$$ds^2 = \frac{|dz|^2}{y^2}, \quad u(z,\xi) = \frac{|z-\bar{\xi}|^2}{4y\eta} = \frac{e^{\rho}+e^{-\rho}+2}{4}$$

$\rho = d(z,\xi)$ invariant distance

$$gz = \frac{az+b}{cz+d}, \text{ put } \varepsilon_g(z) = e^{2i\arg(cz+d)} =$$

$$= \frac{cz+d}{c\bar{z}+d}, \quad f(z) \text{ form for } \Gamma \text{ of weight } k$$

if $f(\gamma z) = \varepsilon_\gamma^k(z) f(z)$ for $\gamma \in \Gamma$.

$$\Delta_k = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2iky\frac{\partial}{\partial x}$$

carries form of weight k into form of weight k.

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for $k \le h$, put

$$D_{h,k} = (2i)^{h-k} y^{1-k} \frac{d^{h-k}}{dz^{h-k}} y^{h-1},$$

for $k \ge h$

$$D_{h,k} = (-2i)^{k-h} y^{k+1} \frac{d^{k-h}}{d\bar{z}^{k-h}} y^{-h-1},$$

so that

$$D_{h,k} = \overline{D_{-h,-k}}.$$

$$D_{h,k}(gz) = \varepsilon_g^{\overset{2}{h}}(z)\, D_{h,k}(z)\ \varepsilon_g^{-k}(z)$$

$D_{h,k}$ carries forms of weight k into forms of weight h. If f is eigenform of Δ_k then $D_{h,k} f$ is eigenform of Δ_h (unless it vanishes identically).

For Δ_0 eigenvalues of form $\lambda = \frac{1}{4} + r^2$ $0 \leq \lambda$, 0 for constant, for Δ_k $k > 0$ finite no of neg eigenvalues $l(1-l)$ where $0 < l \leq k$ whose multiplicity equals the number of analytic forms of weight l. For $k < 0$ corresponding result.

For general k we shall denote by u_r^k a complete orthonormal set of eigenforms of Δ_k with eigenvalue $\frac{1}{4} + r^2$.

Poincare series:

For $h \geq k$

$$L^{k,h}(z,\zeta;s) = \sum_{\gamma\in\Gamma} x(\gamma z,\zeta)^{-s}\left(1-\frac{1}{x(\gamma z,\zeta)}\right)^{\frac{h-k}{2}} \cdot e^{ih\arg\frac{\gamma z-\zeta}{\gamma z-\bar\zeta} + ik\arg\frac{\bar z-\gamma^{-1}\bar\zeta}{z-\gamma^{-1}\bar\zeta}}$$

$$= \sum_{\gamma\in\Gamma} \varepsilon_\gamma^{-k}(z)\, x(\gamma z,\zeta)^{-s}\, \frac{(\gamma z-\zeta)^{h-k}\,|\gamma z-\bar\zeta|^{2k}}{(\gamma z-\bar\zeta)^{h+k}}.$$

abs conv. for $\Re s > 1$.

For $h \le k$

$$L^{k,h}(z,\zeta;s) = \overline{L^{h,k}(\zeta,z;\bar s)}.$$

Expansion

$$L^{k,h}(z,\zeta;s) = \sum \alpha_r^{k,h}(s)\, u_r^k(z)\, \overline{u_r^h(\zeta)},$$

For $h=k$

$$\alpha_r^{k,k}(s) = 4\pi\,\frac{\Gamma(s-\frac{1}{2}-ir)\Gamma(s-\frac{1}{2}+ir)}{\Gamma(s-k)\,\Gamma(s+k)},$$

for $h \neq k$, more complicated but of same general nature.

Pole at $s=1$ (due to constant eigenfunction) occurs only for $k=h=0$, otherwise poles are on line $\Re s = \frac{1}{2}$ at points $s = \frac{1}{2}+ir$ or on interval

2b (3b)

for $h > k$, put $l = h - k$

$\alpha_n^{k,h}(s) =$

$$\varepsilon_n^{k,h} \, 4\pi \frac{\Gamma(2s-1)}{\Gamma(2s+l-1)} \sum_{\nu=0}^{l} (-1)^\nu \binom{l}{\nu} \frac{\Gamma(s-\frac{1}{2}+ir+\nu)\,\Gamma(s\frac{1}{2}-ir+l-\nu)}{\Gamma(s+k+\nu)\,\Gamma(s-k-\nu)}$$

$\downarrow$

$\varepsilon_n^{k} \cdot \overline{\varepsilon_n^{h}}$? (not important for estimates later).

$|\alpha_n^{k,h}(s)|^2$ obtained by integrating

$$\iint_{\mathcal{D}} \left| L^{k,h}(z,\xi;s) \right|^2 d\omega_\xi \qquad + 2^{\|h-k\|}$$

$$= \sum |\alpha_n^{k,h}(s)|^2 \, u_n^k(z) \, \overline{u_n^k(z)}$$

$$\iint_{\mathcal{D}} L^{k,h}(z,\xi,s) \cdot \overline{L^{k,h}(z',\xi,s)} \, d\omega_\xi$$

$$= \sum |\alpha_n^{k,h}(s)|^2 \, u_n^k(z) \, \overline{u_n^k(z')}$$

$\downarrow$

can be determined.

$\cancel{X}$ - $\varkappa(\gamma z, \xi)$

$0 < s < 1$ (if there are eigenvalues between 0 and $\frac{1}{4}$), and at these points shifted to the left by an integral amount.

If no eigenvalue between 0 and $\frac{1}{4}$, we get

$$\sum_{u(\gamma z, \xi) < X} e^{ih \arg \frac{\gamma z - \xi}{\gamma z - \bar{\xi}} + ik \arg \frac{\bar{z} - \gamma^{-1}\xi}{z - \gamma^{-1}\bar{\xi}}} =$$

$$= \frac{4\pi}{A(D)} X + O(X^{\frac{2}{3}}), \text{ for } h = k = 0,$$

$$= O\left(X^{\frac{2}{3}} + (1+|k|)(1+|h|) X^{\frac{1}{2}}\right),$$

for $h^2 + k^2 > 0$.

If eigenvalues between 0 and $\frac{1}{4}$ results modified.