 0 \) along every infinite sequence of \( X_j \)'s?

Sticking to hyperbolic rational homology 3-spheres, the spectrum of \( \Delta_1 \) consists of two parts: the exact and co-exact forms. The exact part is the non-zero eigenvalues off the Laplacian \( \Delta_0 \) on functions and denote by \( \lambda^*_{\text{coexact}}(X) \) the smallest eigenvalue of \( \Delta_1 \) on co-exact forms. Lin and Lipnowski give applications of the \( \lambda^*_{\text{coexact}} \) gap to the Seiberg-Witten equations on \( X \).
This leads to the Bass coexact \( \text{coexact}(\text{hyp}_3) \) question.

Let \( \text{IHS} \) denote the subset of \( \text{hyp}_3 \) which are integral homology spheres.

For \( N \) an integer let \( P(N) \) denote the members of \( \text{hyp}_3 \) for which \( \text{T}^*_1(X) \) is generated by fewer than \( N \) elements.

In their study of \( \text{coexact} \), Ab Durrahman, Adve, Giri, Lowe and Zung show that

- \( \text{Bass} \) \( \text{coexact}(\text{IHS}) \) is not rigid, in fact has infinitely many limit points.
- For any \( N \);
  \[ \text{Bass} \text{coexact}(\text{IHS}) \cap P(N) \text{ is rigid.} \]
RIGIDITY OF THE SPECTRUM:

**Definition:** A sequence $X_j$ of quotients $\mathbb{H}\backslash S$ Benyamin-Schramm converges to $S$ if arbitrary large balls about the random point in $X_j$ are isometric with such a ball in $S$, as $j \to \infty$.

M. Fraczyk has shown that if $\text{Vol}(x) \to \infty$ with $x \in \text{CONG}_{d}$, $d=2,3$, then $X$ B-S converges to $\mathbb{H}^d$.

It follows that both

$$\text{BASS}_{\Delta}(\text{CONG}_3)$$

and

$$\text{BASS}_{\Delta}(\text{CONG}_3)$$

are rigid; they are infinite discrete in $(0,\infty)$ with 0 as the only limit point.
HIGHER RANK $S$:

Various rigidity sets in, leading to spectral rigidity.

1. The family $\mathcal{F}$ of quotients are all arithmetic (Margulis); and usually even congruence so the spectra of any $P$ become part of the general Ramanujan conjectures (discuss in Lecture 4).

2. Any sequence of quotients $x_j$ of $\infty S$ $b$-s converge to $S$ as $\text{Vol}(x_j) \to \infty$.

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