FURTHER NOTES ON THE BASS NOTE SPECTRUM OF $W = D_1 D_2$

(CONTINUED FROM PAGE 4' LECTURE 1)

THE RELATION TO THE MARKOFF SPECTRUM OF $BASS_W(\mathcal{Y}_2)$ COMES FROM COMPUTING THE SPECTRUM OF W ON $X = \mathbb{R}^2/L$. USING THE O.N.B. $e(\langle x, l \rangle)$ FOR $l \in L^*$ THE DUAL LATTICE TO L, AND WHERE $e(z) = e^{2\pi i z}$, WE SEE THAT THIS SPECTRUM IS

$$\{ 4\pi^2 l_1 l_2 : l = (l_1, l_2) \in L^* \}.$$

IF $L = g\mathbb{Z}^2$ FOR $g \in SL_2(\mathbb{R})$ THEN THESE VALUES ARE GIVEN BY THOSE OF A BINARY QUADRATIC FORM $\phi_{L^*}(x_1, x_2)$ AT INTEGER POINTS (x_1, x_2). CHOOSING DIFFERENT g's GIVES THE SAME FORM UP TO $SL_2(\mathbb{Z})$ EQUIVALENCE; THAT IS THE LINEAR ACTION ON FORMS. IT FOLLOWS THAT

$$\mu(\phi) = \inf \{ |\phi(x_1, x_2)| ; x \in \mathbb{Z}^2 \setminus \{0\} \}. \quad \longrightarrow (1)$$

WE NORMALIZE ϕ BY SCALING BY $1/\sqrt{\mathrm{DISC}(\phi)}$. IN THIS WAY IT IS CLEAR THAT

$$\mathrm{BASS}_W(\mathcal{Y}_2) = \mathrm{MARKOFF}$$

WHERE MARKOFF IS DEFINED IN THE STANDARD WAY (SEE CUSICK-FLAHIVE [C-F]).

WE CHOOSE NATURALLY DEFINED ARITHMETICAL SUBSETS OF $\mathcal{Y}_2$ BY RESTRICTING TO ϕ'S ABOVE WHICH ARE RATIONAL. THAT IS

$$\phi(x_1,x_2) = a x_1^2 + b x_1 x_2 + c x_2^2 \quad \text{———} (2)$$

WITH $a,b,c \in \mathbb{Q}$ AND $d(\phi) = b^2 - 4ac > 0$.
WE CAN SCALE (2) UNIQUELY SO THAT a,b,c ARE INTEGERS AND $\mathrm{GCD}(a,b,c) = 1$.
IN THIS FORM $d = d(\phi)$ IS A POSITIVE INTEGER AND

$$d \equiv 0, 1 \ (4) . \quad \text{———} (3)$$

THE $SL_2(\mathbb{Z})$ ACTION PRESERVES INTEGRALITY AND d AND GCD AND HAS A FINITE NUMBER OF ORBITS DENOTED BY $h(d)$. OUR CHOICES OF SUBSETS $\mathcal{Y}$ OF $\mathcal{Y}_2$ DEPEND ON THE FACTORIZATION OF d. WRITE

$$d = D t^2, \text{ WITH } D \text{ FUNDAMENTAL AND } t \geq 1$$
$$\text{(WE ALLOW } D = 1) \quad \text{———} (4)$$

THE $\mathcal{F}$'s WE CONSIDER ARE(*)

1) RATIONAL; IE ALL RATIONAL ϕ's.

2) ISOTROPIC; ALL THE $\mathbb{Q}$ ISOTROPIC RATIONAL ϕ's, THAT IS $D=1$ AND $d=t^2, t\geq 1$.

3) FUNDAMENTAL; ALL d's WITH $t=1$.

4) FIX $D>1$ FUNDAMENTAL AND LET $M(D)$ DENOTE ALL THE FORMS OF DISCRIMINANT $d=Dt^2$ WITH $t\geq 1$.

5). GIVEN D AS IN (4) AND S A FINITE SET OF PRIMES p SATISFYING $\left(\frac{D}{p}\right)=-1$, LET $U_S(D)$ BE THE SUBSET OF $M(D)$ OF THE FORM $d=Dp_1^{e_1}\cdots p_r^{e_r}$, $p_j\in S, e_j\geq 0$ ("S-UNITS").

