Further Notes on the Bass Note

Spectrum of \( W=D_1 \cdot D_2 \) (continued from page 4, lecture 1)

The relation to the Markoff spectrum of \( B_{\omega}(\frac{1}{2}) \) comes from computing the spectrum of \( W \) on \( X=\mathbb{R}^2/L \), using the o.n.b. \( e(\langle x, l \rangle) \) for \( l \in L^* \), the dual lattice to \( L \), and where \( e(z)=e^{2\pi i z} \), we see that this spectrum is

\[
\left\{ 4\pi^2 l_1l_2 : l=(l_1, l_2) \in L^* \right\}.
\]

If \( L=g\mathbb{Z}^2 \) for \( g \in SL_2(\mathbb{R}) \) then these values are given by those of a binary quadratic form \( \phi_{L^*}(x,y) \) at integer points \((x,y)\). Choosing different \( g \)'s gives the same form up to \( SL_2(\mathbb{Z}) \) equivalence, that is the linear action on forms. It follows that

\[
\mu(\phi) = \inf \left\{ |\phi(x_1, x_2)| : x_1, x_2 \in \mathbb{Z}^2 \setminus \{0, 0\} \right\}.
\]

(1)

We normalize \( \phi \) by scaling by \( \frac{1}{\sqrt{\text{disc}(\phi)}} \), in this way it is clear that
\[ \mathbb{B}_n \left( \frac{1}{2} \right) = \text{MARKOFF} \]

where \text{MARKOFF} is defined in the standard way (see Cusick-Flahive [C-F]).

We choose naturally defined arithmetical subsets of \( \frac{1}{2} \) by restricting to \( \phi \)'s above which are rational. That is

\[ \phi(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 \quad (1) \]

with \( a, b, c \in \mathbb{Q} \) and \( d(\phi) = b^2 - 4ac > 0 \). We can scale (2) uniquely so that \( a, b, c \) are integers and \( \gcd(a, b, c) = 1 \).

In this form \( d = d(\phi) \) is a positive integer and

\[ d = 0, 1 (4) \quad (2) \]

The \( SL_2(\mathbb{Z}) \) action preserves integrality and \( d \) and \( \gcd \) and has a finite number of orbits denoted by \( \Phi(d) \). Our choices of subsets \( \mathcal{F} \) of \( \frac{1}{2} \) depend on the factorization of \( d \). Write

\[ d = D t^2 \quad \text{with} \ D \ \text{fundamental and} \ t \geq 1 \]

(we allow \( D = 1 \)) \quad (4)
The \( y's \) we consider are \((\star)\)

1) **Rational**, i.e., all rational \( \phi's \).

2) **Isotropic**, all the \( D \) isotropic rational \( \phi's \), that is \( D = 1 \) and \( d = t^2, t \geq 1 \).

3) **Fundamental**, all \( d's \) with \( t = 1 \).

4) Fix \( D > 1 \) fundamental and let \( M(D) \) denote all the forms of discriminant \( d = Dt^2 \) with \( t \geq 1 \).

5) Given \( D \) as in (4) and \( S \) a finite set of primes \( p \) satisfying \( (\frac{D}{p}) = -1 \), let \( U_S(D) \) be the subset of \( M(D) \) of the form \( d = Dp_1^{e_1} \cdots p_r^{e_r}, p_j \in S, e_j > 0 \) (**\( S \)-units**).

We record what is known about the Bass note spectra of these \( y's \):

(\( \star \)) Note that by Duke [Du], each of these \( y's \) is dense in the space of indefinite binary forms as the \( \Delta(d) \) members of \( y \) are so as \( d \to \infty \).
(1) \[
\text{BASS (RATIONAL)} = \text{LAGRANGE} - (5)
\]

The usual definition of the Lagrange spectrum is the set of values \( \gamma(x) \) for \( x \in \mathbb{C} \), where

\[
\gamma(x) = \lim_{q \to \infty} \| q \cdot x \|.
\]

where \( \| z \| = \text{dist}(z, \mathbb{Z}) \).

