J.

Jessen, Selberg Congress 1946.

Selberg statement about zeros of $\zeta(s) - a$

work of Jessen and for fixed $\sigma > \frac{1}{2}$

(a) Distribution of a function $f(s)$ on lines $s = \sigma + it$; "time" spent in region Ω. $\underset{0<t<T}{m}(f(s) \in \Omega) \to T \int_\Omega D_\sigma(z)\, d\omega_z$

$\iint D_\sigma(z)\, d\omega_z = 1.$

$$\hat{D}_\sigma(w) = \int D_\sigma(z)\, e^{\pi i (w\bar{z} + \bar{w} z)}\, d\omega_z$$

$$= \lim \frac{1}{T} \int_0^T e^{\pi i (w \overline{f(\sigma+it)} + \bar{w} f(\sigma+it))}\, dt$$

$\log \zeta(\sigma + it)$

$$\hat{D}_\sigma(w) = \prod_p \left(1 + \sum_{\nu=1}^{\infty} \binom{\pi i w}{\nu} \binom{\pi i \bar{w}}{\nu} p^{-2\nu\sigma}\right)$$

for small $|w|$ like $e^{-\pi^2 |w|^2 \sum_p \frac{1}{p^{2\sigma}}}$

normalize

$$\frac{\log \zeta(\sigma+it)}{\sqrt{\pi \sum_\sigma p^{-2\sigma}}} \qquad D_\sigma(z) \sim e^{-\pi |z|^2}$$

$$\hat{D}_\sigma(w) \sim \frac{1}{\sqrt{\pi}} e^{-\pi |w|^2}$$

$$\frac{\log \zeta(\sigma+it)}{\sqrt{\pi \log \frac{1}{\sigma - \frac{1}{2}}}}$$ conjecture for $\sigma = \frac{1}{2}$ or very near : $\log t$ replace $\frac{1}{\sigma - \frac{1}{2}}$

② 1.

Old results:

(A) $$\int_T^{2T} \Big| \log \zeta(\tfrac{1}{2}+it) - \sum_{p \le T^{\frac{1}{2k}}} p^{-\frac{1}{2}-it} \Big|^{2k} dt$$

$$= O(T k^{Ak})$$ easily with $A=5$ harder $A=4$

(B) $$\frac{1}{T}\int_T^{2T} (\log \zeta(\tfrac{1}{2}+it))^h \, (\overline{\log \zeta(\tfrac{1}{2}+it)})^k \, dt$$

$$= \delta_{h,k} \, k! \left(\log \frac{\log T}{k}\right)^k$$

(*) $$+ O\left(\Gamma\left(\frac{h+k}{2}+1\right)(h+k)^A \left(\log \frac{2\log T}{h+k}\right)^{\frac{h+k-1}{2}}\right)$$

$$+ O\left((h+k)^{A\frac{h+k}{2}} \right)$$

On R.H. middle term (*) improved to $\frac{h+k}{2}-1$

Distribution properties of

$f(t) = \dfrac{\log \zeta(\frac{1}{2}+it)}{\sqrt{\pi \log\log t}}$; $f_1(t) = \mathrm{R} f(t)$;

distribution f.

$\frac{5}{4} e^{-\pi |z|^2}$

distribution function.

; $e^{-\pi x^2}$

generalization to $f_i(t) = f(\alpha_i t + \beta_i)$

with $\cancel{\begin{vmatrix} \alpha_i & \beta_i \\ \alpha_j & \beta_j \end{vmatrix}} \neq 0$ for $i \neq j$ $(\alpha_i \mp \alpha_j)^2 + (\beta_i \mp \beta_j)^2 \neq 0$

$\underline{\alpha_i \neq 0}$ act.

statistically independent.

in $\underline{T, 2T}$

$f(t)$ and $f(t+\beta)$ statistically indep.

for $\log \frac{1}{\beta} = o(\log\log T)$ (short memory)

$\beta > (\log T)^{-\epsilon}$

Evaluate integrals

$$\frac{1}{T} \int_3^T F\left(\frac{\log \zeta(\frac{1}{2}+it)}{\sqrt{\pi \log\log t}} \right) dt$$

assumpt.

