Harmonic analysis

2. Teil

Vorlesung von Prof. Selberg

Göttingen S.S. 1954

§ 6

The non-compact case, preparations for the general case, and the trace formula in the "non-singular" case.

We have in the earlier ~~paragraphs~~ chapters restricted ourselves to groups with compact fundamental domain and proved our trace-formula in that case. When considering now also groups with non compact fundamental domain, we will have to retain the basic condition that the area of the fundamental domain in the invariant metric shall be finite.

One can then easily show that non-compactness arises only when the group Γ contains parabolic transformations, and the fundamental polygon has "cusps" that is, vertices with the angle zero. The "cusps" are also the ~~same as the~~ fixpoints of parabolic transformations in the group. Two cusps are called equivalent if there is some element in Γ that carries one into the other. One can easily show that the number of inequivalent cusps ~~is~~ is finite.

If we introduce the notion of a 'primitive' parabolic transformation in Γ, as one that is not a power with exponent >1 of any other transformation in Γ, and at the same time with respect to G (the group of all motions of the hyperbolic plane) is equivalent to $z \to z+1$. (It is easily seen that any parabolic transformation with respect to G is equivalent to either $z \to z+1$ or to $z \to z-1$). The number of inequivalent (with respect to Γ) primitive parabolic transformations is finite and equal to that of inequivalent cusps, since a one-to one correspondence can be established. Also any parabolic transformation in Γ is an integral (positive or negative) power of the primitive parabolic transformation which has the same fixpoint.

Our first object will be to ~~consider~~ study the behaviour of our kernel ~~$K(z,z',\chi)$~~

$$(6.1) \quad K(z,z',\chi) = \sum_{M \subset \Gamma} \chi(M)\, k(z, Mz'),$$

under the new circumstances. For simplicity we shall in the following only consider the case that χ is a onedimensional unitary representation of Γ, the general case of representation by $r \times r$ unitary matrices can be handled in a similar way, but the formulas get more complicated.

We first introduce certain notations that we shall keep also through the following chapters. We assume our fundamental ~~domain~~ polygon $\mathcal{D}$ to be chosen such that it has $\varkappa$ cusps and that these are all ~~equ~~ inequivalent and lie at the points $\xi_1, \xi_2 \ldots \xi_\varkappa$ on the real axis. The primitive parabolic transformation that has ξ_i as fixpoint we denote by S_i for $1 \le i \le \varkappa$. We further define for $1 \le i \le \varkappa$,

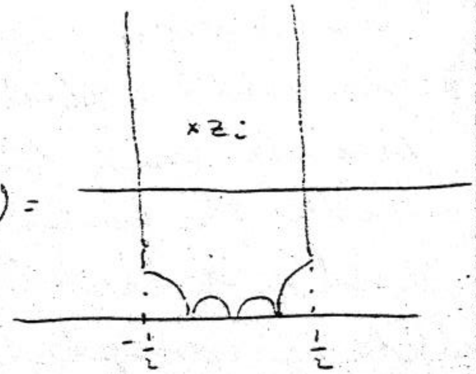
$$z_i = \frac{-\lambda_i}{z - \xi_i},$$

where the $\overbrace{\lambda_i}^{\text{positive real number}}$ is so chosen that $z \to S_i z$ corresponds to $z_i \to z_i + 1$ (If $\xi_i = \infty$, we put instead $z_i = \lambda_i z$). We also shall write $z_i = x_i + i y_i$. Also for $M \subset \Gamma$ we write

$$(Mz)_i = z_{i,M} = x_{i,M} + i y_{i,M}.$$

One easily sees that if z and z' both are restricted to some compact sub-region of $\mathcal{D}$, our previous arguments for the compact case are still valid, as we ~~in our proof of the~~ merely used the fact that the images of $\mathcal{D}$ under the transformations of Γ did not overlap, and not that they exhausted the hyperbolic plane completely. Therefore if $h(r)$ satisfies our previous conditions, $K(z, z'; \chi)$ is still uniformly bounded and continuous, as long as z and z' are restricted to a compact subregion of $\mathcal{D}$. We therefore have only to study what happens if at least one or possibly both points z and z' tend towards the cusps staying within $\mathcal{D}$. Without loss of generality, we may assume that one of the points, say z, is confined to some non-compact subregion of $\mathcal{D}$ that contains only one cusp, say ξ_i. This subregion may be taken as the subregion of $\mathcal{D}$ where $y_i \geq A$, where A is a positive constant, large enough so that the curve $y_i = A$ cuts only the two sides of $\mathcal{D}$ that pass through the cusp ξ_i, (one can show that $A = 1$ will always suffice, but we shall not need that).

Writing now



(62) $K(z, z', \chi) = \sum_{M \subset \Gamma} \chi(M)\, k(z, M z') =$

$$= \sum_{n=-\infty}^{\infty} \chi^n(S_i)\, k(z, S_i^n z') + {\sum_{M\subset\Gamma}}' \chi(M)\, k(z, M z'),$$

where the dash on the second summation indicates that all powers of S_i are missing; - We shall first investigate the second term on the righthand side, and show that this still remains uniformly bounded under our previous assumptions about $h(r)$ or $k(t)$. That is we suppose that

$$|k(t)| \leq \frac{A}{(2+t)^{2+\varepsilon}},$$

for some positive constants A and ε.