That (5) holds was proved by Cusick [C]

(note that for the roots \( \frac{3}{2}, \frac{3}{2}' \) of \( \phi(x, 1) = 0 \) with \( \phi \) rational ; \( \gamma(\frac{3}{2}) = \gamma(\frac{3}{2}') = \mu(\phi) \))

Clearly (from (5)) Lagrange \( \subset \) Markoff.

It is known that these sets are not equal, though they do coincide on \([0, \mu_0]\) and \([\frac{1}{2}, 1]\), see Freiman [F].
(2) For \( \phi \in \text{ISOTROPIC} \), \( \mu(\phi) = 0 \) so that \( \text{BASS (RATIONAL)} = \emptyset \). Hence we look for the smallest non-zero value in the spectrum and set

\[
\mu'(\phi) = \min \left\{ |\phi(x_1, x_2)| : x \in \mathbb{Z}^2 \right\}
\]

Correspondingly we define the \( \text{BASS}' \) spectrum to be

\[
\text{BASS}'(\text{ISOTROPIC}) = \left\{ \mu'(\phi) : \phi \in \text{ISOTROPIC} \right\}
\]

Then

\[
\text{BASS}'(\text{ISOTROPIC}) = \text{ZAREMBA}'
\]

where

\[
\text{ZAREMBA} = \left\{ p(\alpha) : \alpha \in \mathbb{Q}/\mathbb{Z} \right\}
\]

and

\[
p\left( \frac{p}{q} \right) = \min \left\{ y \right\} \frac{\left| y \right|}{q}, 0 < y < q.
\]

is ZAREMBA's function \( \alpha = p/q \in [0, 1] \), \((p,q)=1\)

\( p : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \).

To see this one can use Hecke correspondences to give representatives
OF THE \( h(t^2) \) FORMS WITH \( d = t^2 \); ONE FINDS THAT \( h(t^2) = \sum_{d < t} \) AND THAT THE \( q \)'S OF THESE FORMS ARE A \( P(\frac{a}{b}) \) WHERE \( q \). ONE CHECKS THAT \( P \) AS DEFINED IN (11) IS THE SAME AS THE \( P \)-FUNCTION DEFINED IN ZAREMBA \([2, 1]\) (SEE McMULLEN [MC] FOR A
AN INTERPRETATION IN TERMS OF ORDERS IN THE ALGEBRA \( \mathbb{Q}(x) \)). ZAREMBA EXAMINES THE SET "ZAREMBA" IN THE INTERVAL \([\frac{10}{29}, 1]\). HE SHOWS THAT \( P(x) < \frac{10}{29} \) UNLESS \( x = \frac{\pm 4}{29} \) WHEN \( P(x) = \frac{10}{29} \) OR \( x = \frac{f_{n-2}}{f_n} \) OR \( \frac{f_{n-1}}{f_n} \) WHEN \( P(x) = \frac{f_{n-2}}{f_n} \) WHERE \( f_n \) IS THE \( n \)-TH FIBONACCI NUMBER. HENCE IN \([\frac{10}{29}, 1]\) THE ONLY LIMIT POINT IS \( \frac{2}{3 + \sqrt{5}} = 0.3819 \ldots \).

ONE CAN USE HALL'S METHOD OF PRODUCING A RAY IN MARKOFF TO SHOW FOR EXAMPLE THAT \([0, \frac{1}{16}] \subset ZAREMBA.

IT WOULD INTERESTING TO INVESTIGATE BASS' (ISOTROPIC) IN \( (\frac{1}{16}, \frac{10}{29}) \).
(3) **Bass** (Fundamental) contains infinitely many limit points and in fact a non-trivial Hall Ray at 0.

