

are $O\left(\frac{e^{2\pi y}}{y^2}\right)$ as $y \rightarrow \infty$. Denote by D_p the domain obtained from D by removing a small circular disc C_p with radius p and center z , and consider the integral

$$(7.29) \quad \iint_{D_p} \tilde{G}(z, z') y'^2 \Delta v(z') \frac{dx' dy'}{y'^2} =$$

$$\lim_{p \rightarrow 0} \iint_{D_p} (\tilde{G}(z, z') \Delta v(z') - v(z') \Delta \tilde{G}(z, z')) \frac{dx' dy'}{y'^2},$$

applying here Green's formula we get, taking into account the fact that the integrals over the boundary of D cancel out, and remembering the discontinuity of the normal derivative of $\tilde{G}(z, z')$ along the line $y' = A$ (according to (7.28)), we get that the latter expression equals^{*}

$$= \lim_{p \rightarrow 0} \int_{C_p} \left(\tilde{G}(z, z') \frac{\partial v(z')}{\partial n} - v(z') \frac{\partial \tilde{G}(z, z')}{\partial n} \right) ds$$

$$= 2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} v(x' + iA) dx' = 2\pi v(z') - 2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} v(x' + iA) dx',$$

where we have used (7.17) when taking the limit as $p \rightarrow 0$.

^{*} We have here for simplicity supposed that the line $y' = A$ does not cross the circle C_p .

Thus the operator

(35)

$$(7.30) \quad \frac{1}{2\pi} \iint_D \tilde{G}(z, z') u(z') \frac{dx' dy'}{y'^2}$$

inverts the operator $y^2 \Delta$, conversely we can show by standard methods used in the theory of partial differential equations, that if $u(z')$ is continuous and invariant under P and satisfies the condition $u(z') = O(e^{2\pi y'})$ as $y' \rightarrow \infty$ then the function

$$v(z) = \frac{1}{2\pi} \iint_D \tilde{G}(z, z') u(z') \frac{dx' dy'}{y'^2},$$

which obviously also is invariant under P (since $\tilde{G}(z, z')$ is a function of z), satisfies the relations

$$y^2 \Delta v(z) = u(z) \quad \text{and} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} v(x + iA) dx = 0.$$

We now consider the integral equation of the third kind

$$(7.31) \quad u(z, \lambda) = 1 + \frac{\lambda}{2\pi} \iint_D \tilde{G}(z, z') u(z', \lambda) \frac{dx' dy'}{y'^2},$$

writing it for some small ε with $0 < \varepsilon < \frac{1}{2\pi}$ in the form

$$(7.31') \quad e^{-\varepsilon y} u(z, \lambda) = e^{-\varepsilon y} + \frac{\lambda}{2\pi} \iint_D e^{\varepsilon(y' - y)} \tilde{G}(z, z') e^{-\varepsilon y'} u(z', \lambda) \frac{dx' dy'}{y'^2}$$

(36)

we ^{get} from (7.28), ~~and~~ ^{and} (7.27') that
for $y' \geq A$, $y' > y$, we have

$$(7.32) \quad e^{\varepsilon(y'-y)} \tilde{G}(z, z') = O\left(e^{-\varepsilon|y-y'|} \log\left(2 + \frac{1}{|y-y'|}\right)\right),$$

and for $y \geq A$, $y > y'$, that

$$(7.32') \quad e^{\varepsilon(y'-y)} \tilde{G}(z, z') = O\left(e^{-\varepsilon|y-y'|} \log|y-y'| + \log\left(2 + \frac{1}{|y-y'|}\right)\right).$$

Finally for $y < A$, $y' < A$, we have from (7.28) and (7.24), that

$$(7.32'') \quad e^{\varepsilon(y'-y)} \tilde{G}(z, z') = O\left(1 + |\log|z-z'|| + \sum_{i=1}^N |\log|z-h_i, z'||\right).$$

