On the remainder term in the lattice point problem of the circle.

Introduction.

In the following $A(x)$ denotes the number of lattice points (u, v) which belonge to the circle area $u^2 + v^2 \le x$. Further let: $R(x) = A(x) - \pi x$

A theorem of Hardy then says that

$$\underline{\lim} \frac{R(x)}{x^{\frac{1}{4}} (\log x)^{\frac{1}{4}}} < 0 .$$

In the positive ~~upper~~ direction as far as I know one has only

$$\overline{\lim} \frac{R(x)}{x^{\frac{1}{4}}} > 0 .$$

The aim of this Paper is to prove that

$$\overline{\lim} \frac{R(x)}{x^{\frac{1}{4}} (\log \log x)^{\frac{1}{4}}} > 0 .$$

§ 1

lemmas ~~Auxiliary Theorems.~~

In the following m runs through all positive quadratfrei integers, the numbers $\sqrt{m}$ ~~them~~ obviously then are all linearly independent. We then have

Lemma I

Let $m_1, m_2, \ldots, m_\nu$ be ν numbers of which not two are equal ~~all different from eachother~~, and $m_i \le N$ for $i = 1, 2 \ldots, \nu$. Further ~~let~~ $\lambda_1, \lambda_2, \ldots, \lambda_\nu$ integers not all zero, and $|\lambda_1| + |\lambda_2| + \cdots + |\lambda_\nu| \le 2k$. Then

$$|\lambda_1 \sqrt{m_1} + \lambda_2 \sqrt{m_2} + \cdots + \lambda_\nu \sqrt{m_\nu}| > (2k\sqrt{N})^{-2^{2k}} .$$

Proof: We obviously may suppose that $\nu \le 2k$, we then form the expression

$$P = \prod (\lambda_1\sqrt{m_1} \pm \lambda_2\sqrt{m_2} \pm \cdots \pm \lambda_\nu\sqrt{m_\nu})$$

since ~~this~~ is a symmetrical Polynom of $\sqrt{m_i}$ and $-\sqrt{m_i}$ the value of this expression must be an integer, and because all factors are different from zero, its modulus must be ≥ 1. On the other hand the number of factors is easily seen to be $2^{\nu-1}$, and the modulus of each factor $\le 2k\sqrt{N}$.

~~Hence~~ $P < |\lambda_1\sqrt{m_1}+\cdots+\lambda_\nu\sqrt{m_\nu}|(2k\sqrt{N})^{2^{\nu-1}-1} < |\lambda_1\sqrt{m_1}+\cdots+\lambda_\nu\sqrt{m_\nu}|(2k\sqrt{N})^{2^{2k}}$

We thus get

$$|\lambda_1\sqrt{m_1}+\cdots+\lambda_\nu\sqrt{m_\nu}|(2k\sqrt{N})^{2^{2k}} > 1$$

or

$$|\lambda_1\sqrt{m_1}+\cdots+\lambda_\nu\sqrt{m_\nu}| \geq (2k\sqrt{N})^{-2^{2k}}$$

q.e.d.

Now let

$$f(t) := \alpha_0 + \sum_{m\le N} \alpha_m e^{2\pi i\sqrt{m}\, t},$$

where N is some positive number, and the α's are independent of t. We then have

Lemma V

Let ε be a small positive number, then there exist a $c_1 = c_1(\varepsilon)$, such that there is a t_0 in the Interval $e^{e^{c_1 N}} \le t_0 \le 2e^{e^{c_1 N}}$ for which ~~with~~ we have ~~the property~~

$$|f(t_0)| > (1-\varepsilon)\Big\{|\alpha_0| + \sum_{m\le N}|\alpha_m|\Big\}.$$

Proof: Beside $f(t)$ we also use the auxiliary function

$$f^*(t_m) = f^*(t_1, t_2, \cdots, t_m, \cdots) = \alpha_0 + \sum_{m\le N} \alpha_m e^{2\pi i\, t_m},$$

Also let $S = |\alpha_0| + \sum_{m\le N}|\alpha_m|$, $k = [\frac{c}{2}N]$ and $T = [e^{cN}]$ where c is a constant which will be determined later. We then investigate the expression

(1)

$$(1) = \int_T^{2T} |f(t)|^{2k} dt = \sum \alpha_0^{\lambda} \bar{\alpha}_0^{\lambda'} \alpha_{m_1} \cdots \alpha_{m_{k-\lambda}} \bar{\alpha}_{m_1'} \cdots \bar{\alpha}_{m'_{k-\lambda'}} \int_T^{2T} \exp\big((2\pi i (\sqrt{m_1} + \cdots + \sqrt{m_{k-\lambda}} - \sqrt{m_1'} - \cdots - \sqrt{m'_{k-\lambda'}}) t \big) dt \geq$$