Denoting by $M\ (\mathrm{mod}\ S_i)_{\text{left}}$ that a sum extends over a complete set of transformations M that do not differ by a power of S_i on the left side, and indicating by the dash Σ' still that M is not equal to a power of S_i, we can write

$$(6.3)\quad \sum_{M\in\Gamma}{}' |k(z, Mz')| = \sum_{M(S_i)_{\text{left}}}{}' \sum_{n=-\infty}^{\infty} |k(z, S_i^n Mz')|,$$

~~Introducing~~ Here

$$|k(z, S_i Mz')| = |k(z_i, (S_i^n Mz')_i|$$

$$= |k(z_i, (Mz')_i + n)| =$$

$$= \left|k\left(\frac{y_i}{y'_{i,M}} + \frac{y'_{i,M}}{y_i} - 2 + \frac{(x_i - x'_{i,M} - n)^2}{y'_{i,M}\, y_i}\right)\right|$$

$$\leqslant \frac{A_1 (y'_{i,M})^{1+\varepsilon}}{\left(y_i + \frac{(x_i - x'_{i,M} - n)^2}{y_i}\right)^{1+\varepsilon}}.$$

Thus

$$\sum_{n=-\infty}^{\infty} |k(z, S_i^n Mz')| < A (y'_{i,M})^{1+\varepsilon} \sum_{n=-\infty}^{\infty} \frac{1}{\left(y_i + \frac{(x_i - x'_{i,M} - n)^2}{y_i}\right)^{1+\varepsilon}}$$

$$< 2A (y'_{i,M})^{1+\varepsilon} \sum_{n=0}^{\infty} \frac{1}{\left(y_i + \frac{n^2}{y_i}\right)^{1+\varepsilon}}.$$

Here since $y_i \geqslant A$

$$\sum_{n=1}^{\infty} \frac{1}{\left(y_i + \frac{n^2}{y_i}\right)^{1+\varepsilon}} < y_i^{-1-\varepsilon} + \int_0^{\infty} \frac{du}{\left(y_i + \frac{u^2}{y_i}\right)^{1+\varepsilon}} < A_2 y_i^{-\varepsilon}.$$

Hence

$$\sum_{-\infty}^{\infty} |k(z, S_i^n M z')| < A_\varepsilon\, y_i^{-\varepsilon} (y'_{i,M})^{1+\varepsilon}.$$

Thus from (6.3)

$$\sum_{M \subset \Gamma}' |k(z, Mz')| < A_\varepsilon\, y_i^{-\varepsilon} \sum_{M(S_i)_{\text{left}}}' (y'_{i,M})^{1+\varepsilon}.$$

Anticipating here a result that will be proved in next ~~paragraph~~ chapter (§ 7, Theorem 7.1.), we have here, in the notation of that ~~chapter~~ chapter,

$$\sum_{M(S_i)\,\text{left}}' (y'_{i,M})^{1+\varepsilon} = E_i(z', 1+\varepsilon) - (y'_i)^{1+\varepsilon};$$

and this expression is uniformly bounded for z' in $\mathcal{D}$ (and even tends to zero as $(y'_j)^{-\varepsilon}$ if z' tends to the cusp ξ_j for any $1 \le j \le \varkappa$). Therefore we have

that $\sum_{M \subset \Gamma}' |k(z, Mz')|$ is uniformly bounded for z and z' in $\mathcal{D}$ and $y_i \ge A$ (and actually tending to zero if z approaches ξ_i or z' approaches any cusp ξ_j).

It remains to look at the first term on the right-hand side of (6.2). We shall now introduce a stronger assumption about $k(t)$ namely that $k(t)$ is of bounded total variation or $\int_0^\infty |dk(t)| < \infty$. This is easily shown to be the case if we replace the previous condition $h(r) = O\left(\frac{1}{|r|^{2+\varepsilon}}\right)$ as $|r| \to \infty$ in the strip $|\mathfrak{I}(r)| \leq \frac{1}{2} + \varepsilon$, by the stronger condition

$$(6.4) \qquad h(r) = O\left(\frac{1}{|r|^{3+\varepsilon}}\right).$$

We now consider two cases:

(a) $\chi(S_i) \neq 1$, then

$$(6.5) \sum_{n=-\infty}^{\infty} \chi^n(S_i)\, k(z, S_i^n z') = \sum_{n=-\infty}^{\infty} \chi^n(S_i)\, k\left(\frac{(y_i - y_i')^2 + (x_i - x_i' - n)^2}{y_i y_i'}\right).$$

From this we get

$$\left| (1-\chi(S_i)) \sum_{n=-\infty}^{\infty} \chi^n(S_i)\, k(z, S_i^n z') \right| =$$

$$= \left| \sum_{n=-\infty}^{\infty} \chi^{n}(S_i) \left\{ k\left(\frac{(y_i-y_i')^2+(x_i-x_i'-n)^2}{y_i y_i'}\right) - \right.\right.$$

$$\left.\left. - k\left(\frac{(y_i-y_i')^2+(x_i-x_i'-n+1)^2}{y_i y_i'}\right) \right\} \right| \leq 2\int_0^\infty |d\,k(t)|,$$

so that in this case we have uniform boundedness of the expression (6.5) for z and z' in $\mathcal{D}$. ~~and y~~

