if there are eigenvalues $\sigma_i(1-\sigma_i)$ with $\frac{1}{2} < \sigma_i < 1$, terms

$$c^i_{h,k} \; x^{\sigma_i} \; u_i^k(z) \, \overline{u_i^h(\xi)}$$

occur in finite number.

For $A(\mathcal{D})$ finite but $\Gamma \backslash G$ not compact the results are not much changed, besides the point spectrum there is then a continuous spectrum and in the expansion of $L^{h,k}(z,\xi,s)$ we have beside the series also integrals over the continuous spectrum to deal with so the proofs are longer, but the end result the same.

They show that the two angles entering are <u>equidistributed</u>, which is <u>equivalent to equidistribution</u> (mod $2\pi$) <u>of the angles $\theta_1$ and $\theta_2$</u> in the representation

$$\gamma = \begin{pmatrix} \cos\frac{\theta_1}{2} & \sin\frac{\theta_1}{2} \\ -\sin\frac{\theta_1}{2} & \cos\frac{\theta_1}{2} \end{pmatrix} \begin{pmatrix} \rho & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta_2}{2} & \sin\frac{\theta_2}{2} \\ -\sin\frac{\theta_2}{2} & \cos\frac{\theta_2}{2} \end{pmatrix}$$

where $\rho > 1$, $\rho < x$ as $x \to \infty$.

We now assume that $A(\mathcal{D})$ is finite but $\Gamma\backslash G$ not compact and assume one cusp is at $\infty$, and that $\Gamma$ has the parabolic subgroup $\Gamma_\infty = \{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \}$ where $n$ runs over the integers. It is possible to use the Poincaré series

$$P_\infty^{k,n}(z,s) = \sum_{\Gamma_\infty \backslash \Gamma} y_{\gamma z}^{s}\, e^{2\pi i n x_{\gamma z}}\, e_\gamma^{k}(z),$$

to investigate the representation of

$$\gamma = \begin{pmatrix} \cos\frac{\theta}{2}, \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2}, \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \rho & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

and show that for $1 \le \rho < X$; $\theta \bmod 2\pi$ and $u \bmod 1$ are equidistributed.

Again expansions in terms of eigenforms are the key to the analytic properties of $P$ as a function of $s$, and so to estimations for

$$\sum_{\substack{c^2+d^2<X \\ \Gamma_\infty\backslash\Gamma}} e^{2\pi i n \operatorname{Re}\frac{ai+b}{ci+d}} \left(\frac{ci+d}{-ci+d}\right)^{k}$$

which provide the answer.

We shall look closer at the representation of the $\gamma$'s not in $\Gamma_\infty$ (that is: with $c \neq 0$) as

$$\gamma = \begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{c} \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{pmatrix},$$

and look at the distribution of $\frac{a}{c}, \frac{d}{c} \bmod 1$ for $|c| < X$. A convenient Poincaré series to use is, for $m \geq 0$

$$U^m(z,s) = \sum_{\Gamma_\infty \backslash \Gamma} y_{\gamma z}^s \, e^{2\pi i m \gamma z},$$

and for $m < 0$

$$U^m(z,s) = \sum_{\Gamma_\infty \backslash \Gamma} y_{\gamma z}^s \, e^{2\pi i m \overline{\gamma z}}.$$

We can expand $U^m(z,s)$ for $m \neq 0$ in eigenfunctions of $y^2 \Delta = \Delta_0$ for $m \neq 0$ we get a series over the discrete spectrum and an integral over the continuous spectrum, the Eisenstein series. For $m = 0$, $U^m(z,s)$ is simply the Eisenstein series corresponding to the cusp at $\infty$.

For $m \neq 0$, $U^m(z,s)$ is meromorphic in the whole plane, for $\sigma \geq \frac{1}{2}$ the poles are only at points $\frac{1}{2} + ir$ where $\frac{1}{4} + r^2$ is an eigenvalue and at a finite no of real points on $\frac{1}{2} < s < 1$. For $m = 0$ we have also a pole at $s=1$ and possible poles in $\frac{1}{2} < s < 1$ (simple poles only), and no poles on $\sigma = \frac{1}{2}$.

The study of the $n$'the Fourier-coefficient of $U^m(z,s)$ leads us to the dirichlet-series:

$$Z^{(m,n)}(s) = \sum_{\substack{c \neq 0 \\ 0 \leq a, d \leq |c|}} \frac{e^{2\pi i (m \frac{a}{c} + n \frac{d}{c})}}{|c|^{2s}} ,$$

for $\sigma \geq \frac{1}{2}$, this can have poles only where $U^m(z,s)$ has a pole, but if either $m$ or $n$ equals zero there is no pole on $\sigma = \frac{1}{2}$.

Again we can show equidistribution of $\left(\frac{a}{c}, \frac{d}{c}\right) \bmod 1$ for $|c| < X$ as $X \to \infty$. But if we only look at one of the ratios, the results are rather sharper.

If either $m$ or $n$ (or both) equal zero we can show that

$$\int_{-T}^{T} |Z^{(m,n)}(\tfrac{1}{2}+it)|^2 dt = O(T^2).$$

Using this we get for $m=n=0$,

$$\sum_{|c|<x} (x-|c|) = cx^3 + \sum_{\frac{1}{2}<\sigma_i<1} c_i x^{2\sigma_i+1} + o(x^2),$$

and for $m \neq 0, n=0$,

$$\sum_{\substack{|c|<x \\ 0\leq a<|c|}} (x-|c|)\, e^{2\pi i m \frac{a}{c}} = \sum_{\frac{1}{2}<\sigma_i<1} c_i^{(m)} x^{2\sigma_i+1} + o(x^2).$$

Thus if there are no poles of the Eisenstein series between $\frac{1}{2}$ and $1$, we have

$$\sum_{\substack{|c|<x \\ 0\leq a<|c|}} (x-|c|)\, e^{2\pi i m \frac{a}{c}} = o(x^2).$$

If we specialize to the modular group and take $m=1$, this gives

$$\sum_{0<c<x} (x-c)\mu(c) = o(x^2), \quad \boxed{\mu = \text{Möbius function.}}$$

from which immediately

$$\sum_{n<x} \mu(n) = o(x).$$