(1) $$F(s) = \sum_n \frac{a_n}{n^s},$$

abs. convergent for $\sigma > 1$, $s = \sigma + it$.
Assume $F(s)$ meromorphic, but so that $(s-1)^m F(s)$ is an integral function of finite order for some integer $m \geq 0$.
Also assume

(2) $$\Phi(s) = \overline{\Phi(1-\bar{s})},$$

where

(3) $$\Phi(s) = \varepsilon Q^s \prod_{i=1}^{k} \Gamma(\lambda_i s + \mu_i) \, F(s),$$

and for the constants $\varepsilon, Q, \lambda_i, \mu_i$
$|\varepsilon| = 1$, $Q > 0$, $\lambda_i > 0$, $\mathrm{Re}\, \mu_i \geq 0$.

Follows that $\Phi(s)$ is real for $s = \frac{1}{2} + it$, also that $(s-1)^m F(s)$ is of order 1.
We write

(4) $$\Phi(\tfrac{1}{2} + it) = \varkappa(t) e^{i\vartheta(t)} F(\tfrac{1}{2} + it),$$

where $\varkappa(t)$ and $\vartheta(t)$ are real and $\varkappa(t) > 0$.
Also we put

(5) $$\Lambda = \sum_{i=1}^{k} \lambda_i$$

Zeros of $F(s)$ are divided in two classes: trivial zeros, located at poles of the $\Gamma(\lambda_i z + \mu_i)$; and the rest, which we call non-trivial.

non-trivial zeros all located in some vertical strip $1-A \leq \sigma \leq A$, where A is some positive constant.

If $N(T)$ denotes number of non-trivial zeros in region $0 \leq t \leq T$, counting zeros on boundary with half multiplicity, then:

$$(6) \quad N(T) = \frac{\vartheta(T)}{\pi} + \frac{1}{\pi} \arg F(\tfrac{1}{2}+iT) + c =$$

$$= \frac{1}{\pi} T(\log T + c') + O(\log T),$$

where c and c' are constants. Similar formula holds for zeros with negative imaginary part as one can see by looking at $\overline{F(\bar{s})}$ instead of $F(s)$.

Further assumptions:

$$(7) \quad a_1 = 1, \quad a_n = O(n^\delta)$$

for any fixed $\delta > 0$, and

$$(8) \quad \log F(s) = \sum_n \frac{b_n}{n^s},$$

with $b_n = 0$ except when n is of form p^r where p is a prime and r a positive integer. Also

$$(9) \quad b_n = O(n^\theta),$$

for some $\theta < \frac{1}{2}$.

From (7), (8) and (9) follows that the series

$$\sum_n{}^{*} \frac{|b_n|^2}{n^\sigma}$$

where Σ^* indicates that sum is extended only over the $n = p^r$ with $r > 1$, is convergent for $\sigma > \max(2\theta, \frac{1}{2})$.

Thus

$$(10) \qquad \sum_{n \le x} \frac{|b_n|^2}{n} = \sum_{p \le x} \frac{|a_p|^2}{p} + O(1).$$

<u>Conjecture 1. We have</u>

$$(11) \qquad \sum_{p \le x} \frac{|a_p|^2}{p} = n_F \log\log x + O(1),$$

<u>where n_F is an integer depending on F.</u>

"Primitive" function: We say $F(s)$ is primitive if $F(s)$ can not be written as $F(s) = F_1(s) F_2(s)$ where F_1 and F_2 both satisfy our conditions.

<u>Conjecture 2. For a primitive function F we always have $n_F = 1$, so that</u>

$$(12) \qquad \sum_{p \le x} \frac{|a_p|^2}{p} = \log\log x + O(1).$$

Conjecture 3. For two distinct primitive functions F and F' we have

$$(13) \qquad \sum_{p \leq x} \frac{a_p \, \overline{a'_p}}{p} = O(1).$$

If with $F(s)$ also

$$F^{\chi}(s) = \sum_{n} \frac{a_n}{n^s} \chi(n),$$

satisfies our conditions, where χ is a primitive dirichlet character then F and F^{χ} would have the same n_F so they are either both primitive or both not primitive.

Conjecture 4. If F and F^{χ} are not primitive, we can get the factorization of F^{χ} into primitive factors by putting the character χ into the series defining the primitive factors of F.