(b) $\chi(S_i)=1$, then

$$\sum_{n=-\infty}^{\infty} \chi^{n}(S_i)\, k(z, S_i^n z') = \sum_{n=-\infty}^{\infty}\left\{k(z, S_i^n z') - \int_n^{n+1} k(z, S_i^u z')\,du\right\}$$

$$+ \int_{-\infty}^{\infty} k(z, S_i^u z')\,du,$$

here again the first term on the right-hand side is seen to be uniformly bounded because $k(t)$ is of ~~total~~ bounded total variation. For the second term on the right-hand side we have

$$\int_{-\infty}^{\infty} k(z, S_i^u z')\,du = \int_{-\infty}^{\infty} k(z_i, z_i+u)\,du$$

$$= \int_{-\infty}^{\infty} k\left(\frac{(y_i-y_i')^2+v^2}{y_i y_i'}\right) dv =$$

$$\sqrt{y_i y_i'}\int_{-\infty}^{\infty} k\left(\frac{(y_i-y_i')^2}{y_i y_i'}+x^2\right)dx = \sqrt{y_i y_i'}\; Q\left(\frac{(y_i-y_i')^2}{y_i y_i'}\right) =$$

$$= \sqrt{y_i y_i'}\; g\left(\log \frac{y_i}{y_i'}\right).$$

We can now go back to (6-2) and sum up our results. If $\chi(S_i)=1$, we shall say that χ is singular with respect to the cusp ξ_i, otherwise that it is nonsingular with respect to the cusp. If χ is nonsingular with respect to all cusps ξ_i for $1 \le i \le \kappa$, we call χ nonsingular, ~~with respect~~ and if it is singular with respect to at least one cusp we call it singular.

We then have

Theorem (6.1). If χ is nonsingular then $K(z, z', \chi)$ is uniformly bounded. If χ is singular, let it be singular with respect to the cusps ξ_i $1 \le i \le \kappa_1$, $1 \le \kappa_1 \le \kappa$ and non-singular with respect to the cusps ξ_i $\kappa_1 < i \le \kappa$, then

$$K(z, z', \chi) - \sum_{i=1}^{\varkappa_1} \sqrt{y_i y'_i}\, g\left(\log \frac{y_i}{y'_i}\right)$$

is uniformly bounded for z and z' in $\mathfrak{D}$.

As one easily sees $K(z, z', \chi)$ is not itself uniformly bounded in the latter case, since for instance $\sqrt{y_i y'_i}\, g(\log \frac{y_i}{y'_i})$ tends to infinity if z and z' both tend to ξ_i such that $\log \frac{y_i}{y'_i}$ is a constant α for which $g(\alpha) \neq 0$. however since $g(u) = O(e^{-(\frac{1}{2}+\varepsilon)|u|})$, with ~~these results can easily be extended to~~ some $\varepsilon > 0$, we see that $K(z, z', \chi)$ also in the latter case remains bounded unless both z and z' tend towards the same cusp ξ_i $1 \le i \le \varkappa_1$ (and at about the same rate).

These results could be extended to the case that χ is not a onedimensional representation. In that case χ is nonsingular with respect to ξ_i if no eigenvalue of $\chi(S_i)$ is equal to 1, and singular of degree ν with respect to ξ_i if 1 is an eigenvalue of multiplicity ν. In particular if χ is nonsingular with respect to all ξ_i, $K(z, z', \chi)$ is again uniformly bounded.

From the preceding it is now simple to extend the trace formula to the noncompact case if χ is nonsingular. We have as before

$$(6\text{-}6)\quad \sum h(r_i) = 2\iint_{\mathcal{D}} K(z,z,\chi)\,\frac{dx\,dy}{y^2},$$

under the preliminary assumption that that $h(r)$ can be written as a product of two other functions satisfying (6.4), or what is the same that

$$(6\text{-}7)\quad h(r) = O\left(\frac{1}{|r|^{6+\varepsilon}}\right),$$

with some $\varepsilon > 0$, and is regular in some strip $|\Im(r)| \le \frac{1}{2}+\varepsilon$.

In the computation of the right-hand side of (6.6), we cannot anymore split up $K(z,z)$ completely into the single terms of the series, namely we must for each primitive parabolic transformation S_i, keep the terms

$$(6\text{-}8)\quad \sum_{n=-\infty}^{\infty}{}' \chi^n(S_i)\, k(z, S_i^n z) \qquad \left(\textstyle\sum' \text{ means } n \neq 0\right)$$

together, whereas the other terms (corresponding to identity transformation, and elliptic and hyperbolic transformations) are treated as in the compact case.

Combining with the terms (6.8) all the terms corresponding to transformations that with respect to Γ are equivalent to ~~powers of~~ some S_i^n with $n \neq 0$; and observing that the commuting group for any S_i^n $n \neq 0$ is the group generated by S_i, we find easily that these terms contribute to the right-hand side of (6.6)

(6.9) $$2 \iint_{\substack{|x_i| \leq \frac{1}{2} \\ y_i \geq 0}} \sum_{n=-\infty}^{\infty}{}' \chi(S_i)^n \, k(z_i, z_i + n) \, \frac{dx_i \, dy_i}{y_i^2},$$

where the domain of integration is a fundamental domain of the commuting group Γ_{S_i}.

If we write $\chi(S_i) = e^{\pm \alpha_i i} \neq 1$, (6.9) becomes

$$4 \int_0^{\infty} \sum_{n=1}^{\infty} \cos n\alpha \; k\left(\frac{n^2}{y^2}\right) \frac{dy}{y^2} =$$

$$= 4 \int_0^{\infty} \sum_{n=1}^{\infty} \cos n\alpha \; k(n^2 u^2) \, du,$$

We compute this by writing it as

$$\lim_{\varepsilon \to 0} 4 \int_{\varepsilon}^{\infty} \sum_{n=1}^{\infty} \cos n\alpha \; k(n^2 u^2)\, du ,$$

Here

$$\int_{\varepsilon}^{\infty} \sum_{n=1}^{\infty} \cos n\alpha \; k(n^2u^2)\, du = \sum_{n=1}^{\infty} \cos n\alpha \int_{\varepsilon}^{\infty} k(n^2u^2)\, du$$

$$= \sum_{n=1}^{\infty} \frac{\cos n\alpha}{n} \int_{n\varepsilon}^{\infty} k(u^2)\, du =$$

$$= \int_{0}^{\infty} k(u^2) \Big\{ \sum_{1 \le n \le \frac{u}{\varepsilon}} \frac{\cos n\alpha}{n} \Big\}\, du .$$

