Amalfi lecture

Origin: 1946, 1. work done up to early 50's on distribution of values of certain F&E. dirichlet series (first $\log \zeta(s)$), then generalizations on vertical lines near ~~& results &~~ the critical line $\sigma = \frac{1}{2}$. various conjectures were arrived at.

2. In early 70's work on distribution of "a" points (with $a \neq 0$) of $\zeta(s)$ and F&E. generalizations, leading again to results and conjectures.

3. Finally, ~~quite~~ in recent years, stimulated by the work by Bombieri and Hejhal on the zeros of Epstein zeta functions on binary forms, work on functions that are linear combinations of (F & E) dirichlet series with the same functional equation.

We shall consider series, abs. conv for $\sigma > \sigma_0$,

$$(1) \quad F(s) = \sum \frac{a_n}{n^s}, \quad s = \sigma + it$$

Write

$$(2) \quad \phi(s) = \varepsilon Q^s \prod_{i=1}^{k} \Gamma(\lambda_i s + \mu_i) \; F(s)$$

where $|\varepsilon| = 1$; $Q > 0$; $\lambda_i > 0$, $R\mu_i \geq 0$.

We assume $F(s)$ meromorphic, even so that $(s-1)^m F(s)$ is an integral function of finite order (less would do for our purposes), and that it has a functional equation

$$(3) \qquad \Phi(s) = \overline{\Phi(1-\bar{s})},$$

this implies that $\Phi(s)$ is real for $s = \frac{1}{2}+it$. Phragmén-Lindelöf now gives that $(s-1)^m F(s)$ is of order 1.

The zeros of $F(s)$ can be divided into two classes: trivial zeros: located at poles of $\Gamma(\lambda_i z+\mu_i)$ for $i=1,\dots,k$; and the rest which we refer to as nontrivial, they will be located in some vertical strip $A \geq \sigma \geq 1-A$. writing

$$(4) \qquad \Phi(\tfrac{1}{2}+it) = e^{i\vartheta(t)} F(\tfrac{1}{2}+it),$$

and $\Lambda = \sum_{i=1}^{k} \lambda_i$

we have if $N(T)$ denotes the number of nontrivial zeros in the region $0 \leq t \leq T$ for positive T (counting zeros at the limit with half multiplicity) that

$$(5) \qquad N(T) = \frac{\vartheta(T)}{\pi} + \frac{1}{\pi} \arg F(\tfrac{1}{2}+iT) + c$$

$$= \frac{\Lambda}{\pi} T(\log T + c') + O(\log T),$$

here c and c' are constants, a similar formula holds for nontrivial zeros in ~~with~~ $T \leq t \leq 0$ if T is negative (zeros need not lie symmetrically to the real axis, of course).

We now make further assumptions,

$$(6) \qquad a_1 = 1, \quad a_n = O(n^\delta) \text{ for any } \delta > 0,$$

and

$$(7) \qquad \log F(s) = \sum \frac{b_n}{n^s} \quad \text{with } b_n = 0 \text{ except if } n = p^r, \text{ with } r \geq 0$$

Further assume

(8) $\quad b_n = O(n^\theta)$ for some $\theta < \frac{1}{2}$.

From (6), (7) and (8) follows that

(9) $\quad \sum^* \frac{|b_n|^2}{n^\sigma}$ converges for $\sigma > 2\theta$

where $\sum^*$ means that the sum is extended only over the $n = p^r$ with $r > 1$.

The value distribution of $\log F(s)$ around and on $\sigma = \frac{1}{2}$ will turn out to be governed by the behaviours of sum

(10) $$\sum_{n \le x} \frac{|b_n|^2}{n} = \sum_{p \le x} \frac{|a_p|^2}{p} + O(1).$$

In all cases where this behaviour has been determined it turns out that

(11) $$\sum_{p \le x} \frac{|a_p|^2}{p} = n_F \log\log x + O(1)$$

where n_F is an integer, it is reasonabel to conjecture that this is always so.

We shall call a function $F(s)$ of the kind we consider "primitive" if $F(s)$ can not be written as $F(s) = F_1(s) F_2(s)$ where F_1 and F_2 again satisfy our conditions.

