(1) \( F(s) = \sum \frac{an}{m^s} \),

absolutely convergent for \( \sigma > 1, s = \sigma + it \).

Assume \( F(s) \) meromorphic, but so that \( (s-1)^m F(s) \) is an integral function of finite order for some integer \( m > 0 \).

Also assume

(2) \( \overline{F(s)} = \overline{F(1-s)} \),

where

(3) \( \overline{F(s)} = \sum \frac{1}{\Gamma(\lambda_i s + \mu_i)} F(s) \),

where the constants \( \lambda_i, \mu_i \), and for the constants \( \varepsilon, \alpha, \lambda_i, \mu_i \),

\( |\lambda_i| = 1, Q > 0, \lambda_i > 0, \Re \mu_i \geq 0 \).

(4) \( \overline{F(s)} \) is real for \( s = \frac{1}{2} + it \).

Also assume \( (s-1)^m F(s) \) is of order \( 1 \).

We write

(4) \( \phi(\frac{1}{2} + it) = \chi(t) e^{i \varphi(t) s} F(\frac{1}{2} + it) \),

where \( \chi(t) \) and \( \varphi(t) \) are real and \( \chi(t) > 0 \).

Also, we put

(5) \( \Lambda = \sum \lambda_i \),

Zeros of \( F(s) \) are divided into two classes:

- trivial zeros, located at poles of the \( \Gamma(\lambda_i s + \mu_i) \);
- and the rest, which we call non-trivial.
Non-trivial zeros all located in some vertical strip $1-A \leq \sigma \leq A$, where $A$ is some positive constant.

If $N(T)$ denotes number of non-trivial zeros in region $0 \leq t \leq T$, counting each on boundary with half multiplicity, then

$$N(T) = \frac{\Phi(T)}{\pi} + \frac{1}{i} \arg F\left(\frac{1}{2} + iT\right) + c =$$

$$= \frac{A}{\pi} T \left(\log T + c'\right) + O(\log T),$$

where $c$ and $c'$ are constants. Similar formula holds for zeros with negative imaginary part as one can see by looking at $\overline{F(A)}$ instead of $F(A)$.

Further assumptions:

\begin{align*}
(7) \quad a_1 &= 1, \quad a_m = O\left(m^\delta\right) \\
\text{for any fixed } \delta > 0, \text{ and} \\
(8) \quad \log F(\sigma) &= \sum_{m} \frac{b_m}{m^\sigma},
\end{align*}

where $b_m = 0$ except when $m$ is of form $p^k \pi$ with integer $k$ and $\pi$ a prime and $\delta$ a positive number. Also

$$b_m = O\left(m^\theta\right),$$

for some $\theta < \frac{1}{2}$. 
From (7), (8) and (9) follows that the series

$$\sum_{m} \frac{\log m}{m^s}$$

where \( \sum \) indicates that sum is extended only over the \( m = p^r \) with \( r \geq 1 \), is convergent for \( s > \max (2\sigma, \frac{1}{2}) \).

Thus

$$\sum_{m \leq x} \frac{\log m}{m} = \sum_{p \leq x} \frac{\log p}{p} + O(1).$$

**Conjecture 1.** We have

$$\sum_{p \leq x} \frac{\log p}{p} = (n_F \log \log x + O(1)),$$

where \( n_F \) is an integer depending on \( F \).

"Prime free" function: We say \( F(x) \) is "prime-free" if \( F(x) \) can not be written as \( F(x) = F_1(x) F_2(x) \) where \( F_1 \) and \( F_2 \) both satisfy our conditions.

**Conjecture 2.** For a prime-free function \( F \) we always have \( n_F = 1 \), so that

$$\sum_{p \leq x} \frac{\log p}{p} = \log \log x + O(1).$$
Conjecture 3. For two distinct primitive functions $F$ and $F'$ we have

\[
\sum_{p \leq x} \frac{a_p}{p} = O(1).
\]

If with $F(s)$ also

\[
F^*(s) = \sum \frac{a_m}{m^s} X(m),
\]

satisfies our conditions, where $X$ is a primitive Dirichlet character then $F$ and $F^*$ would have the same $N_F$ so they are either both primitive or both not primitive.

Conjecture 4. If $F$ and $F^*$ are not primitive, we can get the factorization of $F^*$ into primitive factors by putting the character $X$ into the series defining the primitive factors of $F$.