Conjecture 5. The Riemann hypothesis holds for the class of functions $F(s)$ which satisfy our conditions.

For investigation of value-distribution of $\log F(\sigma+it)$ for $\sigma = \frac{1}{2}$ or σ very near to $\frac{1}{2}$ we need, beside (11), either to assume R.H. or some weaker hypothesis. Denote by $N(\sigma, T)$ the number of zeros $\rho = \beta + i\gamma$ with $\beta \geq \sigma$ and $|\gamma| \leq T$.

If we assume that for $\sigma > \frac{1}{2} + \frac{1}{\log T}$ we have

$$(14)\quad N(\sigma, T) = O\left(T^{1-\alpha(\sigma-\frac{1}{2})} \frac{\log T}{\sqrt{\log\log T}}\right),$$

uniformly in σ as $T \to \infty$, where the constant α is positive, we can prove essentially as sharp results as on R.H..

Neither R.H. nor (14) has been proved to hold for any $F(s)$ in our class.

D.H. A slightly weaker form of (14),

$$(14')\quad N(\sigma, T) = O\left(T^{1-\alpha(\sigma-\frac{1}{2})} \log T\right)$$

uniformly for $\sigma > \frac{1}{2}$, can actually be proved for some $F(s)$.

Using (14') we do not in general obtain quite as sharp results as on R.H. or (14) though in some instances we do. For the cases where (14') is proved and (11) holding we have results that are proved unconditionally.

Approximate formulas for $\log F(s)$ in the critical strip:

For $x \geq 2$ define:

$$(15)\quad \theta_x(n) = \begin{cases} 1, & \text{for } 1 \leq n \leq x, \\ 2 - \frac{\log n}{\log x}, & \text{for } x \leq n \leq x^2, \\ 0, & \text{for } n \geq x^2. \end{cases}$$

Write also $b_x(n) = b_n \theta_x(n)$, let $2 \leq x \leq t^2$ and $\sigma_x = \frac{1}{2} + \frac{1}{\log x}$. Then, assuming R.H. for $F(s)$, we have for $\sigma_x \leq \sigma^* \leq \sigma$,

$$(16)\quad \log F(\sigma+it) = \sum_{n<x^2} \frac{b_x(n)}{n^{\sigma+it}} + O\left(\frac{x^{\frac{1}{2}-\sigma}}{(\sigma^*-\frac{1}{2})\log^2 x}\left(\left|\sum_{n<x^2} \frac{b_x(n)\log n}{n^{\sigma^*+it}}\right| + \log|t|\right)\right).$$

Also, for $\frac{1}{2} \leq \sigma \leq \sigma_x$, writing $\eta_t = \min_{\rho} |t-\gamma|$, that

$$(16')\quad \log F(\sigma+it) = \sum_{n<x^2} \frac{b_x(n)}{n^{\sigma_x+it}} + O\left(\frac{1+\overset{+}{\log}\frac{1}{\eta_t \log x}}{\log x}\left(\left|\sum_{n\leq x^2} \frac{b_x(n)\log n}{n^{\sigma_x+it}}\right| + \log|t|\right)\right).$$

The constants implied by the O-symbols depend on F only. If we take the imaginary part of (16') we can drop the term $\overset{+}{\log}\frac{1}{\eta_t \log x}$.

If we assume (14) or (14') instead of R.H., we have similar but somewhat more complicated formulas, where σ_x is replaced by a $\sigma_{x,t}$ defined in a way that depends on the zeros ρ with imaginary parts close to t. The $\theta_x(n)$ also has a more complicated definition.

If we do not assume R.H. put

$$\Phi_x(n) = \begin{cases} 1 & \text{for } 1 \le n \le x, \\ 1 - 2\left(\dfrac{\log \frac{n}{x}}{\log n}\right)^2, & \text{for } x \le n \le x^{\frac{3}{2}}, \\ 2\left(\dfrac{\log \frac{x^2}{n}}{\log x}\right)^2, & \text{for } x^{\frac{3}{2}} \le n \le x^2, \\ 0, & \text{for } n \ge x^2. \end{cases}$$

Also if ρ runs over the zeros $\rho = \beta + i\gamma$ with

$$|t-\gamma| \le \frac{x^{2(\beta-\frac{1}{2})}}{\log x},$$

we define

$$\sigma_{x,t} = \frac{1}{2} + 2 \max_{\rho} \left(\beta - \frac{1}{2}, \frac{2}{\log x}\right).$$

The $\sigma_{x,t}$ plays the role of σ_x in (16) and (16').