But for fixed α, $e^{i\alpha} \neq 1$, we have

$$\sum_{1 \le n \le \frac{u}{\varepsilon}} \frac{\cos n\alpha}{n} = \log \frac{1}{|1 - e^{i\alpha}|} + O\Big(\sqrt{\frac{\varepsilon}{u}}\Big),$$

uniformly, thus

$$\int_{\varepsilon}^{\infty} \sum_{n=1}^{\infty} \cos n\alpha \; k(n^2u^2)\, du = \log \frac{1}{|1-e^{i\alpha}|} \int_{0}^{\infty} k(u^2)\, du$$

$$+ O\Big(\sqrt{\varepsilon} \int_{0}^{\infty} k(u^2) \frac{du}{\sqrt{u}}\Big) = \frac{1}{2}\, g(0) \log \frac{1}{|1-e^{i\alpha}|}$$

$$+ O(\sqrt{\varepsilon}) .$$

Inserting this above and making $\varepsilon \to 0$

we get that the value of (6.9) is

$2g(0)\log\frac{1}{|1-\chi(S_i)|}$, Thus we have

the traceformula under the assumption (6.7) by just adding the expression

$$2g(0)\sum_{i=1}^{\varkappa}\log\frac{1}{|1-\chi(S_i)|},$$

to the terms that occurred on the right-hand side in the compact case, and as before we can easily get rid of the restriction (6.7) and replace it by the original condition

$$h(r) = O\left(\frac{1}{|r|^{2+\varepsilon}}\right)$$

by approximating $h(r)$ by the functions $h(r)\,e^{-\delta r^2}$, $\delta>0$ and letting $\delta\to 0$. Thus we obtain

Theorem 6.2. The traceformula is valid in the case of a nonsingular χ, under the same assumptions about $h(r)$ as in the compact case, and we obtain it by adding the expression

$$2g(0)\sum_{i=1}^{\varkappa}\log\frac{1}{|1-\chi(S_i)|}$$

to the terms that occurred on the right-hand side in the compact case.

In case χ is nonsingular but not one dimensional the same thing holds if we replace the expression $|1-\chi(S_i)|$ above by $\|E-\chi(S_i)\|$ where if χ is a $r \times r$ matrix E is the $r \times r$ Identity matrix.

Consequences about $Z_\Gamma(s,\chi)$ can for nonsingular χ be obtained as in the compact case, the new terms in the traceformula cause only a slight modification of the functional equation, but does not influence the multiplicities of the "trivial" zeros (in contrast to what the terms coming from the elliptic transformations did).

§ 7

Preparations for the case of a singular χ, the Eisenstein series.

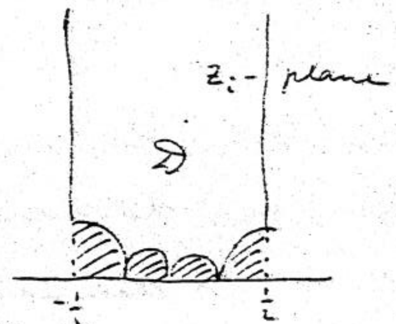
In the remaining case that χ is singular, it is easily seen that we cannot have a traceformula quite like the one we had in the former cases, among other reasons because the integral $\iint_{\mathcal{D}} K(z,z,\chi)\frac{dx\,dy}{y^2}$ in ~~[illegible]~~ general will not exist due to the singular behaviour of the kernel $K(z,z';\chi)$ when both points approach the same cusp ξ_i when $1 \le i \le \varkappa_1$. We shall however later see that it is possible to remove "the singular part" of the kernel, since it is due to the existence of a continuous spectrum of eigenfunctions for our problem, these eigenfunctions are of course not square-integrable over $\mathcal{D}$. Our task will therefore first be to study these eigenfunctions.

We again use the notations of the preceding chapter and suppose in particular that χ is singular with respect to the cusps at ξ_i for $1 \le i \le \varkappa_1$, and non singular with respect to the ξ_i with $\varkappa_1 < \overset{i}{k} \le \varkappa$.

For $1 \le i \le \varkappa_1$, and $s = \sigma + ir$; $\sigma > 1$, we form the series

$$(7.1)\quad E_i(z,s,\chi) = \sum_{M(S_i)_{left}} \overline{\chi}(M)\, y_{i,M}^{s}.$$

We shall first have to prove the absolute convergence of this series, and uniform convergence in any compact domain. This we do as follows; taking absolute values in (7.1) we get the series



$$E_i(z,\sigma) = \sum_{M(S_i)_{\text{left}}} y_{i,M}^{\sigma},$$

with $\sigma > 1$. Because of the invariance of this series if we replace z by Mz where $M \subset \Gamma$, it is enough to consider the case that z lies in $\mathcal{D}$. We consider the integral

$$\iint y_i^{\sigma} \frac{dx_i dy_i}{y_i^2}$$

extended over the shaded area on the figure, namely what remains of the half-strip $|x_i| \le \frac{1}{2}$, $y_i > 0$ if we remove $\mathcal{D}$. The integral obviously exists for $\sigma > 1$. Mapping the shaded area back into $\mathcal{D}$ by suitable transformations from Γ, we find that it equals

$$\iint_{\mathcal{D}} \left(\sum_{M(S_i)_{\text{left}}}' y_{i,M}^{\sigma} \right) \frac{dx\, dy}{y^2}$$