Conjecture 5. The Riemann hypothesis holds for the class of functions $F(s)$ which satisfy our conditions.

For the investigation of value distribution of $\log F(s, \zeta)$ for $\sigma = \frac{1}{2}$ or $\sigma$ very near to $\frac{1}{2}$, we need, beside (11), either to assume RH or some weaker hypothesis. Denote by $N(\sigma, T)$ the number of zeros $\sigma = \beta + i \gamma$ with $\beta > \sigma$ and $|\gamma| \leq T$. 
If we assume that for \( \sigma > \frac{1}{2} \), we have

\[
N(T, T) = O(T^{1 - \alpha(\sigma - \frac{1}{2})} \log T)
\]

uniformly in \( \sigma \) as \( T \to \infty \), where the constant \( \alpha \) is positive, we can prove essentially as sharp results as on R.H.

Neither R.H. nor (14) has been proved to hold for any \( F(s) \) in our class.

A slightly weaker form of (14),

\[
N(T, T) = O(T^{1 - \alpha(\sigma - \frac{1}{2})} \log T)
\]

uniformly for \( \sigma > \frac{1}{2} \), can actually be proved for some \( F(s) \).

Using (14') we do not in general obtain quite as sharp results as on R.H. on (14)

though in some instances we do. For these cases where (14') is proved and (14) holds
we have results that are proved unconditionally.

Approximate formulas for

\[
\log F(s) \text{ in the critical strip:}
\]
For \( x \geq 2 \) define:

\[
\Theta_x(n) = \begin{cases} 
1, & \text{for } 1 \leq n \leq x, \\
2 - \frac{\log(n)}{\log(x)}, & \text{for } x \leq n \leq x^2, \\
0, & \text{for } x \geq x^2.
\end{cases}
\]

While also \( \delta_x(n) = \delta_{m,n} \Theta_x(n) \), let \( 2 \leq x \leq t^2 \) and \( \sigma_x = \frac{1}{2} + \frac{1}{\log x} \). Then, assuming R.H. for \( \Phi(s) \), we have for \( \sigma_x \leq \sigma \leq \sigma^* \leq \sigma \),

\[
\log F(\sigma + it) = \sum_{m < x^2} \frac{\delta_x(m)}{m^{\sigma+it}} + \Theta \left( \frac{x^{1/2} - \sigma}{(\sigma - \frac{1}{2}) \log x} \left( \sum_{m \leq x^2} \frac{\delta_x(m) \log \eta}{m^{\sigma+it}} \right) + \left( \frac{1 + \log \frac{\eta \log x}{\log x}}{\log x} \left( \sum_{m \leq x^2} \frac{\delta_x(m) \log \eta}{m^{\sigma+it}} \right) \right) \right).
\]

Also, for \( \frac{1}{2} \leq \sigma \leq \sigma_x \), writing \( \eta_t = \min \left\{ \eta \right\} \), that

\[
\log F(\sigma + it) = \sum_{m < x^2} \frac{\delta_x(m)}{m^{\sigma+it}} + \Theta \left( \frac{1 + \log \frac{\eta_t \log x}{\log x}}{\log x} \left( \sum_{m \leq x^2} \frac{\delta_x(m) \log \eta}{m^{\sigma+it}} \right) \right).
\]

The constants implied by the \( \Theta \)-symbols depend on \( \Phi \) only. If we take the imaginary part of (1.6), we can drop the term

\[
\log \frac{\eta_t \log x}{\log x}.
\]

If we assume (14) or (14') instead of R.H.,

we have similar but somewhat more complicated formulas, where \( \Phi_x \) is replaced by \( \Phi_x, \) defined in a way that depends on the zeros \( \pi \) with imaginary parts close to \( \eta \). The \( \delta_x(n) \) also has a more complicated definition.
If we do not assume R.H. put
\[ G_n = \begin{cases} 1 & \text{for } 1 \leq n \leq x, \\ 2n(\frac{\log \frac{x}{n}}{\log x})^2 & \text{for } x \leq n \leq x^{\frac{2}{3}}, \\ 2(\frac{\log \frac{x}{n}}{\log x})^2 & \text{for } x^{\frac{2}{3}} \leq n \leq x, \\ 0 & \text{for } n \geq x^2. \end{cases} \]

Also if \( \psi \) runs over the zeros
\[ \psi = \beta + i \gamma \quad \text{with} \]
\[ |t - \gamma| \leq \frac{2(\beta - \frac{1}{2})}{\log x}, \]
we define
\[ G_{x, t} = \frac{1}{2} + 2 \max_{\psi} \left( \beta - \frac{1}{2}, \frac{2}{\log x} \right). \]

The \( G_{x, t} \) plays the role of \( G_x \) in (16) and (16').