Ω 

Boundary $\in$ rectifiable length L.

measure of t in $(3, T)$ for which

$f(t)$ in Ω is

$$T \iint_\Omega e^{-\pi |z|^2} dx\, dy + O\left(\frac{TL}{\sqrt{\log\log T}} \right)$$

measure of t for which $b_1 \le f_1(t) \le b_2$

is $$T \int_{b_1}^{b_2} e^{-\pi x^2} dx + O\left(\frac{T}{\sqrt{\log\log T}} \right)$$

~~Distribution~~ Distribution of zeros of $\zeta(s)-a$.

denote by ρ and $\beta+i\gamma$ (if we have also reference to zeros of $\zeta(s)$ in this connection denote by ρ^{0}; $\beta^{0}+i\gamma^{0}$

Remind of functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \qquad (s \longleftrightarrow 1-s)$$

real on line $\sigma=\frac{1}{2}$

for $s=\frac{1}{2}+it$

write $\arg \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) = \frac{1}{i}\vartheta(s) = \theta(t)$

then $t>0$

$$\theta(t)=\frac{t}{2}\log\frac{t}{2\pi e}+c+O\left(\frac{1}{t}\right) \qquad c=-\frac{1}{8}$$

for $t>A$; $\theta'(t)=\frac{1}{2}\log\frac{t}{2\pi}+O\left(\frac{1}{t}\right)>0$

also

$$\left|\zeta\left(\tfrac{1}{2}-\delta+it\right)\right| = \left(\frac{t}{2\pi}\right)^{\delta}\left|\zeta\left(\tfrac{1}{2}+\delta+it\right)\right|\left(1+O\left(\frac{\delta}{t}\right)\right)$$

for δ ~~fixed~~ small, unif for $t>A$; $0\le\delta\le\delta_0$

assume $a\neq 1$ for convenience.

$$(1)\quad N_a(T)=\frac{T}{2\pi}\log\frac{T}{2\pi e}+O(\log T)$$

$$(2)\quad \sum_{A\le\gamma\le T}\left(\beta-\frac{1}{2}\right)=O(\log T)$$

(1) as for zeros of $\zeta(s)$, (2) by computing with μ large enough to have all zeros in $A \le \gamma \le T$ in $\beta > -\mu$

$$\sum_{A \le \gamma \le T} (\beta + \mu) = (\mu + \tfrac{1}{2}) \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)$$

(use Littlewoods theorem.

$$\sum \beta - \sigma_1 = \frac{1}{2\pi} \int_{T_1}^{T_2} \log |g(\sigma_1 + it)|\, dt$$

$$+ \frac{1}{2\pi} \int_{T_1}^{T_2} \log |g(\sigma_2 + it)|\, dt$$

$$+ \int_{\sigma_1}^{\sigma_2} \arg g(\sigma + i T_1)\, d\sigma + \int_{\sigma_1}^{\sigma_2} \arg g(\sigma + i T_2)\, d\sigma$$

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(3) $$\sum_{A \le \gamma \le T} \beta - \tfrac{1}{2} = \frac{1}{2\pi} \int_A^T \log |\zeta(\tfrac{1}{2} + it) - a|\, dt$$

$$- \frac{1}{2\pi} \int_A^T \log \left| \frac{\zeta(\sigma_2 + it) - a}{1 - a} \right| dt$$

$$- \frac{1}{2\pi} \int_A^T \log |1 - a| + O(\log T)$$

1st Integral divide in 3 parts.

J_1 $|\zeta(\tfrac{1}{2} + it)| \ge \tfrac{3}{2} |a|$; $J_2 = |\zeta(\tfrac{1}{2} + it)| \le \tfrac{1}{2} |a|$

J_3 rest.

(6)

$$m(J_1) = \frac{T}{2} + O\left(\frac{T}{\sqrt{\log\log T}}\right)$$
$$m(J_2) = \frac{T}{2} + O\left(\frac{T}{\sqrt{\log\log T}}\right)$$
$$m(J_3) = O\left(\frac{T}{\sqrt{\log\log T}}\right)$$

In J_2 $\cancel{\zeta}$

$$\log|\zeta(\tfrac{1}{2}+it)-a| = \log|a| + O\left(\frac{|\zeta(\frac{1}{2}+it)|}{a}\right)$$

$$|a|\,2^{-n} \le \zeta(\tfrac{1}{2}+it) \le |a|\,2^{-n+1}$$

$\log 2$.

