TWISTED BASE NOTE SPECTRA:

LET $\underline{X}$ BE ONE OF OUR LOCALLY UNIFORM GEOMETRIES. $\underline{X} = \tilde{\underline{X}}/\pi_1(\underline{X})$ AND IN EXAMINING THE SPECTRA OF INVARIANT DIFFERENTIAL OPERATORS $\underline{P}$ ON $L^2(\underline{X})$ IT IS CONVENIENT TO ALLOW OTHER GLUING CONDITIONS THAN THOSE REALIZING $\underline{X}$. TWISTING BY UNITARY FLAT LINES BUNDLES ON $\underline{X}$ ARE PARTICULARLY USEFUL.

IF $\chi : \pi_1(\underline{X}) \to U(1)$ IS A MORPHISM, LET

$$L^2(\underline{X}, \chi) = \left\{ u : \tilde{\underline{X}} \to \mathbb{C} \;\middle|\; \begin{array}{l} u(\gamma x) = \chi(\gamma) u(x), x \in \tilde{\underline{X}}, \\ \gamma \in \pi_1(\underline{X}) \text{ AND} \\ \int_{\underline{X}} |u(x)|^2 dv < \infty \end{array} \right\} \quad \text{—(1)}$$

THESE χ's ARE USED TO STUDY THE SPECTRA OF $\underline{P}$ ON ABELIAN COVERS OF $\underline{X}$ VIA "BLOCH WAVE THEORY" WHICH WILL BE USED IN LATER LECTURES.

THE SPACE OF SUCH χ's IS THE TORUS $T(\underline{X})$ WHICH IS DUAL TO $H_1(\underline{X})$.

LET $\lambda_0(\underline{P}, \underline{X}, \chi)$ BE THE BOTTOM (IN ABSOLUTE VALUE) OF THE SPECTRUM OF $\underline{P}$ ON $L^2(\underline{X}, \chi)$, THAT IS THE BASS NOTE. WE CONSIDER THE PROBLEM OF MAKING λ_0 AS LARGE AS POSSIBLE BY VARYING χ. DEFINE THE (OPTIMAL) TWISTED BASS NOTE OF $\underline{P}$ ON $\underline{X}$ BY;

$$V_{\underline{P}}(\underline{X}) = \sup_{\chi \in T(\underline{X})} \lambda_0(\underline{P}, \underline{X}, \chi). \qquad (2)$$

THE TWISTED $\underline{P}$- BASS NOTE SPECTRUM OF A SET $\mathcal{Y}$ OF LOCALLY UNIFORM GEOMETRIES IS

$$\mathrm{BASS}_{\underline{P},\,\mathrm{TWIST}}(\mathcal{Y}) = \overline{\{ V_{\underline{P}}(\underline{X}) : \underline{X} \in \mathcal{Y} \}} \qquad (3)$$

FOR OUR TORI $X_L = \mathbb{R}^n/L \in \mathcal{Y}_n$ AND P A HOMOGENEOUS POLYNOMIAL IN $D_1, D_2, \ldots D_n$ ONE COMPUTES ~~THE~~ ITS SPECTRUM ON $L^2(X_L, \chi)$ AS FOLLOWS:

WE IDENTIFY $T(X_L)$ WITH $\mathbb{R}^n/L^*$ WHERE FOR $\xi \in \mathbb{R}^n/L^*$, $\chi_\xi(l) = e(\langle \xi, l \rangle)$ ——— (4)

THEN THE SPECTRUM OF P ON $L^2(X, \chi_\xi)$ IS EQUAL TO

$$\{ P(l^* - \xi) : l^* \in L^* \}, \qquad \text{——— (5)}$$

AND

$$\lambda_0(P, X_L, \chi_\xi) = \inf \{ |P(l^* - \xi)| : l^* \in L^* \} \qquad \text{——— (6)}$$

THUS THE TWISTING BY χ_ξ GIVES A BASS NOTE SPECTRUM INTERPRETATION OF THE INHOMOGENEOUS PROBLEM IN THE GEOMETRY OF NUMBERS.