If we assume (14') and \( \alpha^* < \alpha \), we have for \( (3 x)^2 < T \), that:
\[ \alpha^*, \alpha^* > 0 \]
\[ \int_0^{G_{x, t} - \frac{1}{2}} \frac{2 G_{x, t} e^{-1}}{\zeta(1 + 1)} dt = O(ke^A \frac{T}{(\log T)^2}). \]
We can now prove:

Theorem 1. Let \( T > 0 \), \( 0 < a < 1 \), and \( T^{\frac{a}{2}} \leq x \leq T^{\frac{1}{2}} \), then for \( \sigma \geq \frac{1}{2} \) we have

\[
(17) \quad \frac{1}{T} \int_0^T \log F(\sigma + it) - \sum_{\frac{T}{h} \leq x \leq \sigma T} \frac{a_p}{p} \log T \, dt = O\left( T^{\frac{1}{2}} \right),
\]

where the constant \( A \) depends on \( a \) and \( F \), and if we do not assume R.H., also on the constant \( \alpha \) in (14). If we replace the expression in the \( L \) symbol by its imaginary part we can replace the \( k^4 \) in the \( A \) symbol by \( k^2 \).

From this we can deduce:

Theorem 2. For \( \frac{1}{2} \leq \sigma \leq \frac{1}{2} + (\log T)^{-\delta} \), where \( \delta \) is fixed > 0, the function

\[
\mathcal{U}(\sigma, t) = \mathcal{U}_{\sigma}(\sigma, t) = \left( \log \frac{T}{\sigma} \right) \sum_{p^\sigma \leq t} \frac{1}{p}.
\]

has a normal Gaussian distribution in the complex plane. Also, the real and imaginary part of \( \mathcal{U}(\sigma, t) \) have a normal Gaussian distribution on the line.

Let \( \chi_{\sigma, \delta}(u) \) denote the characteristic function of an interval \( (\alpha, \beta) \), then

\[
\int_0^T \chi_{\sigma, \delta}(u) \mathrm{d}u = T \int_{-\infty}^\infty e^{-\frac{\pi^2 u^2}{T}} \mathrm{d}u + \Theta(T \sqrt{\log(1+\log T)})^2.
\]
\[
\frac{\pi}{2} \int_0^1 \int_0^1 \frac{\partial^2}{\partial x \partial y} \left( \log \log \log t \right) \, dx \, dy + O(1)
\]

Since in the region \( \sigma \) considered in Th. 2, we have

\[
\sum_{p \leq T} \frac{1}{\log p} = \frac{1}{2} \log \log (\min \left( T, \frac{T}{2} \right)) + O(1)
\]

we see that all primitive functions have the same distribution. Also using (18) we get that distinct primitive functions are statistically independent.

Using functional equation we find in the region \( \sigma < \frac{1}{2} \), that for

\[
\sigma = \frac{1}{2} - \mu \frac{\sqrt{\log T}}{\log T}
\]

\( \mu \) is a constant, then for constant \( \mu > 0 \), that

\[
\frac{\log |f(\sigma + it)|}{\sqrt{\pi \log \log T}}
\]

has a centered Gaussian distribution subject to the right by the amount

\[
|f^{(n)}(\sigma + it)| = \frac{\Lambda^{(n)}}{\sqrt{\pi \log T}}
\]
Distribution of $\alpha$-points or zeros of $F(x) - a$.

If we assume R.H or (14) we can obtain

$$\int_0^T \log |F(\frac{1}{2} + it)| \, dt = \frac{\sqrt{\pi} F}{2\sqrt{t}} \sqrt{\log \log T} +$$

$$+ O\left( T \log \log T \frac{1}{\sqrt{\log \log T}} \right) ,$$

D.H.

Notably, if we only use (14) we get instead the larger remainder term $O(T)$.

Similar evaluations can be given for

$$\frac{1}{2} < \sigma < \frac{1}{2} + (\log T)^{-\delta} \text{, (with somewhat better}\,$$

algebraic terms for the larger $\sigma$ in the range), and also for $\sigma < \frac{1}{2}$ and of the form (19). In the latter case main term involves a more complicated constant depending on $\mu$ (or $\mu'$ given by (20)) because of the shift of the Gaussian distribution.