The first was proved by Bourgain and Kontorovich [BK-17]. They seek matrices in \( SL(2, \mathbb{Z}) \) of trace \( t \) and for which \( D = t^2 - 4 \) is fundamental, which lie in subsemigroups corresponding to continued fractions with uniformly bounded coefficients. Their techniques come from the diophantine analysis on “thin” matrix groups (see [SA 37]). Kotsovolis shows how their analysis can be adapted to produce a Hall Ray.

(4) We know little about \( \text{Bass}(\text{M}(D)) \) other than it contains infinitely many limit points. This was proved by Woods [WO] and Wilson [WI] and more geometrically and generally by McMullen [M] who points to the surprising richness of \( \text{Bass}(\text{M}(D)) \) and coined the name “arithmetic chaos.”
(5). \( \text{BASS}(\mathcal{U}_s(D)) \) is rigid.

Dirichlet gave the first examples of such \( s \)-unit \( d \)'s for which \( \mathcal{H}(d) \) is small and even equal to 1. For such it follows from Duke [DU] that the corresponding \( \phi \) is rigid (ie \( \mathcal{N}(\phi) \to 0 \) as \( d \to \infty \)). Aka and Schapira [A-S] give a far reaching extension of this phenomenon to such \( s \)-unit \( d \)'s. Besides showing that the class numbers \( \mathcal{H}(d) \) are small for \( d \in \mathcal{U}_s(D) \) they prove a Duke equidistribution type theorem using Hecke orbits which leads to the rigidity of \( \text{BASS}(\mathcal{U}_s(D)) \).
The set $C(d)$ of integral (primitive) forms of discriminant $d$ form a group of order $h(d)$ under Gauss composition, in order to examine the rigidity properties of the $\phi$s in $C(d)$ set

$$m(d) = \max_{\phi \in C(d)} m(\phi), \quad (12)$$

For isotropic forms $d = t^2$ set

$$m'(t^2) = \max_{d(\phi) = t^2} m'(\phi), \quad (13)$$

If $\mathcal{F}$ is a set of discriminants then $\text{Bass}(\bigcup_{d \in \mathcal{F}} C(d))$ is rigid iff $m(d) \to 0$ as $d \to \infty$, def.

Duke's theorem [Du] implies that the probability measures on $[0,1]$ given by

$$\frac{1}{h(d)} \sum_{\phi \in C(d)} \delta_{\phi} \quad (14)$$

converge to $\delta_0$ (Dirac mass at 0) as $d \to \infty$.

So if $h(d)$ is very small then it implies rigidity.
The size of \( \mathfrak{h}(d) \) is a notoriously difficult problem. From Dirichlet's class number formula

\[
\mathfrak{h}(d) \log E_d \approx \sqrt{d} \quad (15)
\]

where \( E_d \) is the fundamental solution to

\[
e^2 - d\,e^2 = 4 \quad (16)
\]

It is expected that almost all \( d \)'s when ordered by size have \( \mathfrak{h}(d) = O(d^\varepsilon) \) for \( \varepsilon > 0 \) (Hooley [400], [5A 2]). From this and Popa [PO] and the subconvexity [H-M] it would follow that \( \mathfrak{m}(d) \to 0 \) for almost all \( d \)'s. That is to say there is a subset \( \mathcal{F} \) of \( d \)'s of full density for which \( \mathcal{F} \cap \mathcal{C}(d) \) is rigid. In particular this would apply to fundamental discriminants.

While establishing anything like this for fundamental discriminants, the arithmetic chaos problem for \( \mathfrak{m}(d) \) (\( d \) fixed) is much more tractable.
For $d$'s of the form $Dp^2$, $p$ a prime, Golubeva [G0] exploits that the order of the fundamental unit $\epsilon(Dp^2)$ is large if the order of $\epsilon(d)$ is large in the class group $(\text{mod} p)$. This reduces the construction of large units to an Artin primitive root type problem and assuming the Generalized Riemann Hypothesis (GRH) she shows that for many $p$'s $h(5p^2)$ is equal to 2 (its minimal possible value) and hence it follows that $m(5p^2) \to 0$ for these $p$'s. This can be pushed further and Aka [A] and Kurlberg show that under GRH for almost all $d \in M(D)$, $h(d) = O_{\varepsilon}(d^{\varepsilon})$. Hence under the same assumption there is a subset $F$ of $t$'s of full density such that $m(Dt^2) \to 0$ as $t \to \infty$, cf. (17).