$$T \sum_{\sqrt{m_1}+\cdots+\sqrt{m_{k-\lambda}} - \sqrt{m_1'} - \cdots - \sqrt{m'_{k-\lambda'}} = 0} {}_1 \; \alpha_0^{\lambda} \bar{\alpha}_0^{\lambda'} \alpha_{m_1} \cdots \alpha_{m_{k-\lambda}} \bar{\alpha}_{m_1'} \cdots \bar{\alpha}_{m'_{k-\lambda'}} - \frac{1}{2\pi} \sum\nolimits_2$$

$$|\alpha_0|^{\lambda+\lambda'} |\alpha_{m_1}| \cdots |\alpha_{m_{k-\lambda}}| \, |\alpha_{m_1'}| \cdots |\alpha_{m'_{k-\lambda'}}| \frac{2}{|\sqrt{m_1} + \cdots + \sqrt{m_{k-\lambda}} - \sqrt{m_1'} - \cdots - \sqrt{m'_{k-\lambda'}}|}.$$

Where ~~the sum~~ Σ_1 ~~contains~~ contains all the terms for which $\sqrt{m_1} + \cdots + \sqrt{m_{k-\lambda}} - \cdots - \sqrt{m'_{k-\lambda'}} = 0$, while Σ_2 consists of the others.

Now since the numbers $\sqrt{m}$ are linearly independent, we have that $\sqrt{m_1} + \cdots + \sqrt{m_{k-\lambda}} - \cdots - \sqrt{m'_{k-\lambda'}}$ ~~is then and~~ only ~~then~~ is equal to zero when $t_{m_1} + \cdots + t_{m_{k-\lambda}} - t_{m_1'} - \cdots - t_{m'_{k-\lambda'}}$ ~~is identically zero~~ vanishes identically. Hence we easily find that

$$\Sigma_1 = \int_0^1 |\varphi(t_m)|^{2k} \Pi \, dt_m,$$

if ~~this is inserted~~ we insert this in (1) and use lemma I to obtain an upper bound for Σ_2, we get

$$(2) \quad \int_T^{2T} |f(t)|^{2k} dt \geq T \int_0^1 |\varphi(t_m)|^{2k} \Pi \, dt_m - S^{2k} (2k\sqrt{N})^{2^{2k}}.$$

To obtain a lower bound for the integral on the right hand side, we put $t_m^{(0)} = \frac{\arg \alpha_0 - \arg \alpha_m}{2\pi}$, then

$$(3) \quad \int_0^1 |\varphi(t_m)|^{2k} \Pi \, dt_m > \int_{t_m^{(0)} - \frac{\varepsilon}{20}}^{t_m^{(0)} + \frac{\varepsilon}{20}} |\varphi(t_m)|^{2k} \Pi \, dt_m.$$

But for $|t_m - t_m^{(0)}| \leq \frac{\varepsilon}{20}$, we have

$$|\varphi(t_m) - S \cdot e^{i \arg \alpha_0}| = \Big| \sum_{m \leq N} \alpha_m (e^{2\pi i t_m} - e^{2\pi i t_m^{(0)}}) \Big| \leq \sum_{m \leq N} |\alpha_m| \, |e^{2\pi i \frac{\varepsilon}{20}} - 1|$$

$< \frac{2\pi\varepsilon}{20} \sum_{m \leq N} |\alpha_m| < \frac{\varepsilon}{3} S$. Hence for $|t_m - t_m^{(0)}| \leq \frac{\varepsilon}{20}$,

$$|\varphi(t_m)| > (1 - \tfrac{\varepsilon}{3}) S.$$

(3) now gives

$$\int_0^1 |\varphi(t_m)|^{2k} \prod dt_m > \left(\frac{\varepsilon}{10}\right)^N \left(1-\frac{\varepsilon}{3}\right)^{2k} S^{2k},$$

We thus get from (2)

$$\int_T^{2T} |f(t)|^{2k} dt > T\left(\frac{\varepsilon}{10}\right)^N \left(1-\frac{\varepsilon}{3}\right)^{2k} S^{2k} - S^{2k}\left(2k\sqrt{N}\right)^{2^{2k}}.$$

From this we see that there is a t_0 in the Interval $(T, 2T)$ for which

$$|f(t_0)|^{2k} > S^{2k}\left(\frac{\varepsilon}{10}\right)^N\left(1-\frac{\varepsilon}{3}\right)^{2k}\left\{1 - \frac{\left(2k\sqrt{N}\right)^{2^{2k}}\left(\frac{10}{\varepsilon}\right)^N\left(1-\frac{\varepsilon}{3}\right)^{-2k}}{T}\right\},$$

or remembering that $k=[\frac{c}{2}N]$ and $T=[e^{e^{cN}}]$,

$$(4) \quad |f(t_0)| > S\left(1-\frac{\varepsilon}{3}\right)\left(\frac{\varepsilon}{10}\right)^{\frac{1}{c-2}}\left\{1 - \frac{\left(cN^{\frac{3}{2}}\right)^{2^{cN}}\left(\frac{10}{\varepsilon}\right)^N\left(1-\frac{\varepsilon}{3}\right)^{-cN}}{e^{e^{cN}}-1}\right\}^{\frac{1}{c-2}}.$$

We now choose ~~first~~ c_1 so great that for the first

$$\left(\frac{\varepsilon}{10}\right)^{\frac{1}{c_1-2}} > 1-\frac{\varepsilon}{3},$$

and secondly

$$\frac{\left(c_1N^{\frac{3}{2}}\right)^{2^{c_1N}}\left(\frac{10}{\varepsilon}\right)^N\left(1-\frac{\varepsilon}{3}\right)^{-c_1N}}{e^{e^{c_1N}}-1} < \frac{\varepsilon}{3}$$

for all $N \geq 1$.