We use a formula of Littlewood to estimate the sum

$$\sum_{\substack{0 \leq \gamma < T \\ \gamma \in \Gamma_a}} \Gamma(\gamma - \sigma) \text{ where denote zeros of } F(x) - a.$$
one get for \( a \neq 1, 0, \infty, \)

\[
\sum_{0 < y < T} (\beta - \sigma) = \sum_{\beta > \sigma} \sqrt{\log \log T} \left( \frac{e^{-\pi \mu^2}}{2\pi} + \mu - \mu' \int e^{-\pi u^2} du \right) T \sqrt{\log \log T} + \]

\[
+ \int_{\log \log \log T}^{\infty} \left( \frac{\log a - 1}{2\pi} \right) \log \log T + O \left( \frac{\log \log \log T}{\log \log T} \right),
\]

assuming R.H or (14), assuming only (14)

we get a remainder term \( O(T) \). NB!

Defining \( N_a(\sigma, T) \) and \( N_a(T) \)

analogously for \( F(x; a + 0) \) to \( N(\sigma, T) \) and \( N(T) \)

for \( F(x) \), we get for \( \sigma \) of the form (19)

\[
N_a(\sigma, T) = N_a(T) \int_{-\mu'}^{\infty} e^{-\pi u^2} du,
\]

for \( \mu' > 0 \). Shows about half of \( a \)-points
distributed to the left of line \( \sigma = 1/2 \) at
distances of order \( \sqrt{\log T} \) from line with
distances corresponding to half a Gaussian
distribution.
Most of other half of a-points lie very close to \( \sigma = \frac{1}{2} \), at distances of order most more than
\[
\frac{(\log \log \log T)^3}{\log T \sqrt{\log T}}
\]

Assuming RH or (14) (or (14') somewhat weaker conclusions follow).

Difficult to decide how this family of a-points distribute to the right and left of the line \( \sigma = \frac{1}{2} \), without making some strong (but plausible) additional conjectures about the distribution of the zeros of \( F(0) \).

Can however prove that if we consider \( \alpha \) and \( \alpha' \) such that their ratio \( \alpha' / \alpha \) is real and negative, then for almost all values of the argument of \( \alpha \)
\[
(F(\alpha) - \alpha)(F(\alpha') - \alpha')
\]
takes about \( \frac{3}{4} \) of its zeros to the left of \( \sigma = \frac{1}{2} \) and almost \( \frac{1}{4} \) to the right.

One may conjecture that this is true for \( \Re(\alpha) \) itself and for all \( \alpha \neq 0 \).
Linear combinations:

Let

\[ F(x) = \sum_{i=1}^{n} c_i F_i(x), \]

where \( n > 1 \), the \( c_i \neq 0 \) and the \( F_i \) are linearly independent functions of the kind dealt with earlier, and which all satisfy the same functional equation. For simplicity we will assume the \( F_i \) are relatively prime.

Then the \( \log |F_i(\sigma + it)| \) are statistically independent in the region

\[ |(\log T)^{1/\varepsilon} - 1/\varepsilon| < (\log T)^{-1/\varepsilon} \]

where \( \varepsilon \to 0 \) as \( T \to \infty \). For \( i \neq j \) we can show that in this region,

\[ \left| \log |F_i(\sigma + it)| - \log |F_j(\sigma + it)| \right|^2 \]

\[ > (\log \log \log T), \]

besides except for a subset of \((0, T)\) of measure \( O\left( T \frac{(\log \log \log T)}{\sqrt{\log \log T}} \right) \).
Outside of this subset we have in $(0, T)$

\[
\log |F(\sigma + it)| = \max_i \log |c_i F_i(\sigma + it)| + O\left( e^{-\left( \log \log \log T \right)^2} \right).
\]

As a consequence the distribution functions for

\[
\frac{\log |F(\sigma + it)|}{\sqrt{T/ \log \log t}}
\]

can be determined from those of

\[
\frac{\log |F_i(\sigma + it)|}{\sqrt{T/ N \log \log t}}
\]

Another consequence: on the line $\sigma = \frac{1}{2}$ we have that $|F(\frac{1}{2} + it)|$ is large except for a subset of $(0, T)$ of measure asymptotic to $e^{-\alpha T}$.