Conjecture: for a primitive function F we have $n_F = 1$, so that

(12) $$\sum_{p \le x} \frac{|a_p|^2}{p} = \log\log x + O(1).$$

We further conjecture that for two different primitive functions F and F'

we have

$$(13) \qquad \sum_{p \le x} \frac{a_p \overline{a'_p}}{p} = O(1),$$

so that in some sense the primitive functions form an orthonormal system. (they of course do not form a hilbert space, for any dirichlet L function with a ~~nonprincipal~~ primitive character $L(s+ia)$ where a is real is also a primitive function, so their number is not denumerable)

The conjectures (11), (12) and (13) could also have been stated in a stronger form where each term in the sum has an additional factor $\log p$ and $\log\log x$ is replaced by $\log x$.

These conjectures, which by the way are not unrelated to several other conjectures like the Sato-Tate conjecture and Langlands conjectures etc. (one obvious consequence is that the Riemann zeta-function is the only primitive function with a pole, and that any $F(s)$ in our class with a pole of order m at $s=1$ is divisible with $(\zeta(s))^m$), have been verified in a number of cases for the dirichletseries with functional equation and Eulerproduct that occur in numbertheory, by assuming that the factorization we can give is actually into primitive factors. Only in very few cases can we actually show that a function is primitive. One can show that for all such functions $\Lambda \ge \frac{1}{2}$, and only when $\Lambda \le 1$ are we able to verify that a function is primitive, since it is either primitive or of the form $\zeta(s)\, L(s+ia)$ or the form $L_1(s+ia)\,L_2(s+ia')$ with a and a' real and ~~L_1 and L_2~~ the L's dirichlet L functions belonging to primitive characters.

It is finally natural to conjecture that the Riemann hypothesis holds for the class of $F(s)$ we are considering, of course this conjecture has not been verified in even a single instance.

For the investigation of the value distribution of $\log F(\sigma+it)$ on $\sigma=\frac{1}{2}$ or for σ very near to $\frac{1}{2}$, we need besides (11), either the Riemann hypothesis or some weaker hypothesis concerning the nontrivial zeroes.

If we, as usual, denote by $N(\sigma,T)$ the number of zeroes ~~with~~ $\rho=\beta+i\gamma$ with $\beta\geq\sigma$ and γ between 0 and T, and assume that for $\sigma>\frac{1}{2}$, we have

$$(14)\quad N(\sigma,T)=O\left(|T|^{1-\alpha(\sigma-\frac{1}{2})}\frac{\log|T|}{\sqrt{\log\log T}}\right),$$

~~where~~ uniformly, where α is some positive constant, we will be able to prove as sharp results as on the Riemann hypothesis. (14) can not be verified in any single instance either. However, a slightly weaker form of (14), namely

$$(14') \quad N(\sigma, T) = O\left(|T|^{1-\alpha(\sigma-\frac{1}{2})} \log|T|\right)$$

for $\sigma > \frac{1}{2}$, can actually be proved for the Riemann zeta function, for dirichlets L-functions, and using techniques more recently developed by Good, Iwaniec and Hafner now also for the L-functions of quadratic fields and for the dirichlet series derived from cusp forms (analytic or solutions of the hyperbolic wave equations) which are eigenfunctions of the Hecke operators for the modular-group or some congruence subgroup.

Using (14') we do not obtain quite so sharp results as with the R.H. or (14), but for the classes of functions for which (14') can be proved, these results are then proved unconditionally.

Approximate formulas for $\log F(s)$.

We shall state this formula only in the form we use if we assume the R.H., if we only assume (14) or (14') we use a analogous but more complicated formula

For $x \geq 2$, define

$$(15) \quad \Theta_x(n) = \begin{cases} 1, & \text{for } 1 \leq n \leq x, \\ \dfrac{\log \frac{x^2}{n}}{\log x}, & \text{for } x \leq n \leq x^2, \\ 0, & \text{for } n \geq x^2. \end{cases}$$

also write $b_x(n) = b_n \Theta_x(n)$. Further let $2 \le x \le t^2$ and $\sigma_x = \frac{1}{2} + \frac{1}{\log x}$.

Then, for $\sigma \geq \sigma^* \geq \sigma_x$, we have

(16) $$\log F(\sigma+it) = \sum_{n<x^2} \frac{b_x(n)}{n^{\sigma+it}} + O\left(\frac{x^{\frac{1}{2}-\sigma}}{(\sigma^*-\frac{1}{2})\log^2 x}\left(\left|\sum_{n<x^2} \frac{b_x(n)\log n}{n^{\sigma^*+it}}\right| + \log|t|\right)\right),$$

And for $\frac{1}{2} \le \sigma \le \sigma_x$, we have if we write $\eta_t = \min_\rho |t-\gamma|$, that

(16') $$\log F(\sigma+it) = \sum_{n<x^2} \frac{b_x(n)}{n^{\sigma+it}} + O\left(\frac{1+\overset{+}{\log}\frac{1}{\eta_t \log x}}{\log x}\left(\left|\sum_{n<x^2} \frac{b_x(n)\log n}{n^{\sigma_x+it}}\right| + \log|t|\right)\right).$$

The O-constants depend on F only. It should be remarked that if we take the imaginary part of (16') we may drop the term $\overset{+}{\log}\frac{1}{\eta_t \log x}$.