WE RECORD WHAT IS KNOWN ABOUT THE BASS NOTE SPECTRA OF THESE $\mathcal{F}$'s:

(*) NOTE THAT BY DUKE [DU] EACH OF THESE $\mathcal{F}$'s IS DENSE IN THE SPACE OF INDEFINITE BINARY FORMS AS THE $h(d)$ MEMBERS OF $\mathcal{F}$ ARE SO AS $d\to\infty$.

(1) BASS(RATIONAL) = LAGRANGE —(5)

THE USUAL DEFINITION OF THE LAGRANGE SPECTRUM IS THE SET OF VALUES $\nu(\alpha)$ FOR $\alpha \in \mathbb{R}$, WHERE

$$\nu(\alpha) = \varliminf_{q \to \infty} q \| \alpha q \| \qquad \text{—(6)}$$

WHERE $\| \xi \| = \operatorname{dist}(\xi, \mathbb{Z})$.

THAT (5) HOLDS WAS PROVED BY CUSICK[C]

(NOTE THAT FOR THE ROOTS ξ, ξ' OF $\phi(x,1) = 0$ WITH ϕ RATIONAL; $\nu(\xi) = \nu(\xi') = \mu(\phi)$)

CLEARLY (FROM (5)) LAGRANGE $\subset$ MARKOFF. IT IS KNOWN THAT THESE SETS ARE NOT EQUAL, THOUGH THEY DO COINCIDE ON $[0, \mu_0]$ AND $[\frac{1}{3}, 1]$, SEE FREIMAN [F].

(2) FOR $\phi \in$ ISOTROPIC, $\mu(\phi) = 0$ SO THAT BASS (RATIONAL) $= \{0\}$. HENCE WE LOOK FOR THE SMALLEST NON-ZERO VALUE IN THE SPECTRUM AND SET

$$\mu'(\phi) = \mathrm{MIN} \left\{ |\phi(x_1, x_2)| > 0 \, ; \, x \in \mathbb{Z}^2 \right\} \tag{7}$$

CORRESPONDINGLY WE DEFINE THE BASS' SPECTRUM TO BE

$$\mathrm{BASS}'(\mathrm{ISOTROPIC}) = \overline{\{ \mu'(\phi) : \phi \in \mathrm{ISOTROPIC} \}} \tag{8}$$

THEN

$$\boxed{\mathrm{BASS}'(\mathrm{ISOTROPIC}) = \text{"ZAREMBA"}} \tag{9}$$

WHERE

$$\mathrm{ZAREMBA} = \overline{\{ \rho(\alpha) : \alpha \in \mathbb{Q}/\mathbb{Z} \}} \tag{10}$$

AND

$$\rho(p/q) = \min_{0 < y < q} y \left\| y \frac{p}{q} \right\| \tag{11}$$

IS ZAREMBA'S FUNCTION ($\alpha = p/q \in [0,1]$, $(p,q) = 1$)
$\rho : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$.

TO SEE THIS ONE CAN USE HECKE CORRESPONDENCES TO GIVE REPRESENTATIVES

OF THE $h(t^2)$ FORMS WITH $d = t^2$; ONE FINDS THAT $h(t^2) = \sum_{v|t} t$ AND THAT THE μ's OF THESE FORMS ARE OF THE FORM $\rho\left(\frac{a}{q}\right)$ WHERE $q|t$. ~~THAT~~ ONE CHECKS THAT ρ AS DEFINED IN (11) IS THE SAME AS THE ρ-FUNCTION DEFINED IN ZAREMBA [Z 1] (SEE McMULLEN [Mc] FOR AN INTERPRETATION IN TERMS OF ORDERS IN THE ALGEBRA $\mathbb{Q}\times\mathbb{Q}$).