$\int_{m}$

$$\int_{J_2} = \frac{T}{2}\log|a| + O\left(\frac{T}{\sqrt{\log\log T}}\right)$$

In J_3

$$|\zeta(\tfrac{1}{2}+it)-a| = |e^{i\theta(t)}\zeta(\tfrac{1}{2}+it)-ae^{i\theta(t)}|$$
$$\geq |\Im\, a e^{i\theta(t)}| = \tfrac{1}{2}|ae^{i\theta(t)}-\bar{a}e^{-i\theta(t)}|$$
$$= \cancel{\theta(t)}\,|a|\,|\sin(\theta(t)+\alpha)|$$

$\mu(t)$ $\alpha = \arg a$

$$\int_{J_3} = O\left(\frac{T}{\sqrt{\log\log T}}\right) + \int_{J_3} |\log|\mu(t)||\,dt$$

$$\le (m(J_3))^{1-\frac{1}{2k}} \cdot \left(\int_0^T \log^{2k}\mu(t)\,dt\right)^{\frac{1}{2k}}$$

Suitable choice of k as funct of T

$$O\left(T\,\frac{\log\log\log T}{\sqrt{\log\log T}}\right).$$

J_1

$$\log|\zeta(\tfrac12+it)-a| = \log|\zeta(\tfrac12+it)| + O\left(\frac{a}{\zeta(\frac12+it)}\right)$$

$$= \frac{1}{2\pi}\int_A^T \log^+|\zeta(\tfrac12+it)|\,dt + O\left(\frac{T}{\sqrt{\log\log T}}\right)$$

$$\sum_{\substack{\beta>\frac12 \\ A\le\gamma\le T}} (\beta-\tfrac12) = \frac{1}{2\pi}\int_A^T \log^+|\zeta(\tfrac12+it)|\,dt + \frac{1}{4\pi}T\log\frac{|1-a|^2}{|a|} + O\left(\frac{T\log\log\log T}{\sqrt{\log\log T}}\right)$$

Without R.H

$$\int_A^T \log^+|\zeta(\tfrac12+it)|\,dt = T\sqrt{\log\log T}\int_0^\infty x e^{-\pi x^2}\,dx + O(T)$$

with R.H. can show.

$$= T\sqrt{\log\log T}\underbrace{\int_0^\infty x e^{-\pi x^2}\,dx}_{=\frac{1}{2\pi}} + O\left(\frac{T}{(\log\log T)^{\frac12-\epsilon}}\right)$$

$$= \frac{T\sqrt{\log\log T}}{2\sqrt{\pi}} + \begin{cases} O(T) \\ O\left(\frac{T}{(\log\log T)^{\frac12-\epsilon}}\right) & \text{on R.H.}\end{cases}$$

thus.

$$\sum_{\substack{\beta>\frac12 \\ A<\gamma<T}} (\beta-\tfrac12) = \frac{T\sqrt{\log\log T}}{4\pi\sqrt{\pi}} + \frac{T}{2\pi}\log\frac{|1-a|^2}{|a|} + \begin{cases} O(T) \\ O\left(\frac{T}{(\log\log T)^{\frac12-\epsilon}}\right)\end{cases}$$

proves that if $\frac{\Phi(t)}{\sqrt{\log\log t}} \to \infty$ almost all zeros lie in $|\sigma-\tfrac12| < \frac{\Phi(t)}{\log t}$.

$\log f(\sigma+it)$ $\quad w = \log \zeta(\sigma+it)$

$$\iint D_\sigma(z)\, e^{-\pi i(wz+\bar{w}\bar{z})}\, dz\, d\bar{z}$$

$$\frac{1}{T}\iint \left(\overline{\cancel{f}(\cancel{\sigma})+it)}\right)^{\pi i \bar{w}}\left(\zeta(\sigma+it)\right)$$

$$\frac{1}{T}\int_T^{2T} \left(f(\sigma+it)\right)^{\pi i \bar{w}}\left(\overline{f(\sigma+it)}\right)^{\pi i w} dt$$

$$\prod\left(1 + \frac{-\pi^2 w^2}{p^{2\sigma}} + \frac{(\pi i w)(\pi i w - 1)(\pi i \bar{w} - 1)\,\pi i}{\;}\right.$$

$$-\frac{\pi^2 w^2}{2}\left(-\pi^2 w^2 + 1 - \pi i(w+\bar{w})\right)$$

$$g\left(\frac{\log \zeta(\tfrac{1}{2}+it)}{\sqrt{\log\log t}}\right)$$

$$= \frac{1}{\pi}\int g\left(\frac{w}{\sqrt{\pi}}\right) e^{-|w|^2}\, dw$$

$\sqrt{\frac{\pi}{\alpha}}$

$\frac{1}{\sqrt{\pi}}$

$e^{-\alpha x}$

$\frac{1}{\sqrt{\pi}}$ $\quad f(t)$ in Ω

$$\iint e^{-\pi z^2}\, dz = 1 \qquad f($$

$e^{-\pi x^2}$ $\quad \int_{-\infty}^{\infty}$

$$\pi \int\int_0^\infty e^{-\pi r^2}\, r\, dr\, \cancel{d\varphi}$$

$\int$

$\frac{1}{2}$