If we instead of assuming the R.H assume only (14) or (14') we have similar formulas where σ_x is replaced by a $\sigma_{x,t}$ which depends also on t and is defined in a more complicated way, also the $b_x(n)$ have a more complicated definition.

These formulas enable us to prove that if k is a positive integer $0<a<1$ and $T^{\frac{a}{k}} \le x \le T^{\frac{1}{k}}$, then for $\frac{1}{2} \le \sigma$

(17) $$\frac{1}{T}\int_0^T \left|\log F(\sigma+it) - \sum_{p<x} \frac{a_p}{p^{\sigma+it}}\right|^{2k} dt = O(k^{4k} e^{Ak})$$

with a constant A depending on a and F.

If we in (17) replace the expression in the $| \quad |$ sign by its imaginary part we can replace the k^{4k} in the remainder term by k^{2k}.

This enables us to get hold of the distribution of $\log F(\sigma+it)$ in the complex plane or of the distribution of $\log|F(\sigma+it)|$ and $\operatorname{Arg} F(\sigma+it)$ separately on the real line when $\sigma=\frac{1}{2}$ or is $\geq\frac{1}{2}$ and so close to $\sigma=\frac{1}{2}$ that

$$\sum_{p\leq T^{\frac{1}{k}}} \frac{|a_p|^2}{p^{2\sigma}} \quad \text{is of same order of}$$

magnitude as $\sum_{p\leq T^{\frac{1}{k}}} \frac{|a_p|^2}{p}$, which is roughly when

$$\frac{1}{2} \leq \sigma \leq \frac{1}{2} + (\log T)^{-\delta}$$

for some positive δ.

One finds that

$$\frac{\log F(\sigma+it)}{\sqrt{\pi \sum_{p\leq T} \frac{|a_p|^2}{p^{2\sigma}}}}$$

has a normal gaussian distribution in the complex plane, and the real part and imaginary part a normal gaussian distribution on the real line.

Since in the region considered

$$\sum_{p\leq T} \frac{|a_p|^2}{p^{2\sigma}} = n_F \log \min\left(\frac{1}{\sigma-\frac{1}{2}}, \log T\right) + O(1),$$

all primitive functions F have the same distribution. One also finds that distinct primitive functions are statistically independent.

We may look at $\log|F(\sigma+it)|$ also for $\sigma < \frac{1}{2}$ using the functional equation to relate it to $\log|F(1-\sigma+it)|$, we see that in the region $|\sigma-\frac{1}{2}| \le (\log T)^{-1+\varepsilon}$ with $\varepsilon \to 0$, where,

$$\sum_{p\le T} \frac{|a_p|^2}{p^{2\sigma}} \sim n_F \log\log T$$

we get

$$\frac{\log|F(\sigma+it)|}{\sqrt{\pi n_F \log\log T}} = \frac{\log|F(1-\sigma+it)|}{\sqrt{\pi n_F \log\log T}} + \Lambda \frac{(\frac{1}{2}-\sigma)\log(t)}{\sqrt{\pi n_F \log\log T}} + O(1)$$

Thus in the region $\sigma = \frac{1}{2} - \frac{\varphi(T)}{\log T}$ with $\varphi(T) = \mu\sqrt{\log\log T}$, ~~the gaussian distribution of~~ $\frac{\log|F(\sigma+it)|}{\sqrt{\pi n_F \log\log T}}$ has a gaussian distribution that has been shifted by an amount $\frac{\Lambda\mu}{\sqrt{\pi}\, n_F}$. Results with rather precise remainder terms can be obtained. This makes it possible to evaluate various integrals involving $\log F(\sigma+it)$