The story with $D=1$, i.e. the isotropic forms is quite different.
ZAREMBA [Z2] conjectures that there is $\eta > 0$ such that
\[ m'(t^2) \geq \eta \quad \text{for } t \geq 1. \tag{18} \]
Equivalently, in terms of his $p$-function, for all $q \geq 1$
\[ \max_{1 < a < q \atop (a,q) = 1} p\left(\frac{a}{q}\right) \geq \eta. \tag{19} \]

McMullen [Mc] gives a quantified version of (18) (see his Conjecture 6.2).

Mercat [Me] relates the isotropic $\phi$s to the anisotropic ones by explicit constructions of $\phi$s with large $M(\phi)$, using rationals with large $p$. This allows him to produce limit points in $\text{Bass}(M(D))$ which are large (corresponding to continued fractions having $a$'s in $\{1,2\}$). From his construction it follows that
\[ m\left((t^2-1)t^2\right) \geq \eta > 0 \quad \text{for } t \geq 1 \]
if (18) is true. \tag{20}
The discriminants \( d = (t^2 - 1)t^2 \) have small units \((E_d = O(d))\) which might be necessary for (20) to hold. While the optimistic conjecture by Einsiedler-Lindenstrauss-Michel-Venkatesh [E-L-M-V] that if \( \varphi(d) = O(d^{1/2 - \alpha}) \) for some \( \alpha > 0 \), then each geodesic in \( \text{SL}(2, \mathbb{Z})/\text{SL}_2(\mathbb{R}) \) corresponding to a \( \phi \in C(d) \) becomes equidistributed, is false as shown by Solan and Yifrach [S-Y]; that \( m(d) \to 0 \) under this assumption may still hold. One might even quantify this and postulate that

\[
\frac{m(d) d^{1/2}}{\varphi(d)} \sim d^{O(1)} \quad \text{as} \quad d \to \infty \tag{21}
\]

The smallest possible units come from \( d^1 \)'s of the form \( t^2 - 4 \) \((E_d = \sqrt{d})\) and Bourgain and Kontorovich [B-K 27] conjecture something slightly weaker (but quantitative) than the following analogue of (20)

There is \( \gamma' > 0 \) such that

\[
m(t^2 - 4) \geq \gamma' \quad \text{for} \quad t > 2. \tag{22}
\]
The Bass viewpoint suggests the following problems:

(a) For a given D show that \( \text{Bass}(M(D)) \) has a Hall ray at 0.

(b) Remove the GRH assumption on page 11, that is prove that for each \( D > 1 \) there is a subset of \( M(D) \) of full density which is rigid.

(c) For any \( D > 1 \), \( \text{Bass}(M(D)) \cap (\frac{1}{3}, 1) \) is finite. This is equivalent to showing that the square-free part of \( 9m^2 - 4 \) goes to infinity as \( m \) goes over Markoff numbers. See Corvaja-Zannier [C-Z] and Luca [L] for related problems.

(d) \( \text{Bass}(\text{Fund}) \cap (\frac{1}{3}, 1) \) is infinite. This is equivalent to \( 9m^2 - 4 \) being square-free for infinitely many Markoff numbers. Unfortunately the recent progress on divisibility of Markoff triples (Bourgain-Gamburd-Sarnak [B-G-S], Chen [Ch]) is still very far from addressing this.

(e) Prove that there is an (infinite) rigid sequence of fundamental \( D \)'s.
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