We then get from (4)

$$|f(t_0)| > S\left(1-\frac{\varepsilon}{3}\right)^3 > S(1-\varepsilon).$$

q. e. d.

Lemma III

Let $g(t) = \sum_{m \leq N} \beta_m \cos(2\pi\sqrt{m}\,t - \vartheta_m)$, where the β_m and ϑ_m are all real, let ε be a small positive number then there exist a $c_2 = c_2(\varepsilon)$, such that there is a t_0 ~~in~~ the with $e^{e^{c_2 N}} \le t_0 \le 2e^{e^{c_2 N}}$, with the property

$$g(t_0) > (1-\varepsilon) \sum_{m \le N} |\beta_m| .$$

Proof. Let $f(t) = \sum_{m \le N} |\beta_m| + \sum_{m \le N} \beta_m e^{2\pi i \sqrt{m}\, t - i\vartheta_m}$.

Lemma II then says that there exist a $c_2 = c_1(\frac{\varepsilon}{4})$ such that there is a t_0, with $e^{e^{c_2 N}} \le t_0 \le 2e^{e^{c_2 N}}$, for which

$$|f(t_0)| > 2\left(1-\frac{\varepsilon}{4}\right) \sum_{m \le N} |\beta_m|$$

But

$$|f(t_0)|^2 = \Big(\sum_{m \le N} |\beta_m|\Big)^2 + \Big|\sum_{m \le N} \beta_m e^{2\pi i \sqrt{m}\, t_0 - \vartheta_m}\Big|^2 + 2g(t_0) \sum_{m \le N} |\beta_m|$$

$$\le 2\Big\{\sum_{m \le N} |\beta_m|\Big\}^2 + 2g(t_0) \sum_{m \le N} |\beta_m| ,$$

hence

$$2\Big\{\sum_{m \le N} |\beta_m|\Big\}^2 + 2g(t_0) \sum_{m \le N} |\beta_m| > 4\left(1-\frac{\varepsilon}{2}\right)^{\cancel{2}} \Big\{\sum_{m \le N} |\beta_m|\Big\}^2 ,$$

or

$$g(t_0) > (1-\varepsilon) \sum_{m \le N} |\beta_m| .$$

q.e.d.

In the following $U(m)$ denotes the number of ~~solutions~~ representations of ~~the number~~ m in the form $u^2 + v^2$, where (u, v) are integers. It is well known that $U(m) = 4 \sum_{d/m} \chi(d)$, ~~where~~ χ denoting the primitive Character mod. 4. We also have $\sum_{0 \le m \le N} U(m) = \pi N + O(N^{\frac{3}{4}})$.

Lemma $\overline{IV}$

We have $\sum_{m \le N} U(m) = c_3 N + o(N)$,

where $c_3 > \frac{\pi}{2}$.

Proof: For $\mathcal{R}(s) > 1$, we put

$$\varphi(s) = \sum \frac{u(m)}{m^s} = 4(1+2^{-s}) \prod_p (1+2p^{-s}),$$

where here and in the following p runs over the prime numbers of the form $4n+1$, while q runs through those of the form $4n+3$. Now

$$\varphi_1(s) = \sum_1^\infty \frac{u(n)}{n^s} = \frac{4}{1-2^{-s}} \prod \frac{1}{(1-p^{-s})^2(1-q^{-2s})},$$

hence we get

$$\varphi(s) = \varphi_1(s)\,(1-2^{-2s}) \prod (1-3p^{-2s}+2p^{-3s})(1-q^{-2s}).$$

Putting

$$(1-2^{-2s}) \prod (1-3p^{-2s}+2p^{-3s})(1-q^{-2s}) = \sum_1^\infty \frac{\gamma_n}{n^s},$$

we ~~have~~ that the series on the right converges absolutely for $\mathcal{R}(s) > \frac{1}{2}$. The Identity

$$\sum \frac{u(m)}{m^s} = \left(\sum_1^\infty \frac{u(n)}{n^s}\right)\left(\sum_1^\infty \frac{\gamma_n}{n^s}\right),$$

now gives

$$\sum_{m \le N} u(m) = \sum_{1 \le n \le N} \gamma_n \sum_{r=1}^{\frac{N}{n}} u(r) = \pi N \sum_{1 \le n \le N} \frac{\gamma_n}{N} +$$

$$\sum_{1 \le n \le N} |\gamma_n|\, O\left(\frac{N^{\frac{3}{4}}}{n^{\frac{3}{4}}}\right) = \pi N \sum_{1 \le n \le N} \frac{\gamma_n}{n} + O(N^{\frac{3}{4}}) = \pi N \sum_1^\infty \frac{\gamma_n}{n} + o(N)$$

$$= \pi N \left(1-\frac{1}{4}\right) \prod \left(1-\frac{3}{p^2}+\frac{2}{p^3}\right)\left(1-\frac{1}{q^2}\right) + o(N) = c_3 N + o(N).$$