where the dash implies that no M that is a power of S_i occurs in the summation; since the terms are positive, this shows that

$$\sum_{M(S_i)_{\text{left}}} y^{\sigma}_{i,M} = y^{\sigma}_{i} + \sum_{M(S_i)_{\text{left}}}' y^{\sigma}_{i,M},$$

converges almost everywhere to an integrable function. The convergence in every point, and also the uniform convergence in any compact domain, then follows because each term is an eigenfunction of $y^2\Delta$ and our class of integral operators. Namely consider a point $z^{(0)}$ and denote by C_ρ the interior of a geodesic circle with radius ρ (in the hyperbolic metric) and center at $z^{(0)}$. Then

$$y^{(0)\sigma}_{i,M} = \mu(\rho,\sigma) \iint_{C_\rho} y^{\sigma}_{i,M} \cdot \frac{dx\,dy}{y^2}$$

where $\mu(\rho,\sigma)$ is a positive constant depending on ρ and σ only, from this the rest follows easily.

Returning now to $E_i(z,s,\chi)$, we see from (7.1) easily that

$$E_i(Mz,s,\chi) = \chi(M)\,E_i(z,s,\chi), \tag{7.2}$$

and

(7.3) $\quad y^2\Delta E_i(z,s,\chi) + s(1-s)E_i(z,s,\chi) = 0.$

From (7.2) we have in particular that $E_i(z,s,\chi)$ is periodic in z_j with period 1 for $1 \le j \le \varkappa_1$ and that it is multiplied by $\chi(S_j) = e^{i\beta_j}$ with $0<|\beta_j| \le \pi$, for $\varkappa_1 < j \le \varkappa$ if z_j is increased by 1. It is therefore possible to expand $E_i(z,s,\chi)$ in certain fourier series ~~of~~ in terms of x_j.

Let us first look at the fourier expansion of $E_i(z,s,\chi)$ in terms of x_j, where $1 \le j \le \varkappa_1$. We write

$$y_{i,M} = \frac{y_j}{|c_{i,j} z_j + d_{i,j}|^2},$$

and observe that the $c_{i,j}$ and $d_{i,j}$ have the following properties: They depend on M, and i and j only; if $i=j$, there occurs the pair $c_{i,j}=0$, $d_{i,j}=1$; if $i \ne j$ all $c_{i,j} \ne 0$; finally ~~with~~ if the pair $c_{i,j}$, $d_{i,j}$ corresponds to M, the pair $c_{i,j}$, $d_{i,j}+mc_{i,j}$ corresponds to MS_j^m, and these since $\chi(S_j)=1$ have the same value for χ.

Therefore we can write for $1 \le j \le \varkappa$,

$$(7.4)\quad E_i(z,s,\chi) = \delta_{i,j}\, y_j^{\,s} +$$

$$+ \sum_{\substack{c_{i,j}\neq 0 \\ 0\le d_{i,j}<|c_{i,j}|}} \overline{\chi}(M) \sum_{n=-\infty}^{\infty} \frac{y_j^{\,s}}{|c_{i,j}(z+n)+d_{i,j}|^{2s}},$$

where $\delta_{i,j}=1$ if $i=j$, and $\delta_{i,j}=0$ otherwise.
Writing now

$$(7.5)\quad E_i(z,s,\chi) = \sum_{n=-\infty}^{\infty} \alpha_n^{(i,j)}(y_j,s,\chi)\, e^{2\pi i n x_j},$$

we have

$$(7.6)\quad \alpha_n^{(i,j)}(y_j,s,\chi) = \int_0^1 E_i(z,s,\chi)\, e^{-2\pi i n x_j}\, dx_j.$$

Taking first $n=0$, we obtain using (7.4), easily

$$\alpha_0^{(i,j)}(y_j,s,\chi) = \delta_{i,j}\, y_j^{\,s} +$$

$$+ \sum_{\substack{c_{i,j}\neq 0 \\ 0\le d_{i,j}<|c_{i,j}|}} \frac{\overline{\chi}(M)}{|c_{i,j}|^{2s}} \int_{-\infty}^{\infty} \frac{y_j^{\,s}}{(x_j^2+y_j^2)^s}\, dx_j,$$

Here

$$\int_{-\infty}^{\infty} \frac{y_j^{\,s}}{(x_j^2+y_j^2)^s}\, dx_j = y_j^{\,1-s}\int_{-\infty}^{\infty} \frac{du}{(u^2+1)^s} = \sqrt{\pi}\,\frac{\Gamma(s-\frac12)}{\Gamma(s)}\, y_j^{\,1-s},$$

so that

(7.7) $\alpha_0^{(i,j)}(y_j, s, x) = \delta_{i,j}\, y_j^{s} + \varphi_{i,j}(s,x)\, y_j^{1-s},$

where we have put

(7.8) $\varphi_{i,j}(s,x) = \sqrt{\pi}\,\dfrac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, L_0^{(i,j)}(s,x),$

and for integral m

(7.9) $$L_m^{(i,j)}(s,x) = \sum_{c_{i,j}\neq 0}\ \sum_{0\le d_{i,j}<|c_{i,j}|} \frac{\overline{\chi(M)}}{|c_{i,j}|^{2s}}\, e^{2\pi m i \frac{d_{i,j}}{c_{i,j}}}.$$

Next we take $m \neq 0$ in (7.6), then we obtain from (7.4) easily

$$\alpha_m^{(i,j)}(y_j, s, x) = \cancel{\sum_{c_{i,j}\neq 0}}\ L_m^{(i,j)}(s,x)\int_{-\infty}^{\infty} \frac{y_j^{s}\, e^{-2\pi n i x_j}}{(x_j^2+y_j^2)^s}\, dx_j.$$