THE CORRESPONDING TWISTED BASS NOTE IS THEN GIVEN BY

$$\nu_{P}(X_L) = \sup_{\xi \in \mathbb{R}^n} \inf_{l^* \in L^*} \{ |P(l^* - \xi)| \} \qquad \text{---} (7)$$

THE TWO P's THAT HAVE BEEN STUDIED EXTENSIVELY W.R.T. THE TWISTED BASS NOTE SPECTRA ARE $P(x_1, .., x_n) = x_1^2 + \cdots + x_n^2$ ~~AND~~ $:= \Delta_n$ AND $P = x_1 x_2 \cdots x_n := W_n$.

(1) Δ_n : THE FIRST THING TO NOTE IS THAT $V_{\Delta_n}(X)$ IS INVARIANT UNDER THE ACTION OF $K = SO_n(\mathbb{R})$ ACTING ON THE LEFT IN $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$. SO IT IS A FUNCTION OF THE RIEMANNIAN FLAT TORUS X_L, AND IT IS THE DIAMETER OF X_L. IT CAN ALSO BE INTERPRETED AS THE COVERING RADIUS BY SPHERES ~~OF / $\mathbb{R}^n$~~ CENTERED AT THE LATTICE POINTS L, OF $\mathbb{R}^n$. THIS LAST INTEPRETATION IS THE FORM IN WHICH THERE IS A COMPREHENSIVE

STUDY (SEE ROGERS [RO]). FROM THE DIAMETER POINT OF VIEW IT FOLLOWS THAT $V_{\Delta_n}(X) \to \infty$ AS $X \to \partial(\mathcal{Y}_n)$ AND SINCE V_{Δ_n} IS CONTINUOUS WE HAVE THAT

$$\mathrm{BASS}_{\Delta_n,\mathrm{TWIST}}(\mathcal{Y}_n) = [\theta_n, \infty) \quad \text{WITH } \theta_n > 0. \tag{8}$$

θ_2 AND θ_3 ARE KNOWN AS WELL AS THEIR CORRESPONDING UNIQUE MINIMIZERS AND MUCH OF THE STUDY IS CONCERNED WITH THE ASYMPTOTICS OF θ_n AS $n \to \infty$ [RO].

IN THE SPIRIT OF RESTRICTING TO SUBSETS $\mathcal{F}$ OF $\mathcal{Y}_n$, WOODS [WO 2] CONJECTURED THAT FOR WELL ROUNDED LATTICES L (IE ONES FOR WHICH THE SET OF MINIMAL LENGTH NON-ZERO VECTORS SPAN $\mathbb{R}^n$) WE HAVE THAT

$$\mathrm{BASS}_{\Delta_n,\mathrm{TWIST}}(\text{WELL ROUNDED}) \subset \left[\sigma_n, \frac{n}{4}\right],$$

WITH EQUALITY AT THE TOP IFF $L \in K\mathbb{Z}^4$. —— (9)

CONJECTURE (9) HAS BEEN PROVEN FOR $n \leq 10$ (KATHURIA AND RAKA [K-R]), BUT RELATIVELY RECENTLY WAS SHOWN TO BE FALSE FOR $n \geq 30$ (SEE REGEV-SHAPIRA-WEISS [R-S-W]).

(2) $\underline{W_n}$: $\nu_{W_n}(X)$ IS INVARIANT UNDER THE DIAGONAL $A = \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \in SL_n(\mathbb{R}) \right\}$ ACTION ON $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$, WHICH BEING NON-COMPACT ALLOWS FOR AN EFFECTIVE APPLICATION OF HOMOGENEOUS DYNAMICS TO STUDY ν_{W_n}.

THE HOLY GRAIL IS THE ELEGANT CONJECTURE OF MINKOWSKI:

$$\nu_{W_n}(X_L) = \sup_{\xi \in \mathbb{R}^n} \inf_{\ell \in L} \left\{ |\xi_1 - \ell_1| \cdots |\xi_n - \ell_n| \right\} \leq 2^{-n}$$

WITH EQUALITY IFF $L \in A\mathbb{Z}^n$, —— (10)

(AND FOR $\mathbb{Z}^n$; $\xi = (\frac{1}{2}, \cdots \frac{1}{2})$).