We still have to prove that $c_3 > \frac{\pi}{2}$, or that

$$\left(1-\frac{1}{4}\right) \prod \left(1-\frac{3}{p^2}+\frac{2}{p^3}\right)\left(1-\frac{1}{q^2}\right) > \frac{1}{2},$$

now

$$\zeta(2) = \frac{1}{1-\frac{1}{4}} \prod \frac{1}{\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{q^2}\right)} = \frac{\pi^2}{6},$$

hence

$$\left(1-\frac{1}{4}\right) \prod \left(1-\frac{3}{p^2}+\frac{2}{p^3}\right)\left(1-\frac{1}{q^2}\right) = \frac{6}{\pi^2} \prod \left(1-\frac{2}{p(p+1)}\right) >$$

$$> \frac{6}{10}\left(1-\frac{2}{5\cdot 6}\right)\left(1-\frac{2}{13\cdot 14}\right)\left(1-\frac{2}{17\cdot 18}\right)\left(1-\sum_{n\geq 7}\frac{2}{4n(4n+2)}\right) =$$

$$\frac{3}{5}\cdot\left(1-\frac{1}{15}\right)\left(1-\frac{1}{91}\right)\left(1-\frac{1}{153}\right)\left(1-\frac{1}{28}\right) > \frac{1}{2}.$$

Q. e. d.

Lemma V

Let $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ be a ~~sequence~~ sequenze of numbers and

$$\sum_{n\leq N} \lambda_n = c_4 N + o(N), \text{ then}$$

$$\sum_{n\leq N} \frac{\lambda_n}{n^{\frac{3}{4}}}\left(1-\sqrt{\frac{n}{N}}\right) = \frac{8}{3} c_4 N^{\frac{1}{4}} + o(N^{\frac{1}{4}}).$$

Proof: Put $\sum_{n\leq x} \lambda_n = S(x)$, then

$$\sum_{n\leq N} \frac{\lambda_n}{n^{\frac{3}{4}}}\left(1-\sqrt{\frac{n}{N}}\right) = \int_0^N x^{-\frac{3}{4}}\left(1-\sqrt{\frac{x}{N}}\right) d\,S(x) = \int_0^N S(x)\left(\frac{3}{4}x^{-\frac{7}{4}} - \frac{1}{4}\frac{x^{-\frac{5}{4}}}{\sqrt{N}}\right)dx =$$

$$= \int_0^N c_4 x\left(\frac{3}{4}x^{-\frac{7}{4}} - \frac{1}{4}\frac{x^{-\frac{5}{4}}}{\sqrt{N}}\right)dx + o\left(\int_0^N x\left(\frac{3}{4}x^{-\frac{7}{4}}\right)dx\right) = \frac{8}{3}c_4 N^{\frac{1}{4}} + o(N^{\frac{1}{4}}).$$

Q. e. d.

§2
Proof of the Theorem

We start from the formula

$$(5) \qquad \int_0^y R(t)\,dt = \sum_1^\infty \frac{u(n)}{\sqrt{n}} \int_0^y \sqrt{t}\, J_1(2\pi\sqrt{nt})\,dt,$$

where J_1 is a Bessel function. Now let $\lambda(y)$ be continuous and differentiable in the interval $0 \leq \alpha \leq y \leq \beta$, and $\lambda'(y)$ continuous and bounded in the same interval. We then have

$$\int_\alpha^\beta R(y)\lambda(y)\,dy = \left[\lambda(y)\int_0^y R(t)\,dt\right]_\alpha^\beta - \int_\alpha^\beta \lambda'(y)\left\{\int_\alpha^y R(t)\,dt\right\}dy,$$

if the series from (5) is ~~introd~~ inserted on the right, we easily get

7 reverse

$U(m\, n^2)$

$\sum \quad U(m$

$U(m\,n) = \sum \mu(d)\, \chi$

$U(m)\, U(n) = \sum \chi(d)\, U\left(\frac{m\,n}{d^2}\right)$

$U(m\,n) = \sum c_d. \quad U\left(\frac{m}{d}\right) U\left(\frac{n}{d}\right)$

$- \chi(d)$ saa er Lars i Lund og Lid

$\mu(d)\, \chi(d)\, U\left(\frac{m}{d}\right) U\left(\frac{n}{d}\right)$

folglich aber.

$\sum_{d/m} \sum_{d^2/n} \frac{U\left(\frac{m\, n^2}{d}\right)}{m^{\frac{3}{2}}} \left(1 - \sqrt{\frac{m}{N}}\; m\right) \cos\left(2\pi\, x_m - \frac{3}{4}\pi\right)$. folglich aber.