Here

$$\int_{-\infty}^{\infty} \frac{y^s e^{-2\pi m i x}}{(x^2+y^2)^s}\, dx = \frac{1}{\Gamma(s)}\int_0^{\infty} dt \int_{-\infty}^{\infty} t^{s-1} e^{-t(y+\frac{x^2}{y})-2\pi m i x}\, dx$$

$$= \frac{\sqrt{\pi y}}{\Gamma(s)}\int_0^{\infty} t^{s-\frac{1}{2}}\, e^{-ty-\frac{\pi^2 m^2}{t}y}\,\frac{dt}{t} =$$

$$= \frac{\pi^s |m|^{s-\frac{1}{2}}\sqrt{y}}{\Gamma(s)}\int_0^{\infty} t^{s-\frac{1}{2}}\, e^{-\pi|m|y(t+\frac{1}{t})}\,\frac{dt}{t}.$$

Thus we get for $m\neq 0$,

(7.10) $\alpha_m^{(i,j)}(y_j, s, \chi) = \varphi_{i,j}^{(m)}(s,\chi)\sqrt{y_j}\int_0^\infty t^{s-\frac{1}{2}} e^{-\pi|m|y_j(t+\frac{1}{t})}\frac{dt}{t}$,

where we have put

(7.11) $\varphi_{i,j}^{(m)}(s,\chi) = \dfrac{\pi^{s}|m|^{s-\frac{1}{2}}}{\Gamma(s)} L_m^{(i,j)}(s,\chi)$.

The expansion (7.5) is thus determined.

If finally $\kappa_1 < j \le \kappa$, we put $\chi(S_j) = e^{2\pi i\alpha_j} \neq 1$, with $0 < |\alpha_j| \le \frac{1}{2}$. Then there is an expansion

(7.12) $E_i(z,s,\chi) = \sum_{m=-\infty}^{\infty} \alpha_m^{(i,j)}(y_j,s,\chi)\, e^{2\pi i(m+\alpha_j)x_j}$.

We find that here for all m

(7.13) $\alpha_m^{(i,j)}(y_j,s,\chi) = \varphi_{i,j}^{(m+\alpha_j)}(s,\chi)\sqrt{y_j}\int_0^\infty t^{s-\frac{1}{2}} e^{-\pi|m+\alpha_j|y_j(t+\frac{1}{t})}\frac{dt}{t}$,

where

(7.14) $\varphi_{i,j}^{(m+\alpha_j)}(s,\chi) = \dfrac{\pi^{s}|m+\alpha_j|^{s-\frac{1}{2}}}{\Gamma(s)} L_{m+\alpha_j}^{(i,j)}(s,\chi)$,

and

(7.15) $L_{m+\alpha_j}^{(i,j)}(s,\chi) = \sum_{c_{i,j}\neq 0}\ \sum_{0\le d_{i,j}<|c_{i,j}|} \dfrac{\overline{\chi}(M)\, e^{2\pi i(m+\alpha_j)\frac{d_{i,j}}{c_{i,j}}}}{|c_{i,j}|^{2s}}$.

From these fourier expansions, valid for $\sigma > 1$, the behaviour of the function $E_i(z,s,\chi)$ is easily determined. From the form of the terms (7.10) and (7.13), we see easily that the contribution in the fourier expansion of all these goes exponentially to zero (that is it is $O(e^{-\beta y_j})$ with some $\beta > 0$), as $y_j \to \infty$, or what is the same z approaches the cusp ξ_j within $\mathcal{D}$. We therefore have

Theorem 7.1. If z approaches ξ_j within $\mathcal{D}$, where $1 \le j \le \varkappa_1$, we have

$$E_i(z, s, \chi) = \delta_{i,j}\, y_j^{s} + \varphi_{i,j}(s,\chi)\, y_j^{1-s} + O(e^{-\beta y_j}),$$

and if $\varkappa_1 < j \le \varkappa$, we have

$$E_i(z, s, \chi) = O(e^{-\beta y_j}).$$

Actually in the first case β can be taken as any constant $< 2\pi$, and in the second as any constant $< |\alpha_j|$.

Theorem 7.2. For $1 \le i, j \le \varkappa_1$, we have

$$\varphi_{i,j}(s,\chi) = \varphi_{j,i}(s,\overline{\chi}).$$

This ~~could have been~~ can be seen directly in the following way from the dirichlet series $L_0^{i,j}(s,\chi)$ and $L_0^{j,i}(s,\chi)$. Denoting by T_i the transformation such that $z_i = T_i z$, we have that

$$z_{i,M} = (Mz)_i = T_i M T_j^{-1} z_j ,$$

and similarly,

$$z_{j,M^{-1}} = T_j M^{-1} T_i^{-1} z_i ,$$

so that in

$$y_{i,M} = \frac{y_j}{|c_{i,j} z_j + d_{i,j}|^2} , \quad y_{j,M^{-1}} = \frac{y_i}{|c'_{j,i} z_i + d'_{j,i}|^2}$$

we have $c_{i,j} = c'_{j,i}$, since also $\chi(M^{-1}) = \overline{\chi}(M)$, we obtain that $L_0^{i,j}(s,\chi) = L_0^{j,i}(s,\overline{\chi})$. Another way of proving the theorem is to use Greens theorem on the expression

$$0 = \iint_{\overline{D}} \left(E_i(z,s,\chi) \Delta E_j(z,s,\overline{\chi}) - E_j(z,s,\overline{\chi}) \Delta E_i(z,s,\chi) \right) dx\,dy$$

where $\overline{D}$ denotes what remains of ~~the~~ D when we remove the parts where $y_k > A_k$ for $1 \le k \le \varkappa$, where the A_k are large positive numbers that we later make tend to infinity independently of each other.