ZAREMBA EXAMINES THE SET "ZAREMBA" IN THE INTERVAL $\left[\frac{10}{29}, 1\right]$. HE SHOWS THAT $\rho(\alpha) < \frac{10}{29}$ UNLESS $\alpha = \pm\frac{12}{29}, \pm\frac{17}{29}$ WHEN $\rho(\alpha) = \frac{10}{29}$ OR $\alpha = \frac{f_{n-2}}{f_n}$ OR $\frac{f_{n-1}}{f_n}$ WHEN $\rho(\alpha) = \frac{f_{n-2}}{f_n}$ WHERE f_n IS THE n-TH FIBONACCI NUMBER. HENCE IN $\left[\frac{10}{29}, 1\right]$ THE ONLY LIMIT POINT OF ZAREMBA IS

$$\frac{2}{3+\sqrt{5}} = 0.3819\ldots$$

ONE CAN USE HALL'S METHOD OF PRODUCING A RAY IN MARKOFF TO SHOW FOR EXAMPLE THAT $[0, 1/6] \subset$ ZAREMBA.

IT WOULD INTERESTING TO INVESTIGATE BASS' (ISOTROPIC) IN $(1/6, 10/29)$.

(3) BASS (FUNDAMENTAL) CONTAINS INFINITELY MANY LIMIT POINTS AND ~~MORE~~ IN FACT A NON-TRIVIAL HALL RAY AT 0.

THE FIRST WAS PROVED BY BOURGAIN AND KONTOROVICH [BK-1]. THEY SEEK MATRICES IN $SL(2,\mathbb{Z})$ OF TRACE t AND FOR WHICH $D = t^2 - 4$ IS FUNDAMENTAL, WHICH LIE IN SUB SEMIGROUPS CORRESPONDING TO CONTINUED FRACTIONS WITH UNIFORMLY BOUNDED COEFFICIENTS. THEIR TECHNIQUES COME FROM THE DIOPHANTINE ANALYSIS ON "THIN" MATRIX GROUPS (SEE [SA 3]). KOTSOVOLIS SHOWS HOW THEIR ANALYSIS CAN BE ADAPTED TO PRODUCE A HALL RAY.

(4) WE KNOW LITTLE ABOUT BASS $(M(D))$ OTHER THAN IT CONTAINS INFINITELY MANY LIMIT POINTS. THIS WAS PROVED BY WOODS [WO] AND WILSON [WI] AND MORE GEOMETRICALLY AND GENERALLY BY McMULLEN [M] WHO POINTS TO THE SURPRISING RICHNESS OF BASS $(M(D))$ AND COINED THE NAME "ARITHMETIC CHAOS".

(5). BASS($U_S(D)$) IS RIGID.

DIRICHLET GAVE THE FIRST EXAMPLES OF SUCH S-UNIT d's FOR WHICH $h(d)$ IS SMALL AND EVEN EQUAL TO 1. FOR SUCH IT FOLLOWS FROM DUKE [DU] THAT THE CORRESPONDING ϕ IS RIGID (IE $\mu(\phi) \to 0$ AS $d \to \infty$). AKA AND SCHAPIRA [A-S] GIVE A FAR REACHING EXTENSION OF THIS PHENOMENON TO SUCH S-UNIT d's. BESIDES SHOWING THAT THE CLASS NUMBERS $h(d)$ ARE SMALL FOR $d \in U_S(D)$ THEY PROVE A DUKE EQUIDISTRIBUTION TYPE THEOREM USING HECKE ORBITS WHICH LEADS TO THE RIGIDITY OF BASS($U_S(D)$).

THE SET $C(d)$ OF INTEGRAL (PRIMITIVE) FORMS OF DISCRIMINANT d FORM A GROUP OF ORDER $h(d)$ UNDER GAUSS COMPOSITION. IN ORDE TO EXAMINE THE SPECTRAL RIGIDITY PROPERTIES OF THE ϕ's IN $C(d)$ SET

$$m(d) = \max_{\phi \in C(d)} \mu(\phi) . \qquad (12)$$

FOR ISOTROPIC FORMS $d = t^2$ SET

$$m'(t^2) = \max_{d(\phi)=t^2} \mu'(\phi) \qquad (13)$$

IF F IS A SET OF DISCRIMINANTS THEN $\mathrm{BASS}\left(\bigcup_{d \in F} C(d) \right)$ IS RIGID IFF $m(d) \to 0$ AS $d \to \infty$, $d \in F$.