$d/m \quad d/m$

$\frac{U(m\, n^2)}{n^{\frac{3}{2}}} = \sum_{d/m} \mu(d)\, \chi(d)\, U\left(\frac{m}{d}\right) U\left(\frac{n^2}{d}\right)$

$d/m.$

$1 - \frac{\chi(p^{\alpha+1})}{p^{(\alpha+1)s}} \quad m^{\frac{3}{2}}$

$\sum \frac{U(m\, n^2)}{n^{\frac{3}{2}}} = \sum_{d/m} \mu(d)\, \chi(d)\, \frac{U\left(\frac{m}{d}\right)}{d^{\frac{3}{2}}} \sum_{m_1} \frac{U(m_1^2\, d)}{m_1^{\frac{3}{2}}}$

$n = m_1\, d.$

$d\, m_1^2 \qquad f_m(x)^{\,m}$ recursive retrospective.

$\prod$

$f_m(x) = \sum_{d/m} \mu(d)\, \chi(d)\, \frac{U\left(\frac{m}{d}\right)}{d^{\frac{3}{2}}}\, f_d(x^d).$

$f_m(x) - \frac{\mu(m)\, \chi(m)}{m^{\frac{3}{2}}}\, f_m(x^m) = \sum.$

$f_m(s) = \sum$

$\frac{1}{\prod_{p/m} \left(1 + \cdots + \frac{\chi(p^\alpha)}{p^{\alpha s}}\right)}.$

$\sum_{\text{coll}} \mathcal{L}(s) \quad \sum_{\text{coll}} \mathcal{L}(s)$ folglich $\sum_d \mathcal{L}(s)$.

$\frac{U\left(\frac{m}{d}\right)}{d^s}$

es noch einmal einmal immer mehr.

$\sum_{d/m} \frac{\chi(d)\, U\left(\frac{m^2}{d}\right)}{m^s}$

d/m

$\sum \mu(d)\, \chi(d)\, \frac{U(n^2)\, U(m)}{m^s}$

$f_m(s) = U(m)\, \mathcal{L}(s) \sum_{d/m} \frac{\chi(d)}{d^s}$

def. $f_m(s) = \left(1 - \mu(m) \frac{\chi(m)}{m^s}\right) \mathcal{L}(s)$.

$f_m(s)\, \mathcal{L}(s) \sum \frac{U\left(m \frac{(s)}{n^3}\right)}{n^{\frac{3}{2}}}$

$\sum_{d/m} \frac{\chi(d)}{d^s}$

$U \sum_{d/m} \frac{\chi(d)\, U(m^2)}{d/m}$

$U(m)\, f_1(s) = \sum_{d/m} \frac{\chi(d)}{d^s}$

$U(n^2)\, f_1(s) =$

$$\int_\alpha^\beta R(y)\lambda(y)\,dy = \sum_1^\infty \frac{u(n)}{\sqrt{n}}\left\{\left[\lambda(y)\int_0^y \sqrt{t}\,J_1(2\pi\sqrt{nt})\,dt\right]_\alpha^\beta -\right.$$

$$\left. - \int_\alpha^\beta \lambda'(y)\left\{\int_0^y \sqrt{t}\,J_1(2\pi\sqrt{nt})\,dt\right\}dy\right\} = \sum_1^\infty \frac{u(n)}{\sqrt{n}}\int_\alpha^\beta \sqrt{y}\,\lambda(y)\,J_1(2\pi\sqrt{ny})\,dy,$$

Now putting $\alpha = (t-1)^2$, $\beta = (t+1)^2$ where $t \geq 2$, $\lambda(y) = y^{-\frac{3}{4}}\left(\frac{\sin \pi\sqrt{N}(\sqrt{y}-t)}{\sqrt{y}-t}\right)^2$, and substituting $y = (t+h)^2$, where h is the new variable of integration, we get

$$\frac{1}{\pi^2\sqrt{N}}\int_{-1}^1 \frac{R((t+h)^2)}{\sqrt{t+h}}\left(\frac{\sin\pi\sqrt{N}h}{h}\right)^2 dh = \sum_1^\infty \frac{u(n)}{\sqrt{n}}\cdot\frac{1}{\pi^2\sqrt{N}}\int_{-1}^1 \sqrt{t+h}\,J_1(2\pi\sqrt{n}(t+h))\left(\frac{\sin\pi\sqrt{N}h}{h}\right)^2 dh$$