Lemma 7.1. A function $f(z)$, which for some s with $\sigma > 1$, satisfies the equation

(1) $\quad y^2 \Delta f + s(1-s) f = 0,$

and the relations

(2) $f(Mz) = \chi(M) f(z)$

for all M in Γ, and which furthermore in $\mathcal{D}$ as z tends towards a cusp ξ_j, $1 \le j \le \varkappa$ satisfies the condition

$$(3) \qquad f(z) = O(e^{\varepsilon y_j}),$$

for every fixed positive ε, is necessarily a linear combination of the functions $E_i(z, s, \chi)$ for $1 \le i \le \varkappa_1$.

To prove this we look at the fourier expansions of $f(z)$ in terms of x_j, since each term has to be annihilated by the operator $y^2\Delta + s(1-s)$ and also satisfy the condition (3) as $y_j \to \infty$, one easily shows that for $1 \le j \le \varkappa_1$, the fourier expansion has to be of the form

(7.16)

$$f(z) = a_j y_j^s + b_j y_j^{1-s} + \sum_{m=-\infty}^{\infty}{}' c_m^{(j)} e^{2\pi i m x_j} \sqrt{y_j} \int_0^\infty t^{s-\frac{1}{2}} e^{-\pi |m| y_j (t+\frac{1}{t})} \frac{dt}{t};$$

and if $\varkappa_1 < j \le \varkappa$ of the form

(7.16')

$$f(z) = \sum_{m=-\infty}^{\infty} c_m^{(j)} e^{2\pi i (m+\alpha_j) x_j} \sqrt{y_j} \int_0^\infty t^{s-\frac{1}{2}} e^{-\pi |m+\alpha_j| y_j (t+\frac{1}{t})} \frac{dt}{t}.$$

This holds even without the assumption $\sigma > 1$.*)

Assuming now $\sigma > 1$, we form

$$\tilde{f}(z) = f(z) - \sum_{j=1}^{\varkappa_1} a_j E_j(z, s, \chi),$$

which also satisfies (1) and (2). But from the fourier expansions of f and the E_j it follows that $\tilde{f}(z)$ is uniformly bounded in $\mathcal{D}$, so that

$$\iint_{\mathcal{D}} |\tilde{f}|^2 \frac{dx\,dy}{y^2} < \infty .$$

Since $y^2\Delta$ is an elliptic operator however, a square integrable function satisfying (1) and (2) and not vanishing identically, cannot exist unless $s(1-s)$ is real and non-negative. Because the latter is never the case for $\sigma > 1$, $\tilde{f}$ has to vanish identically, which proves our lemma.

For use in a later chapter we note that since each term in the expansions (7.16) and (7.16') is an eigenfunction of the operator $\iint_H k(z, z') f(z') \frac{dx'\,dy'}{y'^2}$ with the eigenvalue

*) If $s = \frac{1}{2}$, the leading terms $a_j y_j^{s} + b_j y_j^{1-s}$ would be replaced by $(a_j \log y + b_j)\sqrt{y_j}$ in (7.16).

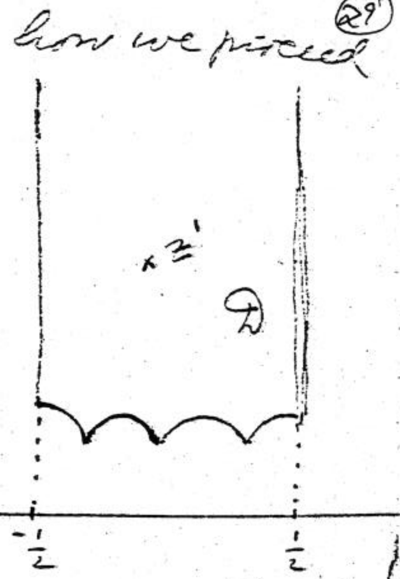
equal to $h\left(\frac{s-\frac{1}{2}}{i}\right)$ if $0 \leq \sigma \leq 1$, the same holds for the function $f(z)$. Therefore we have

Lemma 7.2. If $f(z)$ satisfies the conditions (1), (2), (3) of Lemma 7.1 for some s, we have

$$\iint_{\mathfrak{D}} K(z, z', \chi) f(z') \frac{dx'dy'}{y'^2} = h\left(\frac{s-\frac{1}{2}}{i}\right) f(z),$$

provided ~~the s is equal in the strip~~ $0 \leq \sigma \leq 1$.

Up to now we have only studied the functions $E_i(z, s, \chi)$ for $\sigma > 1$, where the series by which they were defined converge. For our later purposes it is necessary to show that they can be analytically continued in the whole s-plane, except for poles and so actually are meromorphic functions of s. ~~In order~~ For reasons of simplicity we shall give this proof only for the case that χ is identically $=1$, and that $\varkappa = 1$, so that we have only one cusp, which we suppose to be placed at infinity and furthermore we suppose that the primitive parabolic transformation S leaving ∞ fixed is $Sz = z+1$. We shall afterwards briefly indicate how we proceed in the general case.



Let z' be a point in D, which we at first for simplicity assume to be in the interior. By means of the Dirichlet principle one can then show the existence of a function $G(z, z')$ which is harmonic in z throughout D, except for a logarithmic singularity at $z = z'$, such that

(7.17) $\quad G(z, z') - \log \dfrac{1}{|z - z'|}$ = regular at $z = z'$;

and with another logarithmic singularity (in terms of the local uniformizing variable $w = e^{2\pi i z}$), at $z = i\infty$ (or $w = 0$), such that

(7.18) $\quad G(z, z') + 2\pi y$ = regular in w at $w = 0$.