For instance the integral

$$\int_0^T \overset{+}{\log} |F(\sigma+it)|\, dt$$

can be asymptotically evaluated for $\sigma = \frac{1}{2}$ or near $\frac{1}{2}$, and this enables us to evaluate integrals

$$(18) \qquad \int_0^T \log|F(\sigma+it)-a|\, dt$$

where $a \ne 0$, Essentially as

integrals over the sets in $(0,T)$ where $|F(\sigma+it)| > \frac{3}{2}|a|$, $|F(\sigma+it)| < \frac{1}{2}|a|$ and finally the exceptional set where $\frac{1}{2}|a| \le |F(\sigma+it)| \le \frac{3}{2}|a|$. This exceptional set can be shown to have measure which is $O\left(T \frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right)$. The two first integrals are easily computed from our knowledge of the distribution function of $\log|F(\sigma+it)|$. The first integral gives the dominating part of order $T\sqrt{\log\log T}$, the second a correction term of order T, but the third integral which can be easily evaluated for $\sigma = \frac{1}{2}$ and is then of order $O\left(T \frac{(\log\log\log T)^3}{\sqrt{\log\log T}}\right)$, gives ~~some problems~~ trouble in the region $\sigma < \frac{1}{2}$ if we do not assume the R.H and in the region $\sigma > \frac{1}{2}$ even if we assume the R.H. However even in these cases we can show that the estimation $O\left(T \frac{(\log\log\log T)^3}{\sqrt{\log\log T}}\right)$ holds for almost all a (in a strong sense!).

Using a theorem by Littlewood the evaluation of the integrals (18) enable us to make some statements concerning the distribution of the a-points of $F(s)$. In the region $0 < t < T$; $|\sigma| < A$ there are about as many a-points as zeroes, about half of them lie to the left of the critical line statistically well distributed at distances of order $\frac{\mu\sqrt{\log\log T}}{\log T}$ from the line $\sigma = \frac{1}{2}$, of the remaining half must lie close in to the line at distances of the order $\frac{1}{\log T \sqrt{\log\log T}}$, presumably about half of these are to the left and half to the right of the critical line. This last we can not prove without making some (rather plausible) conjecture about the distribution of the zeroes of $F(s)$ on the line, however we can prove that if we take two values of a such that their ratio is real and negative then $(F(s)-a)\cdot(F(s)-a')$ for almost all values of the argument of a has about $\frac{3}{4}$ of its zeroes on the left side and about $\frac{1}{4}$ of its zeroes on the right side of the critical line. Presumably this is true for $F(s)-a$ itself and for all $a \neq 0$.

We shall now look at functions which are linear combinations of the kind of functions we have considered so far, say

$$(19) \qquad F(s) = \sum_{i=1}^{n} c_i F_i(s) \quad \text{, with } n > 1$$

where we assume that all the F_i satisfy the same functional equation we shall not assume them all to be primitive, but distinkt,

We shall need some properties that are easily established. For simplicity we shall assume that the F_i are relatively prime, then it is easily shown that the $\log F_i(s)$ are statistically independent. ~~also~~

In the region $|\sigma - \frac{1}{2}| < (\log T)^{-1+\varepsilon}$ where $\varepsilon \to 0$ with T, we can show that if $i \neq j$

$$\left| \log|F_i(\sigma+it)| - \log|F_j(\sigma+it)| \right| > o(\sqrt{\log_2 T})$$

outside of a set in $(0,T)$ of measure $o(T)$; thus outside this set we have

$$\log|F(\sigma+it)| \approx \max_i \log|F_i(\sigma+it)|.$$

On the line $\sigma = \frac{1}{2}$ for instance, this implies that $\log|F(\frac{1}{2}+it)|$ is large except for a set in $(0,T)$ of measure $\sim 2^{-n}T$.

Also for "almost all" sets of c_i we have

$$(20) \quad \int_0^T \log|F(\sigma+it)|\,dt \sim \int_0^T \max_i \log|F_i(\sigma+it)|\,dt$$

and, similarly,

$$(20') \quad \int_0^T \log|F(\sigma+it)-a|\,dt \sim \int_0^T \max(\log F_i(\sigma+it), \log|a|)\,dt$$

From our knowledge of the distribution functions of the $\log|F_i(\sigma+it)|$, we are able to evaluate the integrals on the righthand side of (20) and (20'). It is evident that the right hand sides tend to increase as n increases, indicating a movement to the right of the zeroes of $F(s)$ and $F(s)-a$, by considering Littlewoods formula.

one is led to the conjecture that for such a function $F(s)$ and $a \neq 0$ about $\frac{1+2^{-n}}{2}$ of the zeros lie to the left of the critical line and ca. $\frac{1-2^{-n}}{2}$ of them to the right. All one can prove is that for almost all such $F(s)$ and almost all arguments of a, $(F(s)-a)\cdot(F(s)-a')$, where $\frac{a'}{a}$ is negative, has these proportions of its zeros on the two sides of the critical line.