But: $J_1(z) = \sqrt{\frac{2}{\pi}}\cdot\frac{\cos(z-\frac{3}{4}\pi)}{\sqrt{z}} + O\left(\frac{1}{z^{\frac{3}{2}}}\right)$,

hence the formula above gives

$$\frac{1}{\pi^2\sqrt{N}}\int_{-1}^1 \frac{R((t+h)^2)}{\sqrt{t+h}}\left(\frac{\sin\pi\sqrt{N}h}{h}\right)^2 dh = \sum_1^\infty \frac{u(n)}{n^{\frac{3}{4}}}\frac{\sqrt{2}}{\pi^{\frac{5}{2}}\sqrt{N}}\int_{-1}^1 \cos\left(2\pi\sqrt{n}(t+h)-\tfrac{3}{4}\pi\right)\left(\frac{\sin\pi\sqrt{N}h}{h}\right)^2 dh$$

$$+\, O\left(\frac{1}{\sqrt{N}}\int_{-1}^1\left(\frac{\sin\pi\sqrt{N}h}{h}\right)^2 dh\right) = \sum_1^\infty \frac{u(n)}{n^{\frac{3}{4}}}\frac{\sqrt{2}}{\pi^{\frac{5}{2}}\sqrt{N}}\int_{-1}^1 \cos\left(2\pi\sqrt{n}(t+h)-\tfrac{3}{4}\pi\right)\left(\frac{\sin\pi\sqrt{N}h}{h}\right)^2 dh + O(1).$$

Now it is easy to show by integration by part, that

$$\frac{1}{\pi^2\sqrt{N}}\int_1^\infty \cos(a+bh)\left(\frac{\sin\pi\sqrt{N}h}{h}\right)^2 dh = O\left(\frac{1}{b}\right),$$

thus we get

$$\frac{1}{\pi^2\sqrt{N}}\int_{-1}^1 \frac{R((t+h)^2)}{\sqrt{t+h}}\left(\frac{\sin\pi\sqrt{N}h}{h}\right)^2 dh = \sum_1^\infty \frac{u(n)}{n^{\frac{3}{4}}}\frac{\sqrt{2}}{\pi^{\frac{5}{2}}\sqrt{N}}\int_{-\infty}^\infty \cos\left(2\pi\sqrt{n}(t+h)-\tfrac{3}{4}\pi\right)\left(\frac{\sin\pi\sqrt{N}h}{h}\right)^2 dh$$

$$+\, O(1).$$

But we have

$$\frac{1}{\pi^2\sqrt{N}}\int_{-\infty}^\infty \cos(a+bh)\left(\frac{\sin\pi\sqrt{N}h}{h}\right)^2 dh = \begin{cases}\left(1-\left|\frac{b}{\sqrt{N}}\right|\right)\cos 2\pi a, & \text{for } |b|\leq\sqrt{N},\\ 0, & \text{for } |b|\geq\sqrt{N}.\end{cases}$$

Hence we get

8 reverse

$$\mathcal{U}(2^i)\,\mathcal{U}(2^j) = 2\cdot\sum c_s\,\mathcal{U}(2^{i+j-2s})$$

$8 \cdot 4+4.$ $c_0 = 1$ 1^2+0^2 $c_2 = 0.$

$\chi(2^i) = 1+0+0+\cdots$ $2.8.$ $nm^2 < N$

1 $v^2+u^2.$ $4\sum\chi(3)\,u^2+v^2 + u + -$

$$\sum \frac{\mathcal{U}(nm^2)}{n^{\frac{3}{4}}m^{\frac{3}{2}}}\left(1-\sqrt{\frac{n}{N}}\,m\right)\cos\left(2\pi\sqrt{x}\,m - \frac{3}{4}\pi\right)$$

$0+q^2$ i gr. j ger.

$$\sum_{m=1}^{\infty}$$

$0+q$ q^2+0^2 q^2+0^2 $0^2+(-q)$ $\mathcal{U}(q^i)\,\mathcal{U}(q^j) = \sum c_s\,\mathcal{U}(q^{i+j-2s})$

usw eeile gilt

i = liha $\underline{i+j-2s = \text{lehe.}}$

$\dfrac{\mathcal{U}(nm^2)}{n^{\frac{3}{4}}m^{\frac{3}{2}}}$ $8.$ $h+h$ $(-h)^2+h^2$ $i+j$ = lihe. i = lihe. , 1

solve $h^2+(-h)^2$ $(-h)^2+(-h)^2$ $\chi(q^i).$ $(i+j+1)(i+1)$

$$\mathcal{U}(m)\cdot\mathcal{U}(n) = 8\sum_{d/m,\ d/n}\chi(d)\,\mathcal{U}\left(\frac{mn}{d^2}\right)$$

h^2+0^2 $- \cancel{i}\,i(i+1)$

$\mathcal{U}(ab) = \sum$ $\mathcal{U}($ a fache

vierfach $\mathcal{U}\left(\dfrac{ab}{d^2}\right)\chi(d).$

$\mathcal{U}(a)\cdot\mathcal{U}(b) = \sum c_d$ $\sum\chi(n)$

4 vierfach $\mathcal{U}($

4 $\chi\ h$

$$\mathcal{U}(p^i)\,\mathcal{U}(p^j) = 4\cdot\sum \quad \mathcal{U}(p^{i+j-2s})$$