From these estimations we see that our kernel in (7.31') is square integrable in the sense that

$$\iint_D \iint_D |e^{\varepsilon(y'-y)} \tilde{G}(z, z')|^2 \frac{dx dy}{y^2} \frac{dx' dy'}{y'^2} < \infty,$$

also the function $e^{-\varepsilon y}$ is square integrable,

$$\iint_D |e^{-\varepsilon y}|^2 \frac{dx dy}{y^2} < \infty. \text{ Therefore the Fredholm}$$

solution of our equation (7.31') or (7.31)

converges, and therefore we have a

solution $u(z, \lambda)$ of (7.31) which is a

meromorphic function of λ , and actually can be written as $\frac{D(z, \lambda)}{D(\lambda)}$, where $D(z, \lambda)$ and $D(\lambda)$ are respectively the numerator and the denominator of the Fredholm solution, and are integral functions of λ of order 2 at most. Furthermore $u(z, \lambda)$ is invariant under P . We also deduce ~~that~~, since the function $e^{-\varepsilon y} u(z, \lambda)$ is square integrable over D for any $\varepsilon > 0$, that because of (7.31) we have even that $u(z, \lambda) = O(e^{\varepsilon y})$ for all positive ε . Finally since the operator (7.30) inverts the operator $y^2 \Delta$, we get from (7.31) that

$$7.34) \quad y^2 \Delta u(z, \lambda) - \lambda u(z, \lambda) = 0.$$

Now put $\lambda = -s(1-s)$, where s is a complex variable and suppose that $-s(1-s)$ is not a zero of $D(\lambda)$, and that $\Re s = \sigma > 1$. Lemma 7.1 gives then, because $u(z, -s(1-s))$ satisfies all conditions, that

$$7.35) \quad u(z, -s(1-s)) = a(s) E(z, s),$$

we write here $E(z, s)$ since χ is identically 1 and there is only one cusp, and $a(s)$ is some function of s , $a(s)$ can be determined explicitly from the fact that

$$\int_{-1}^1 u(x+iA, -s(1-s)) dx = 1,$$

using the Fourier expansion for $E(z, s)$ this gives

$$a(s) (A^s + \varphi(s) A^{1-s}) = 1,$$

But (where we again omit x and indices in $\varphi; (s, x)$, since x is identically 1 and there is only one cusp), but this explicit expression is not necessary. It is enough that we observe that in the Fourier expansion of $u(z, -s(1-s))$ the "constant" term (that is the term independent of x), for each y is a meromorphic function of s of order at most 4 (we shall say that a meromorphic function is of order $\leq \rho$, if it can be written as a quotient of two integral functions each of which is of order $\leq \rho$), in terms of s . This "constant" term is

$$a(s) (y^s + \varphi(s) y^{1-s}),$$

therefore by taking two values for y and eliminating $\varphi(s)$, we get that $a(s)$ itself is a meromorphic function of s of order ≤ 4 .

From (7.35) we then obtain that $E(z, s)$ itself can be continued in the whole s -plane except for poles, and that $E(z, s)$ can be written as a quotient, where the denominator is independent of z , and is an integral function of s of order ≤ 4 , while the numerator

is a function of z and s , which in s is an integral analytic function also of order ≤ 4 , and in z is invariant under Γ , and also it is annihilated by the operator $y^2 \Delta + s(s-1)$.

Before we continue let us briefly indicate how one proceeds in the case when there are more than one cusp, and when $\chi(\Gamma)$ is not necessarily identically 1 but still singular with respect to one or more cusp.

Take first the case that $\chi(\Gamma)$ is identically 1. We can again show the existence of a $G(z, z')$ which is harmonic in z and in D has a logarithmic singularity at z' and another at say the cusp at ξ_1 , and which is symmetric in both variables, and invariant under Γ in both. Letting z' tend to ξ_j for $2 \leq j \leq n$, we obtain a function $g_j(z)$, and put further $g_1(z) = 1$. We can now form an expression

$$\tilde{G}(z, z') = G(z, z') + \sum_{j=1}^n \omega_j(z') g_j(z)$$

where the functions $\omega_j(z')$ for $1 \leq j \leq n$, are automorphic invariant under Γ , continuous and with continuous partial derivatives up to second order, except for a finite number of curves in the compact part of D , and

furthermore chosen so that $\tilde{G}(z, z')$ if z' tends to the cusp ξ_j (or $y_j' \rightarrow \infty$) vanishes sufficiently strongly.

Then we consider the equations for $1 \leq j \leq n$

$$u_j(z, \lambda) = g_j(z) + \frac{\lambda}{2\pi} \iint_D \tilde{G}(z, z') u_j(z', \lambda) \frac{dx' dy'}{y'^2},$$

and observe that the solutions $u_j(z, \lambda)$ are linearly independent (since the $g_j(z)$ clearly are so), and use Lemma 7.1, in a similar way as before, and we obtain finally that the functions $E_i(z, \lambda)$ for $1 \leq i \leq n$, have the same properties that we above established for $E(z, \lambda)$.