If we assume (14') and $\alpha' < \alpha$, we have for $(\xi x)^2 < T^{\alpha'}$ and $x > T^{\alpha''}$ that: $\alpha', \alpha'' > 0$

$$\int_0^T \left(\sigma_{x,t} - \frac{1}{2}\right)^k \xi^{2\sigma_{x,t}-1}\, dt = O\left(k!\, e^{Ak} \frac{T}{(\log T)^k}\right).$$

We can now prove:

Theorem 1. Let k be a positive integer, $0<a<1$, and $T^{\frac{a}{k}} \le x \le T^{\frac{1}{k}}$, then for $\sigma \ge \frac{1}{2}$ we have

$$(17) \quad \frac{1}{T}\int_0^T \Big| \log F(\sigma+it) - \sum_{p<x} \frac{a_p}{p^{\sigma+it}} \Big|^{2k} dt = O\left(k^{4k} e^{Ak}\right),$$

where the constant A depends on a and F; and if we do not assume R.H., also on the constant α in (14'). If we replace the expression in the $|\ \ |$ symbol by its imaginary part we can replace the k^{4k} in the O-symbol by k^{2k}.

From this we can deduce:

Theorem 2. For $\frac{1}{2} \le \sigma \le \frac{1}{2} + (\log T)^{-\delta}$, where δ is fixed >0, the function

$$\varkappa(\sigma,t) = \varkappa_F(\sigma,t) = \frac{\log F(\sigma+it)}{\sqrt{\pi \sum_{p<t} \frac{|a_p|^2}{p^{2\sigma}}}}$$

has a normal Gaussian distribution in the complex plane. Also, the real and imaginary part of $\varkappa(\sigma,t)$ have a normal Gaussian distribution on the line.

Let $\chi_{a,b}(u)$ denote the characteristic function of an interval (a,b), then

$$\int_0^T \chi_{a,b}(\mathrm{Re}\,\varkappa(\sigma,t))\,dt = T\int_a^b e^{-\pi u^2}du + O\left(T\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right).$$

and

$$(18') \quad \int_0^T \chi_{a,b}(\operatorname{Im} \varkappa(\sigma,t))\,dt = T\int_a^b e^{-\pi u^2}\,du + O\left(T\,\frac{\log\log\log T}{\sqrt{\log\log T}}\right).$$

Since in the region for σ considered in Th 2. we have

$$\sum_{p<T} \frac{\|a_p\|^2}{p^{2\sigma}} = n_F \log\left(\min\left(\frac{1}{\sigma-\frac{1}{2}}, \log T\right)\right) + O(1),$$

we see that all primitive functions F have the same distribution. Also using (13) we get that distinct primitive functions are statistically independent.

Using functional equation we find in the region $\sigma < \frac{1}{2}$, that for

$$(19) \quad \sigma = \frac{1}{2} - \mu\,\frac{\sqrt{\log\log T}}{\log T}$$

where μ is constant >0, that

$$\frac{\log|F(\sigma+it)|}{\sqrt{\pi\, n_F \log\log t}}$$

has a normal Gaussian distribution shifted to the right by the amount

$$(20') \quad \mu' = \frac{\Lambda\,\mu}{\sqrt{\pi}\, n_F}.$$

9

Distribution of a-points or zeros of $F(s)-a$.

If we assume R.H or (14) we can show

(21) $$\int_0^T \overset{+}{\log} |F(\tfrac{1}{2}+it)|\,dt = \frac{\sqrt{\varkappa_F}}{2\sqrt{\pi}} T\sqrt{\log\log T} + O\left(T \frac{(\log\log\log T)^3}{\sqrt{\log\log T}}\right),$$

while if we only use (14') D.H. we get instead the larger remainder term $O(T)$.

Similar evaluations can be given for $\frac{1}{2} < \sigma < \frac{1}{2} + (\log T)^{-\delta}$, (with somewhat better remainder terms for the larger σ in the range), and also for $\sigma < \frac{1}{2}$ and of the form (19). In the latter case main term involves a more complicated constant depending on μ (or μ' given by (20)) because of the shift of the Gaussian distribution.