2 also $0.$ $i \le j$ $\sum_{s\le i}(i+j+1-2s)$

$4m+1.$ $\delta.$ $(i+1)(j+1) = \sum 1\cdot(i+j-2s)+1)$

$\sum_{i=1}^{4m+1}\ \sum_{i=1}^{\delta}\chi(1) =$

$\sum_{i,j}^{4m+1}\chi(p^i)$ $1 = \sum c_0\cdot(i+j-0$

$n = 4m+1$ $c_0 = 1$

p $1+1+\cdots$

$2\cdot 1 = \sum c_0\,(2$ $c_1 = 1$

$2\cdot 2 = \sum c_0\,3 + c_1\,1$

$3\cdot 3 = \sum c_0\cdot 5 + c_1\,3 + c_2$

(6) $$\frac{1}{\pi^2\sqrt{N}}\int_{-1}^{1}\frac{R((t+h)^2)}{\sqrt{t+h}}\left(\frac{\sin\pi\sqrt{N}h}{h}\right)^2 dh = \sqrt{\frac{2}{\pi}}\sum_{1\le m\le N}\frac{u(m)}{m^{\frac34}}\left(1-\sqrt{\frac{m}{N}}\right)\cos\left(2\pi\sqrt{m}\,t-\tfrac{3}{4}\pi\right)+O(1)$$

$$\geq \sqrt{\frac{2}{\pi}}\left\{2\sum_{m\le N}\frac{u(m)}{m^{\frac34}}\left(1-\sqrt{\frac{m}{N}}\right)\cos\left(2\pi\sqrt{m}\,t-\tfrac34\pi\right)-\sum_{1\le m\le N}\frac{u(m)}{m^{\frac34}}\left(1-\sqrt{\frac{m}{N}}\right)\right\}-O(1).$$

We now choose a positive ε such that $2(1-\varepsilon)c_3-\pi \geq c_6>0$, this is possible by lemma IV. By lemma III we then can find a t_0, $e^{e^{c_2N}}\le t_0\le 2e^{e^{c_2N}}$, such that

$$\sum_{m\le N}\frac{u(m)}{m^{\frac34}}\left(1-\sqrt{\frac{m}{N}}\right)\cos\left(2\pi\sqrt{m}\,t_0-\tfrac34\pi\right)>(1-\varepsilon)\sum_{m\le N}\frac{u(m)}{m^{\frac34}}\left(1-\sqrt{\frac{m}{N}}\right).$$

~~(6) suppose~~ But by lemma IV and lemma V we have

$$\sum_{1\le m\le N}\frac{u(m)}{m^{\frac34}}\left(1-\sqrt{\frac{m}{N}}\right)=\frac{8}{3}\pi N^{\frac14}+o(N^{\frac14}),$$

and
$$\sum_{m\le N}\frac{u(m)}{m^{\frac34}}\left(1-\sqrt{\frac{m}{N}}\right)=\frac{8}{3}c_3 N^{\frac14}+o(N^{\frac14}),$$

hence (6) gives for $N>N_0$

$$\frac{1}{\pi^2\sqrt{N}}\int_{-1}^{1}\frac{R((t_0+h)^2)}{\sqrt{t+h}}\left(\frac{\sin\pi\sqrt{N}h}{h}\right)^2 dh > \sqrt{\frac{2}{\pi}}\cdot\frac{8}{3}\left(2(1-\varepsilon)c_3-\pi\right)N^{\frac14}-o(N^{\frac14})$$

$$> c_6N^{\frac14}-o(N^{\frac14})>c_7N^{\frac14}.$$

Now let $\frac{R(x_0)}{x_0^{\frac14}}$ be the maximum value of $\frac{R((t_0+h)^2)}{\sqrt{t_0+h}}$ in the interval $-1\le h\le 1$, where $(t_0-1)^2\le x_0\le(t_0+1)^2$, then we have

$$\frac{R(x_0)}{x_0^{\frac14}}\frac{1}{\pi^2\sqrt{N}}\int_{-\infty}^{\infty}\left(\frac{\sin\pi\sqrt{N}h}{h}\right)^2 dh > c_7N^{\frac14},$$

or

(7) $$\frac{R(x_0)}{x_0^{\frac14}}>c_7N^{\frac14}.$$

But, since $e^{e^{c_2N}}\le t_0\le 2e^{e^{c_2N}}$ and, $(t_0-1)^2\le x_0\le(t_0+1)^2$, we get that

$$N < c_8 \log\log x_0$$

hence (7) gives

$$\frac{R(x_0)}{x_0^{\frac{1}{4}}(\log\log x_0)^{\frac{1}{4}}} > c_7 .$$

Now if N increases towards infinity, so does also x_0, hence we get the

Theorem

$$\overline{\lim} \frac{R(x)}{x^{\frac{1}{4}}(\log\log x)^{\frac{1}{4}}} > 0 .$$

10 reverse

$$\frac{1}{2\pi N}\int_{-1}^{1} \frac{R(e^{t+u})}{e^{\frac{t+u}{2}}} \left(\frac{\sin\frac{Nu}{2}}{\frac{u}{2}}\right)^2 du$$

$$= -2\sum_{0<\gamma<N} \frac{\sin\gamma t}{\gamma}\left(1-\frac{\gamma}{N}\right) + O(1).$$

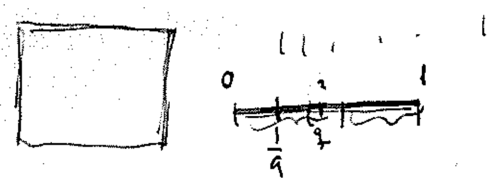
$\{\gamma t\}$ $\quad 0<\gamma<N$; $M < \frac{N}{\cancel{2\pi}}\log\frac{N}{2\pi}$ -

$$\left|\{\gamma t\}\right|_{0<\gamma<N} \le \frac{1}{q}.$$

q^M verdier.