Finally, if $\chi(M) \neq 1$ for some M , we assume χ to be singular with respect to the n cusps ξ_1, \dots, ξ_n , and nonsingular with respect to the eventual others. In this case we can show the existence of a harmonic function ^{of z} $G(z, z', \chi)$ which has a logarithmic singularity at z' (but in this case we need not put one at a cusp), for which

$$G(Mz, z', \chi) = \chi(M) G(z, z', \chi)$$

and

$$G(z, z', \chi) = \overline{G(z', z, \chi)}$$

Letting z' go into the cusp ξ_i for $1 \leq i \leq \kappa_1$, we obtain a harmonic function $g_i(z, \lambda)$ for $1 \leq i \leq \kappa_1$. (For $j > \kappa_1$, $G(z, z', \lambda)$ can be seen to tend to zero as $z' \rightarrow \xi_j$, so that we get no functions $g_j(z)$ from these cusps).

We then as in the previous case construct an expression

$$\tilde{G}(z, z', \lambda) = G(z, z', \lambda) + \sum_{i=1}^{\kappa_1} \omega_i(z') g_i(z, \lambda)$$

with suitably chosen $\omega_i(z')$, and consider the equations

$$\mu_i(z, \lambda, \lambda) = g_i(z, \lambda) + \frac{\lambda}{2\pi} \iint_{\mathcal{D}} \tilde{G}(z, z', \lambda) \mu_i(z', \lambda, \lambda) \frac{dx' dy'}{y'^2}$$

for $1 \leq i \leq \kappa_1$, and obtain again the same conclusions about the $E_i(z, s, \lambda)$ for $1 \leq i \leq \kappa_1$, as above for $E(z, s)$.

Returning now to the general case, we find that if we take for s a value such that $\rho \neq 0$, and ρ is not a pole for the functions $E_i(z, s, \lambda)$ ^{any of} $\sigma > 0$, and $1-s$ is not a pole for the functions E_i $1 \leq i \leq \kappa_1$, then the

functions $E_i(z, 1-s, \chi)$, $1 \leq i \leq \kappa$, satisfy all conditions of Lemma 7.1. Therefore they can be expressed linearly in terms of the functions $E_i(z, s, \chi)$. The connection is easily found by looking at the terms independent of x_j (the "constant" term) in the Fourier expansions of the functions $\bar{E}_i(z, s, \chi)$ and $E_i(z, 1-s, \chi)$ for $1 \leq i \leq \kappa$, ~~and~~ for in terms of x_j for $1 \leq j \leq \kappa$. These terms are given by (7.7). Writing $E(z, s, \chi)$ for the column vector with χ components $E_1(z, s, \chi), E_2(z, s, \chi), \dots, E_\kappa(z, s, \chi)$, and $\phi(s, \chi)$ for the matrix $(\phi_{ij}(s, \chi))$ defined by (7.8) and (7.9), we find

$$(7.36) \quad E(z, 1-s, \chi) = \phi(1-s, \chi) E(z, s, \chi).$$

Changing s into $1-s$ and comparing, we get

$$(7.37) \quad \phi(s, \chi) \phi(1-s, \chi) = E,$$

where E is the κ, κ identity matrix.

Theorem 7.2 implies that

(7.38)

$$\Phi(\rho, \bar{x}) = \Phi'(\rho, x),$$

where Φ' denotes the transpose of Φ . For $\rho = \frac{1}{2} + i\alpha$, α real, we also have

$$\Phi\left(\frac{1}{2} - i\alpha, \bar{x}\right) = \overline{\Phi\left(\frac{1}{2} + i\alpha, x\right)},$$

since

$$E(z, \rho, \bar{x}) = \overline{E(z, \rho, x)},$$

from the definition of the functions E_i , combining this with (7.38) we get

$$\Phi\left(\frac{1}{2} - i\alpha, \bar{x}\right) = \overline{\Phi'\left(\frac{1}{2} + i\alpha, x\right)}.$$

Thus from (7.37) we obtain that

$$\Phi\left(\frac{1}{2} + i\alpha, x\right)$$

is a unitary matrix for real α . It is easily also shown that for ρ real, $\Phi(\rho, x)$ is a Hermitian matrix.