McMULLEN [Mc 2] OPENED THE DOOR TO RESOLVING (10) BY SHOWING THAT EVERY COMPACT ORBIT CLOSURE $\overline{A\xi} \in SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ CONTAINS A WELL ROUNDED L.

THIS TOGETHER WITH THE ANALYSIS IN BIRCH AND SWINNERTON-DYER [B-SD] SHOW THAT IF WOODS CONJECTURE (9) IS VALID FOR ~~[illegible]~~ $n \leq N$ THEN SO IS (10) FOR $n \leq N$, LEADING TO A PROOF OF (10) FOR $n \leq 10$.

WHILE (9) FAILS FOR LARGE n, SHAPIRA AND WEISS SHOW THAT A MODIFICATION IN (9), THAT L BE WELL ROUNDED AND "STABLE" (SEE [S-W]), WOULD SUFFICE TO DEDUCE (10), LEADING TO THE POSSIBILITY FOR ~~THAT~~ THIS APPROACH TO PROVE ~~[illegible]~~ (10) FOR ALL n.

ACCORDING TO (10)

$$\mathrm{BASS}_{W_n, \mathrm{TWIST}}(\mathcal{F}_n) = \mathrm{MINKOWSKI}_n \subset [0, 2^{-n}]. \tag{11}$$

CASSELS [CA] SHOWS THAT 2^{-n} IS NOT ISOLATED IN $\mathrm{MINKOWSKI}_n$ AND SHAPIRA [SH] GIVES A HOMOGENEOUS DYNAMICS INTERPRETATION / PROOF OF THE SAME.

THE BEHAVIOR AT THE BOTTOM OF MINKOWSKI$_n$ IS VERY DIFFERENT IN THE CASES $n=2$ AND $n>2$.

- FOR $n=2$ KHINTCHINE [KH] AND DAVENPORT [DA] SHOW THAT $\gamma_2 = \text{INF}(\text{MINKOWSKI}_2)$ IS POSITIVE (IN FACT $> 1/128$). ONE CAN SHOW THAT MINKOWSKI$_2$ CONTAINS INFINITELY MANY LIMIT POINTS IN $(\gamma_2, 1/4)$ AND IS VERY FAR FROM BEING RIGID.

- FOR $n>2$, SHAPIRA [SH 2] USES THE DYNAMICS OF A ON THE HOMOGENEOUS SPACE OF FLAT LINE BUNDLES $SL_n(\mathbb{R}) \ltimes \mathbb{R}^n / SL_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$ TO SHOW THAT

$$\nu_{W_n}(X) = 0 \quad \text{FOR ALMOST ALL } X. \qquad (12)$$

HENCE $\otimes$ IS A VERY HIGH MULTIPLICITY LIMIT POINT OF MINKOWSKI$_n$ FOR $n>2$.

WHETHER MINKOWSKI$_n$ IS RIGID (THAT IS DISCRETE IN $(0, 2^{-n})$) THANKS TO THE MEASURE RIGIDITY OF THE A-ACTION, IS AN INTERESTING QUESTION.

[B-SD] BIRCH AND SWINNERTON-DYER MATHEMATIKA 3 (1956) 25-39

[CA] CASSELS JNL LONDON MATH SOC. (4) 485-492 (1952)

[K-R] KATHURIA AND RAKA PROC. IND. ACAD. SCI (2022) 132-145

[Mc2] McMULLEN "MINKOWSKI'S CONJECTURE, WELL-ROUNDED LATTICES AND TOPOLOGICAL DIMENSION" 2004 JAMS 18 2005 711-734

[R-S-W] REGEV AND SHAPIRA AND WEISS DMJ 166, 13, 2017, ~~2443~~-2446

[Ro] ROGERS "PACKING AND COVERING" CUP 1964

[SH] SHAPIRA IMRN 2017 (15) 4704-4731

[S-W] SHAPIRA-WEISS JEMS 18 (2016) 1753-1767

[WO2] WOODS JNL NUMBER TH. 4 157-180 (1972)

[KH] KHINTCHINE RENDICONTI DI PALERMO 50 (1926) 170-195

[SHA] SHAPIRA ANNALS 173 (2011) 543-557.

[DA] DAVENPORT PROC. LONDON MATH SOC. 53 (1951) 65-82.