DUKE'S THEOREM [DU] IMPLIES THAT THE PROBABILITY MEASURES ON $[0,1]$ GIVEN BY

$$\frac{1}{h(d)} \sum_{\phi \in C(d)} \delta_\phi \qquad (14)$$

CONVERGE TO δ_0 (DIRAC MASS AT 0) AS $d \to \infty$. SO IF $h(d)$ IS VERY SMALL THEN IT IMPLIES RIGIDITY.

THE SIZE OF $h(d)$ IS A NOTORIOUSLY DIFFICULT PROBLEM. FROM DIRICHLET'S CLASS NUMBER FORMULA

$$h(d) \log \varepsilon_d \approx \sqrt{d} \qquad (15)$$

WHERE ε_d IS THE FUNDAMENTAL SOLUTION TO

$$t^2 - du^2 = 4 , \qquad (16)$$

IT IS EXPECTED THAT ALMOST ALL d's WHEN ORDERED BY SIZE HAVE $h(d) = O_\varepsilon(d^\varepsilon)$ FOR $\varepsilon > 0$ (HOOLEY [HOO], [SA 2]). FROM THIS AND POPA [PO] AND THE SUBCONVEXITY [H-M] IT WOULD FOLLOW THAT $m(d) \to 0$ FOR ALMOST ALL d's. THAT IS TO SAY THERE IS A SUBSET F OF d's OF FULL DENSITY FOR WHICH THE SPECTRUM OF $\bigcup_{d \in F} C(d)$ IS RIGID. IN PARTICULAR THIS WOULD APPLY TO FUNDAMENTAL DISCRIMINANTS.

WHILE ESTABLISHING ANYTHING LIKE THIS FOR FUNDAMENTAL DISCRIMINANTS SEEMS HOPELESS, THE ARITHMETIC CHAOS PROBLEM FOR $M(D)$ (D FIXED) IS MUCH MORE TRACTABLE.

FOR d's OF THE FORM Dp^2, p A PRIME, GOLUBEVA [GO] EXPLOITS THAT THE ORDER OF THE FUNDAMENTAL UNIT $\varepsilon(Dp^2)$ IS LARGE IF THE ORDER OF $\varepsilon(D)$ IS LARGE IN THE CLASS GROUP (mod p). THIS REDUCES THE CONSTRUCTION OF LARGE UNITS TO AN ARTIN PRIMITIVE ROOT TYPE PROBLEM AND ASSUMING THE GENERALIZED RIEMANN HYPOTHESIS (GRH) SHE SHOWS THAT FOR MANY p's $h(5p^2)$ IS EQUAL TO 2 (ITS MINIMAL POSSIBLE VALUE) AND HENCE IT FOLLOWS THAT $m(5p^2) \to 0$ FOR THESE p's. THIS CAN BE PUSHED FURTHER AND AKA [A] AND KURLBERG SHOW THAT UNDER GRH FOR ALMOST ALL $d \in M(D)$, $h(d) = O_\varepsilon(d^\varepsilon)$. HENCE UNDER THE SAME ASSUMPTION THERE IS A SUBSET F OF t's OF FULL DENSITY SUCH THAT

$$m(Dt^2) \to 0 \quad \text{AS } t \to \infty, t \in F. \qquad (17)$$

THE STORY WITH $D=1$, IE THE ISOTROPIC FORMS IS QUITE DIFFERENT.

ZAREMBA [Z2] CONJECTURES THAT THERE IS $\eta > 0$ SUCH THAT

$$m'(t^2) \geq \eta \quad \text{FOR} \quad t \geq 1. \qquad (18)$$

EQUIVALENTLY IN TERMS OF HIS ρ-FUNCTION FOR ALL $q \geq 1$

$$\max_{\substack{1 < a < q \\ (a,q)=1}} \rho\left(\frac{a}{q}\right) \geq \eta . \qquad (19)$$

MCMULLEN [Mc] GIVES A QUANTIFIED VERSION OF (18) (SEE HIS CONJECTURE 6.2).