If we only assume the $F_i(s)$ distinct but not relatively prime, the results are similar, but the numerical factors entering above as 2^{-n}, $\frac{1+2^{-n}}{2}$ and so forth become more complicated to compute.

If we assume in (19) that the c_i are all real, we have that

$$\varepsilon Q^s \prod_{i=1}^{k} \Gamma(\lambda_i s+\mu_i)\, F(s)$$

is real on the line $\sigma = \frac{1}{2}$.

We may then expect that there are zeroes of $F(s)$ on the critical line, though for $n > 1$ the fact that ~~for~~ $|F(\frac{1}{2}+it)|$ is more often large than small shows that the sum

$$\sum_{\substack{0<\gamma<T \\ \beta>\frac{1}{2}}} (\beta-\tfrac{1}{2})$$

is large (actually of the order $T\sqrt{\log\log T}$) so quite many zeros do lie to the right of the critical line.

Utilizing the fact that the $\log|F_i(\frac{1}{2}+it)|$ are not independent for values of t in $(0,T)$ that differ by an amount which is

$$< (\log T)^{-1+\varepsilon},$$

where ε tends to zero with T (the region of inertia, we might call it), and assuming that the zeros of the $F_i(s)$ are well spaced, say that

$$\frac{\#\left(|\gamma_{n+1}-\gamma_n| < \frac{\delta}{\log T},\ 0 < \gamma_n < T\right)}{T\log T} = \sigma(\delta)$$

as $\delta \to 0$. (For the zeta function for instance the pair correlation hypothesis gives that this ratio is $O(\delta^3)$).

We may divide $(0,T)$ in intervals of length $H = \frac{\varphi(T)}{\log T}$ where $\phi(T) = e^{(\log\log T)^\alpha}$, $0<\alpha<1$.

We find except for $o\left(\frac{T}{H}\right)$ of these intervals that in each I_{t_n, t_n+H} one $\log|F_i(\tfrac{1}{2}+it)|$ dominates over the others except for set of measure $o(H)$, and if its zeros are well spaced its oscillations will also dominate so that $F(\tfrac{1}{2}+it)$ has about as many zeros in I_{t_n,t_n+H} as $F_i(\tfrac{1}{2}+it)$ has, which is $\frac{\Lambda}{\pi} H\log T + O((H\log T)^{\frac{3}{4}})$ except for the thin set of I numbering $o\left(\frac{T}{H}\right)$. Thus the hypothesis of well spacing gives almost all zeros of $F(s)$ are on the critical line.

For the case of the Epstein zeta-function of a rational binary form this result was conjectured by H. Montgomery and then proved on the hypothesis of well-spacing by Bombieri and Hejhal.

We may finally speculate on what is the case if we have an $F(s)$ which is such that $F(s)$ ~~[illegible]~~ ~~can be approximated by expressions~~ are the limits of expressions like (19) or like (19) multiplied with some factor N^s where N may grow as n tends to infinity. Examples would be for instance the Epstein zeta-function of a definite binary but irrational form.

It seems a reasonable conjecture that such a function is almost always large on the critical line, and that its a-~~values~~ points for $a \neq 0$ are about equally many on each side of the critical line.

It is rather more dubious, whether if the c_i in the approximations are real, it would have almost all zeros on the critical line. This might be dependent on how well it could be approximated compared with the growth of n. Numerical calculations are unlikely to give any indications of the truth since we are dealing with very slow convergence. Factors like $\sqrt{\lg \lg T}$ hardly change much in the range accessible by computation.

Finally let me mention another conjecture, also beyond any numerical evidence that can be expected in a forseeable time.

If $F(s)$ is a function of the kind considered in the early part ~~with~~ which is a product of distinct primitive factors then the number of signchanges of $\arg F(\frac{1}{2}+it)$ in $(0,T)$ is asymptotic to

$$\frac{2\sqrt{\pi}}{\sqrt{n_F \log\log T}} \; N(T)$$

$$\sim \quad \frac{2 \Lambda T \log T}{\sqrt{\pi n_F \log \log T}} \; ,$$

We can prove an upper bound which differs from this only by a factor $(\log\log\log T)^2$ in order of magnitude, from below we can not get that close.