Before we formulate our results as a theorem, we shall obtain some information about the locations of the poles of the $E_i(z, \rho, x)$, or what is the same the vector $E(z, \rho, x)$. For simplicity we again consider the case with only one cusp of \mathbb{D} , placed at $i\infty$, and such

that the primitive parabolic transformation that leaves the cusp fixed is $Sz = z+1$. Furthermore we take X to be identically 1, so we omit it from our formulas.

It is clear that $E(z, s)$ can not have a pole in the halfplane $\Re s = \sigma > \frac{1}{2}$, unless it be located on the segment $1 \leq s < \frac{1}{2}$ of the real line. Suppose that there was such a pole s_0 of order $\nu > 0$, with $\Re s_0 > \frac{1}{2}$ and s_0 not in the above mentioned segment.

Then $\lim_{s \rightarrow s_0} (s - s_0)^\nu E(z, s) = u(z)$, would

be a function with the following properties:

- (1) $u(z)$ is invariant under Γ , (2) $u(z)$ has a Fourier expansion of the form

$$u(z) = \alpha_0 y^{1-s_0} + \sum_{n=-\infty}^{\infty} \alpha_n e^{2\pi i n x / y} \int_0^{\infty} t^{s_0 - \frac{1}{2} - \pi i n y (t + \frac{1}{t})} \frac{dt}{t},$$

- (3) $u(z)$ satisfies the equation

$$y^2 \Delta u(z) + s_0(1-s_0) u(z) = 0.$$

- (4) $u(z)$ does not vanish identically.

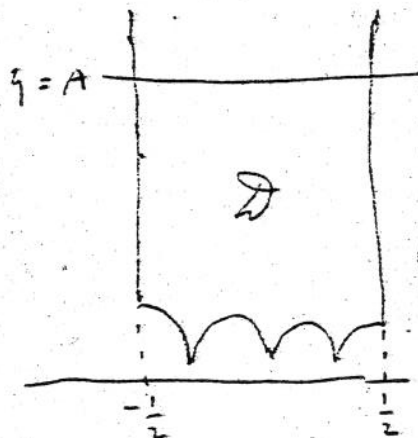
However since $\Re(1-s_0) < \frac{1}{2}$, the Fourier series shows that $u(z)$ is square integrable over the fundamental domain, therefore since $y^2 \Delta$ is an elliptic operator, ^{and} since $u(z)$ does not vanish identically, we would have that $s_0(1-s_0)$ would have to be real and

non-negative, which contradicts the assumption made about ρ .

To obtain more information, we shall first develop a formula involving two functions $E(z, \Delta)$ and $E(z, \Delta')$. Choosing again the positive number A so large that D contains

the domain $|x| \leq \frac{1}{2}$, $y \geq A$, and defining for z in D

$$(7.39) \quad \tilde{E}(z, \Delta) = \begin{cases} E(z, \Delta) & \text{for } y < A, \\ E(z, \Delta) - y^\Delta - \phi(\Delta) y^{1-\Delta} & \text{for } y \geq A; \end{cases}$$



we have

$$(7.40) \quad (\Delta' - \Delta)(\Delta + \Delta' - 1) \iint_D \tilde{E}(z, \Delta) \tilde{E}(z, \Delta') \frac{dx dy}{y^2} \\ = \iint_D (\tilde{E}(z, \Delta) \Delta \tilde{E}(z, \Delta') - \tilde{E}(z, \Delta') \Delta \tilde{E}(z, \Delta)) dx dy.$$

Applying here Green's formula, taking into account the discontinuity along the line $y = A$, and observing that the integrals over the boundary of D cancel out, and that the contribution coming from the line segment $y = A$, $|x| \leq \frac{1}{2}$ becomes

$$\begin{aligned}
 & (y^s + \varphi(s)y^{1-s}) \frac{d}{dy} (y^{s'} + \varphi(s')y^{1-s'}) \\
 & - (y^{s'} + \varphi(s')y^{1-s'}) \frac{d}{dy} (y^s + \varphi(s)y^{1-s}) = \\
 & = (s' - s) (y^{s+s'-1} - \varphi(s)\varphi(s')y^{1-s-s'}) \\
 & + (s+s'-1) (\varphi(s)y^{s'-s} - \varphi(s')y^{s-s'}),
 \end{aligned}$$

with $y = A$, we get from (7.40), for $s \neq s'$, $s+s' \neq 1$.