MERCAT [ME] RELATES THE ISOTROPIC ϕ's TO THE ANISOTROPIC ONES BY EXPLICIT CONSTRUCTIONS OF ϕ's WITH LARGE $\mu(\phi)$, USING RATIONALS WITH LARE ρ. THIS ALLOWS HIM TO PRODUCE LIMIT POINTS IN BASS$(M(D))$ WHICH ARE LARGE (CORRESPONDING TO CONTINUED FRACTIONS HAVING a's IN $\{1,2\}$). FROM HIS CONSTRUCTION IT FOLLOWS THAT

$$m((t^2-1)t^2) \geq \eta > 0 \quad \text{FOR } t > 1 \qquad (20)$$

IF (18) IS TRUE.

THE DISCRIMINANTS $d=(t^2-1)t^2$ HAVE SMALL UNITS ($\varepsilon_d = O(d)$) WHICH MIGHT BE NECESSARY FOR (20) TO HOLD. WHILE THE OPTIMISTIC CONJECTURE BY EINSIEDLER–LINDENSTRAUSS–MICHEL–VENKATESH [E-L-M-V] THAT IF $h(d) = O(d^{1/2-\alpha})$ FOR SOME $\alpha > 0$, THEN EACH CLOSED GEODESIC IN $SL(2,\mathbb{Z})\backslash SL_2(\mathbb{R})$ CORRESPONDING TO A $\phi \in C(d)$ BECCOMES EQUIDISTRIBUTED, IS FALSE AS SHOWN BY SOLAN AND YIFRACH [S-Y]; & THAT $m(d) \to 0$ UNDER THIS ASSUMPTION MAY STILL HOLD. ONE MIGHT EVEN QUANTIFY THIS AND POSTULATE THAT

$$\frac{m(d)\, d^{1/2}}{h(d)} \sim d^{o(1)} \quad \text{AS} \quad d \to \infty \tag{21}$$

THE SMALLEST POSSIBLE UNITS COME FROM d's OF THE FORM t^2-4 ($\varepsilon_d = \sqrt{d}$) AND BOURGAIN AND KONTOROVICH [B-K 2] CONJECTURE SOMETHING SLIGHTLY WEAKER (BUT QUANTITATIVE) THAN THE FOLLOWING ANALOGUE OF (20)

THERE IS $\eta' > 0$ SUCH THAT

$$m(t^2-4) \geq \eta' \quad \text{FOR } t > 2. \tag{22}$$

THE BASS NOTE VIEW POINT SUGGESTS THE FOLLOWING PROBLEMS

(a) FOR A GIVEN D SHOW THAT $\mathrm{BASS}(M(D))$ HAS A HALL RAY AT 0.

(b) REMOVE THE GRH ASSUMPTION ON PAGE 11; THAT IS PROVE THAT FOR EACH $D>1$ THERE IS A SUBSET OF $M(D)$ OF FULL DENSITY WHICH IS RIGID.

(c) FOR ANY $D>1$, $\mathrm{BASS}(M(D)) \cap (\frac{1}{3}, 1)$ IS FINITE. THIS IS EQUIVALENT TO SHOWING THAT THE SQUARE-FREE PART OF $9m^2-4$ GOES TO INFINITY AS m GOES OVER MARKOFF NUMBERS. SEE CORVAJA-ZANNIER [C-Z] AND LUCA [L] FOR RELATED PROBLEMS.

(d) $\mathrm{BASS}(\mathrm{FUND}) \cap (\frac{1}{3}, 1)$ IS INFINITE. THIS IS EQUIVALENT TO $9m^2-4$ BEING SQUARE-FREE FOR INFINITELY MANY MARKOFF NUMBERS. UNFORTUNATELY THE RECENT PROGRESS ON DIVISIBILITY OF MARKOFF TRIPLES (BOURGAIN-GAMBURD-SARNAK [B-G-S], CHEN [CH]) IS STILL VERY FAR FROM ADDRESSING THIS.

(e) PROVE THAT THERE IS AN (INFINITE) RIGID SEQUENCE OF FUNDAMENTAL D'S.

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