Furthermore $G(z, z')$ and its normal derivative agrees in corresponding points of corresponding pairs of sides of D (or what is the same $G(z, z')$ is continuous with its first derivatives on the closed surface we get by identifying corresponding sides of the boundary). $G(z, z')$ will actually be the real part of an abelian integral of third kind with logarithmic singularities at z' and $i\infty$ and purely imaginary periods.

Obviously $G(z, z')$ can be continued throughout the interior of the upperhalfplane, except at the points Mz' where $M \subset \Gamma$, where we again have logarithmic singularities, also we have

$$(7.19) \qquad G(Mz; z') = G(z, z')$$

for M in Γ. Since any function of z' only, may be added to $G(z, z')$ without changing any of the abovementioned properties, we norm $G(z, z')$ so as to make it unique, by the requirement

$$(7.20) \qquad G(z, z') + 2\pi y \to 0, \text{ as } z \to i\infty \,(\text{or } w \to 0).$$

Now let z' and z'' be two distinct points in $\mathcal{D}$, and consider the domain $\mathcal{D}^*$ left by ~~omitting~~ removing the part where $y > A$, for some large positive A, and two small circular discs around the points z' and z'' from $\mathcal{D}$. Consider the integral

$$\iint_{\mathcal{D}^*} \left(G(z, z') \Delta G(z, z'') - G(z, z'') \Delta G(z, z') \right) dx\,dy = 0,$$

applying Greens formula to the lefthand side, we find that the integrals over the boundary cancel out, except for the integrals taken along the two small circles and over the line segment $-\frac{1}{2} \le x \le \frac{1}{2}$, $y = A$. From (7.20) we see making $A \to \infty$ that the contribution from the line segment also vanishes. Making finally the radii of the two circles tend to zero, and using (7.17), we finally obtain

$$G(z'', z') = G(z', z'')$$

or writing z for z''

(7.21) $$G(z, z') = G(z', z).$$

We can easily remove the restriction that z' lie in the interior of $\mathcal{D}$, the only change being that if z' is at a fixpoint of a primitive elliptic transformation of order m, (7.17) is modified to

(7.22) $$G(z, z') - m \log \frac{1}{|z-z'|} = \text{regular at } z = z';$$

all the above properties then continue to hold.

Defined in this way $G(z, z')$ is now, because of the symmetry (7.21) harmonic in both variables in the interior of the upper half-plane, except when z and z' are equivalent under Γ, and we have also

(7.23) $$G(Mz, M'z') = G(z, z'),$$

if M and M' belong to Γ.

We shall now obtain certain estimates for $G(z,z')$ as the points z and z' range over $\mathcal{D}$. Of most interest is the behavior in the neighborhood of the cusp (that is if one or both points approach the cusp). Let A be a positive constant so large that the line $y=A$ cuts the boundary of $\mathcal{D}$ only in the two sides passing through the cusp.

Suppose first that z and z' both lie in $\mathcal{D}$ and below the line $y=A$, so that $y \le A$, $y' \le A$; denoting by $M_1, M_2, \ldots, M_\nu$ the transformations of Γ for which $M_i\mathcal{D}$ $(i=1,2\ldots\nu)$ has at least one boundary point (not counting the cusp as a boundary point) in common with $\mathcal{D}$. Then we obviously have

$$(7.24)\quad G(z,z') = \log\frac{1}{|z-z'|} + \sum_{i=1}^{\nu} \log\frac{1}{|z-M_i z'|} + g_1(z,z')$$

where $g_1(z,z')$ is regular and uniformly bounded for z and z' in $\mathcal{D}$ and $y \le A$, $y' \le A$.

Next suppose that at least one point lies above the line $y=A$. Then we see from (7.17) (7.18), (7.20), (7.21) and the fact that $G(z,z')$ is periodic in both z and z' with period 1, that

$$(7.25)\quad G(z,z') = \log\frac{1}{|e^{-2\pi i z} - e^{-2\pi i z'}|} + g_2(z,z'),$$

where $g_2(z,z')$ is a symmetric regular harmonic function of $w = e^{2\pi i z}$ and $w' = e^{2\pi i z'}$, as long as z and z' lie in $\mathcal{D}$ (the cusp included) and at least one of the numbers $|w|$ and $|w'|$ is $\le e^{-2\pi A}$. Also $g_2(z, z') = 0$ if w or w' equals zero. Therefore

(7.26) $$g_2(z, z') = O(|w\, w'|) = O(e^{-2\pi(y+y')}),$$

under our assumptions about z and z'. For $y > y'$, $y \geq A$, (7.25) and (7.26) easily give

(7-27) $$G(z, z') = -2\pi y + O\left(e^{-2\pi|y-y'|}\log\left(2 + \frac{1}{|y-y'|}\right)\right),$$

and because of the symmetry if $y' > y$, $y' \geq A$, we have

(7-27') $$G(z, z') = -2\pi y' + O\left(e^{-2\pi|y-y'|}\log\left(2 + \frac{1}{|y-y'|}\right)\right).$$

We now define for z' in $\mathcal{D}$,

(7-28) $$\tilde{G}(z, z') = \begin{cases} G(z, z') + 2\pi y' & \text{for } y' \geq A, \\ G(z, z') + 2\pi A & \text{for } y' \leq A. \end{cases}$$

Suppose that $v(z)$ is continuous and invariant under Γ ($v(Mz) = v(z)$) in the interior of the upper half-plane, and has continuous partial derivatives up to second order there, and that these functions satisfy the condition that they