$t = 1, 2, \ldots, q^M + 1$ $\qquad 1 \le \{\gamma t\} \le q^M$

q deler.



q^M

$\cancel{t \to 0}$

$$\sum_{0<\gamma<N} \frac{\sin\gamma t}{\gamma}\frac{1}{N}$$

Lattice point problems.

circle

$A(x)$ no of points in circle radius $\sqrt{x}$.

$u(n) =$ solution $x^2+y^2=n$

$$R_c(x) = A(x) - \pi \cancel{x}$$

Hardys identity

$$\int_0^x R_c(y)\,dy = \frac{x}{\pi} \sum_{n=1} \frac{u(n)}{n} J_2(2\pi\sqrt{nx}) \text{ for } x \geq 0$$

$$\left| J_\nu(z) - \sqrt{\frac{2}{\pi}} \frac{\cos\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)}{\sqrt{z}} \right| \leq \frac{c}{z^{\frac{3}{2}}} \text{ for}$$

$z > 0$; $0 \leq \nu \leq A$.

$$|J_\nu(z)| \leq c\,|z|^\nu \text{ for } |z| \leq A.$$

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+\nu)!} \left(\frac{z}{2}\right)^{2m+\nu}$$

$$= \int t^{\nu-1} e^{\frac{z}{2}\left(t - \frac{1}{t}\right)}\,dt$$

$$\int \frac{x^s}{s} \frac{\cancel{\Gamma(1-s)}\ \zeta^2(1-s)}{\Gamma^2(s)}\,ds \ldots$$

Better ?.

γ Eulers const.

$$R_d(x) = \sum_{n \leq x} d(n) - x \log x - (2\gamma - 1)x - \frac{1}{4}$$

$$R_d(x) = \cancel{\tfrac{1}{4}} - \sum_{n=1}^{\infty} \frac{d(n)}{} \left(\frac{x}{n}\right)^{\frac{1}{2}} F_1(4\pi\sqrt{nx})$$

where $F_1(x) = Y_1(x) + \frac{2}{\pi} K_1(x)$ | $F_1(x) \sim \frac{c}{\sqrt{x}} \cos\left(x - \frac{\pi}{4}\right)$

$$\int_0^x R_d(y)\,dy = -\frac{x}{2\pi} \sum_{n=1}^{\infty} \frac{d(n)}{n} \cdot F_2(4\pi\sqrt{nx})$$

$$F_2(x) = Y_2(x) - \frac{2}{\pi} K_2(x).$$

$$\int_1^\infty t^{-\frac{3}{2}} e^{-a(t+\frac{1}{t})}\,dt + \int_1^\infty t^{-\frac{1}{2}} e^{-a(t+\frac{1}{t})}\,dt$$

$$= \int_1^\infty t^{-\frac{1}{2}}\left(1+\frac{1}{t}\right) e^{-a(t+\frac{1}{t})}\,dt$$

$$t+\frac{1}{t} = y; \qquad \left(1 \mp \frac{1}{t^2}\right)dt = dy$$

$$t^{-\frac{1}{2}}\left(1+\frac{1}{t}\right)dt = \frac{dy}{t^{\frac{1}{2}} - \frac{1}{t^{\frac{1}{2}}}} = \frac{dy}{\sqrt{y-2}}$$

$$\sqrt{y-2} \qquad = \int_2^\infty \frac{e^{-ay}}{\sqrt{y-2}}\,dy = e^{-2a}\int_0^\infty \frac{e^{ay}}{\sqrt{y}}\,dy$$

$$= \underline{\Gamma\left(\tfrac{1}{2}\right)}\ \frac{e^{-2a}}{\sqrt{a}}.$$

$$d\sqrt{u}$$

$$16\pi^2 \int_0^\infty \frac{1}{\sqrt{u}}\, e^{-2\pi\sqrt{2u}}\, du$$

$$\frac{16\pi^2}{2\pi\sqrt{2}}.$$

$$\boxed{\sum_{n=1}^\infty r(n)\left\{\frac{1}{(z+n)^{\frac{1}{2}}} - \frac{1}{n^{\frac{1}{2}}}\right\} = C - \frac{1}{\sqrt{z}} - 2\pi\sqrt{z} + \sum_1^\infty \frac{r(n)}{\sqrt{n}}\, e^{-2\pi\sqrt{zn}}}$$