We use a formula of Littlewood to estimate the sum

$$\sum_{\substack{0<\gamma<T \\ \beta>\sigma}} (\beta-\sigma)$$

where $\rho = \beta + i\gamma$ denote zeros of $F(s)-a$

For $\sigma \le \frac{1}{2}$ and of form (19)

$$\sigma = \frac{1}{2} - \mu \frac{\sqrt{\log\log T}}{\log T},$$

we get for $a \neq 1, 0, \infty$,

$$\sum_{\substack{0<\gamma<T \\ \beta>\sigma}} (\beta - \sigma) = \tag{22}$$

$$= \frac{\sqrt{[illegible]}}{2\sqrt{\pi}} \left(\frac{e^{-\pi \mu'^2}}{2\pi} + \mu' - \mu' \int_{\mu'}^{\infty} e^{-\pi u^2} du \right) T \sqrt{\log\log T}$$

$$+ T \frac{\log |a|}{2\pi} \int_{\mu''}^{\infty} e^{-\pi u^2} du - T \frac{\log |a-1|}{2\pi} +$$

$$+ O\left(T \frac{(\log\log\log T)^3}{\sqrt{\log\log T}} \right),$$

assuming R.H or (14), assuming only (14) we get a remainderterm $O(T)$. N.B.!

Defining $N_a(\sigma, T)$ and $N_a(T)$ analogously for $F(s) - a$ to $N(\sigma, T)$ and $N(T)$ for $F(s)$, we get for σ of the form (19)

$$N_a(\sigma, T) \sim N_a(T) \int_{-\mu'}^{\infty} e^{-\pi u^2} du, \tag{23}$$

for $\mu > 0$. Shows about half of a-points distributed to the left of line $\sigma = \frac{1}{2}$ at distances of order $\frac{\sqrt{\log\log T}}{\log T}$ from line with densities corresponding to half a Gaussian distribution.

Most of other half of a-points lie very close to $\sigma = \frac{1}{2}$, at distances of order not more than

$$\frac{(\log\log\log T)^3}{\log T\sqrt{\log\log T}}$$

assuming R.H. or (14) (or (14') somewhat weaker conclusions follow).

Difficult to decide how this family of a-points distribute to the right and left of the line $\sigma = \frac{1}{2}$, without making some strong (but plausible) additional conjecture about the distribution of the zeros of $F(s)$.

Can however prove that if we consider a and a' such that their ratio is real and negative, then for almost all values of the argument of a

$$(F(s)-a)(F(s)-a')$$

has about $\frac{3}{4}$ of its zeros to the left of $\sigma =\frac{1}{2}$ and about $\frac{1}{4}$ to the right.

One may conjecture that this is true for $F(s) - a$ itself and for all $a \neq 0$.

Linear combinations:

Let

$$F(s) = \sum_{i=1}^{n} c_i F_i(s) , \tag{24}$$

where $n > 1$, the $c_i \neq 0$ and the F_i linearly independent functions of the kind dealt with earlier, and which all satisfy the <u>same functional equation</u>. For simplicity we will assume the F_i are relatively prime.

Then the

$$\log |F_i(\sigma+it)|$$

are statistically independent in the region

$$|\sigma - \tfrac{1}{2}| < (\log T)^{-1+\varepsilon},$$

where $\varepsilon \to 0$ as $T \to \infty$. For $i \neq j$ we can show that in this region,

$$\big|\log |F_i(\sigma+it)| - \log |F_j(\sigma+it)|\big| >$$

$$> (\log\log\log T)^2,$$

holds except for a subset of $(0, T)$ of measure

$$O\left(T \frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right).$$

Outside of this subset we have in $(0,T)$

$$\log|F(\sigma+it)| = \max_i \log|c_i F_i(\sigma+it)| + O\left(e^{-(\log\log\log T)^2}\right).$$

As a consequence the distribution functions for

$$\frac{\log|F(\sigma+it)|}{\sqrt{\pi \log\log t}},$$

can be determined from those of the

$$\frac{\log|F_i(\sigma+it)|}{\sqrt{\pi\, n_{F_i} \log\log t}}.$$

Another consequence: on the line $\sigma = \frac{1}{2}$ we have that $|F(\frac{1}{2}+it)|$ is large except for a subset of $(0,T)$ of measure asymptotic to $2^{-m}T$.