We can also show that for "almost all" sets of $c_i$ we have

\[
\int_0^T \log |F(\sigma + it)| \, dt = \\
\int_0^T \log \left( \max_i |c_i F_i(\sigma + it)| \right) \, dt + \\
+ O\left( T \left( \log \log \log T \right)^2 \right)
\]
Also, if $a$ is a constant $\neq 0$,
\[
\sum_{i} \log \left| \max \left| \epsilon_i F_i (\sigma + i \epsilon) \right|, |a| \right| \text{dt} + \sum_{i} (\log \log \log T)^2 \left( \frac{T}{\log \log T} \right),
\]
again holds for "almost all" sets of $\epsilon_i$ and $a$.

The integrals on the left-hand side of these equations can be asymptotically evaluated from our knowledge of the distribution functions of the $\frac{\log |F_i (\sigma + i \epsilon)|}{\sqrt{T \log \log T}}$.

Obviously they increase as $n$ increases, indicating a shift of zeros of $F (\sigma)$ and $F (\sigma) - a$ to the right with increasing $n$.

If we try to assess how large a proportion of the zeros of $F (\sigma) - a$ lie to the left of the line $\sigma = \frac{1}{2}$, and what proportion lies to the right, we are led to the conjecture that for $a \neq 0$ about
\[
\frac{2^{\sigma + 1} - 2^{\sigma - 1}}{2^{\sigma + 1} - 2^{-\sigma - 1}} \text{ of the total number in } 0 < \epsilon < T
\]
lie to the left of $\sigma = \frac{1}{2}$ and about
\[
\frac{1}{2} - 2 - 2^{-\sigma + 1} \text{ to the right.}
\]
A weaker result, similar to that given for \( n = 1 \) can be proved in this case too. If we only assume the \( F(x) \) to be linearly independent, but not all relatively prime, similar results are obtained but the numerical factors occurring in our earlier conclusion \( 2^{-m}, \frac{1}{2} + 2^{-m-1} \) and \( \frac{1}{2} - 2^{-m-1} \), become rather more complicated.

If we make the additional assumption that the coefficients \( c_i \) in the expression for \( F(x) \) all are real, we get that

\[
\sum_{\lambda \in \mathbb{R}} \frac{1}{1 + \pi i} F(\lambda) \]

is real on the critical line \( s = \frac{1}{2} \).

We may then expect that there could be quite ordinary zeros of \( F(s) \) on the line \( s = \frac{1}{2} \).

It is easily established that for "almost all" pairs \( t, t' \) in \( (0, T) \) satisfying

\[
|t - t'| < (\log T)^{-1 + \varepsilon}
\]

(2:5)

where \( \varepsilon > 0 \) tends to zero as \( T \to \infty \),
we have

\[(26) \quad \left| \log |F_i(\frac{1}{2} + it)| - \log |F_i(\frac{1}{2} + iT')| \right| = O \left( \sqrt{\frac{\log \log T}{T}} \right).\]

In order to utilize this fact to show that \( F(s) \) has many zeros on \( \sigma = \frac{1}{2} \), we need to make an additional, but very plausible, hypothesis. We assume that the zeros of the \( F_i(s) \) are "well spaced", more specifically that:

\[(27) \quad \limsup_{T \to \infty} \frac{\# \left( |\chi_m+1| - \chi_m < \frac{\delta}{T \log T}, 0 < \chi_m < T \right)}{T \log T} = O \left( \delta^0 \right),\]

uniformly as \( \delta \to 0 \), for some positive \( \delta \).

This, together with (26) and (25), enables us to show that the interval \((0, T)\) can be divided in intervals of length

\[H = \frac{1}{\log T} \exp \left( \sqrt{\log \log T} \right),\]

such that in almost all of these one triangle

\[e^{i \omega(t)} F_i(\frac{1}{2} + it)\]

dominates over the other terms in
$e^{i\theta(t)} F(\frac{1}{2} + it)$,

so that $F(\frac{1}{2} + it)$ in such an interval has about as many zeros as the $F_c(\frac{1}{2} + it)$ which dominates in that interval. In this way the hypothesis (27) of well spacing leads to the conclusion:

Almost all zeros of $F(s)$ are on the line $\sigma = \frac{1}{2}$.

For the case of the Epstein zeta-function of a rational binary form, this result was first conjectured by H.L. Montgomery and then proved on a somewhat stronger hypothesis of well spacing by E. Bombieri and D. Hejhal.