We can also show that for "almost all" sets of c_i we have

$$\int_0^T \log|F(\sigma+it)|\,dt =$$

$$\int_0^T \log\left(\max_i |c_i F_i(\sigma+it)|\right) dt + O\left(T \frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right)$$

Also, if a is a constant $\neq 0$,

$$\int_0^T \log|F(\sigma+it)-a|\,dt =$$

$$\int_0^T \log\left(\max_i |c_iF_i(\sigma+it)|, |a|\right) dt + O\left(T\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right),$$

again holds for "almost all" sets of c_i and a.

The integrals on the ~~left~~ right hand side of ~~these~~ equations can be asymptotically evaluated from our knowledge of the distribution functions of the $\frac{\log|F_i(\sigma+it)|}{\sqrt{\pi n_{F_i}\log\log t}}$.

Obviously they increase as n increases, indicating a shift of zeros of $F(s)$ and $F(s)-a$ to the right with increasing n.

If we try to assess how large a proportion of the zeros of $F(s)-a$ lie to the left of the line $\sigma=\frac{1}{2}$, and what proportion lies to the right, we are led to the conjecture that for $a\neq 0$ about $\frac{1}{2}+2^{-n-1}$ of the total number in $0<t<T$ lies to the left of $\sigma=\frac{1}{2}$ and about $\frac{1}{2}-2^{-n-1}$ to the right.

a weaker result, similar to that given for $n=1$ can be proved in this case too.

If we only assume the $F_i(s)$ to be linearly independent, but not all relatively prime, similar results are obtained but the numerical factors occuring in our earlier conclusions 2^{-n}, $\frac{1}{2}+2^{-n-1}$ and $\frac{1}{2}-2^{-n-1}$, become rather more complicated.

If we make the additional assumption that the coefficients c_i in the expression for $F(s)$ all are real, we get that

$$\varepsilon Q^{s} \prod_{i=1}^{k} \Gamma(\lambda_i s+\mu_i)\, F(s)$$

is real on the critical line $\sigma=\frac{1}{2}$.

We may then expect that there could be quite many zeros of $F(s)$ on the line $\sigma=\frac{1}{2}$.

It is easily established that for "almost all" pairs t, t' in $(0,T)$ satisfying

$$|t-t'| < (\log T)^{-1+\varepsilon}, \tag{2.5}$$

where $\varepsilon>0$ tends to zero as $T\to\infty$, we have

$$(26)\quad \left| \log |F_i(\tfrac{1}{2}+it)| - \log |F_i(\tfrac{1}{2}+it')| \right| = o\left(\sqrt{\log\log T}\right).$$

In order to utilize this fact to show that $F(s)$ has many zeros on $\sigma = \frac{1}{2}$, we need to make an additional, but very plausible, hypothesis. We assume that the zeros of the $F_i(s)$ are "well spaced", more specifically that:

$$(27)\quad \limsup_{T\to\infty} \frac{\#\left(|\gamma_{n+1}-\gamma_n| < \frac{\delta}{\log T},\ 0<\gamma_n<T\right)}{T\log T} = O(\delta^{\theta}),$$

uniformly as $\delta \to 0$, for some positive θ.

This, together with (26) and (25) enables us to show that the interval $(0, T)$ can be divided in intervals of length

$$H = \frac{1}{\log T}\exp\left(\sqrt{\log\log T}\right),$$

such that in almost all of these one single

$$e^{i\vartheta(t)}F_i(\tfrac{1}{2}+it)$$

dominates over the other terms in

$$e^{i\vartheta(t)}F(\tfrac{1}{2}+it),$$

so that $F(\frac{1}{2}+it)$ in such an interval has about as many zeros as the $F_i(\frac{1}{2}+it)$ which dominates in that interval. In this way the hypothesis (27) of well spacing leads to the conclusion:

<u>Almost all zeros of $F(s)$ are on the line $\sigma=\frac{1}{2}$.</u>

For the case of the Epstein zeta-function of a rational binary form, this result was first conjectured by H. L. Montgomery and then proved on a somewhat stronger hypothesis of wellspacing by E. Bombieri and D. Hejhal.