$$\begin{aligned}
 (7.41) \quad & \iint_D \tilde{E}(z, s) \tilde{E}(z, s') \frac{dx dy}{y^2} = \\
 & = \frac{A^{s+s'-1} - \varphi(s)\varphi(s')A^{1-s-s'}}{s+s'-1} + \\
 & + \frac{\varphi(s)A^{s'-s} - \varphi(s')A^{s-s'}}{s'-s}.
 \end{aligned}$$

Taking first $s = \sigma + i\tau$, $s' = \sigma - i\tau$, σ, τ real and $\sigma \neq \frac{1}{2}$, $\tau \neq 0$, we get

$$\begin{aligned}
 (7.42) \quad & \iint_D |\tilde{E}(z, \sigma + i\tau)|^2 \frac{dx dy}{y^2} = \\
 & = \frac{A^{2\sigma-1} - |\varphi(\sigma + i\tau)|^2 A^{1-2\sigma}}{2\sigma-1} + \frac{\overline{\varphi(\sigma + i\tau)} A^{2i\tau} - \varphi(\sigma + i\tau) A^{-2i\tau}}{2i\tau}.
 \end{aligned}$$

Observing that since $|\varphi(\frac{1}{2} + i\tau)| = 1$, $\varphi(s)$ is necessarily regular at the point $s = \frac{1}{2} + i\tau$

we find that as $\sigma \rightarrow \frac{1}{2}$, the right-hand side of (7.42) tends to

$$2 \log A - \frac{\varphi'}{\varphi}(\frac{1}{2} + i\tau) + \frac{\overline{\varphi(\frac{1}{2} + i\tau)} A^{2i\tau} - \varphi(\frac{1}{2} + i\tau) A^{-2i\tau}}{2i\tau},$$

and in particular remains bounded. From this we can conclude that $E(z, s)$ is regular at $s = \frac{1}{2} + i\tau$, since otherwise, if $E(z, s)$ had a pole there say of order $\nu > 0$, we ^{would} get that

$$\lim_{s \rightarrow \frac{1}{2} + i\tau} (s - \frac{1}{2} + i\tau)^{\nu} E(z, s) = u(z), \text{ would}$$

satisfy

$$\iint_D |u|^2 \frac{dx dy}{y^2} = 0,$$

and since u clearly is continuous, it would vanish identically, which gives a contradiction.

From (7.42) we get for $\sigma = \frac{1}{2}$,

$$(7.42') \quad \iint_D |\tilde{E}(z, \frac{1}{2} + i\tau)|^2 \frac{dx dy}{y^2} = 2 \log A - \frac{\varphi'}{\varphi}(\frac{1}{2} + i\tau) + \frac{\overline{\varphi(\frac{1}{2} + i\tau)} A^{2i\tau} - \varphi(\frac{1}{2} + i\tau) A^{-2i\tau}}{2i\tau}.$$

Next, letting in (7.42) $\sigma > \frac{1}{2}$ fixed and making $\tau \rightarrow 0$, we obtain (assuming that σ is not a pole of $\varphi(s)$, similarly

$$(7.42'') \quad \iint_D |\tilde{E}(z, \sigma)|^2 \frac{dx dy}{y^2} = \frac{A^{2\sigma-1} - |\varphi(\sigma)|^2 A^{1-2\sigma}}{2\sigma-1} + 2\varphi(\sigma) \log A - \varphi'(\sigma).$$

From this we cannot quite exclude the existence of poles on the segment $1 \leq \sigma \leq \frac{1}{2}$, but we see at first that $\tilde{E}(z, \sigma)$ can only have a pole at a point σ_0 if $\varphi(\sigma)$ has a pole of the same order there. Furthermore since the righthand side of (7.42'') ~~by necessity~~ must be positive, $|\varphi(\sigma)|$ cannot grow large as $\sigma \rightarrow \sigma_0$, unless $-\varphi'(\sigma)$ becomes positive and large at least of the same order as $|\varphi(\sigma)|^2$. Therefore if poles occur, they can be poles of first order only, and the residue of $\varphi(\sigma)$ at the pole must be positive (actually in the case we are now dealing with, when $\chi(H)$ is identically 1, $\sigma=1$ is always a pole). If σ_0 is such a pole, then, writing

$$\lim_{\sigma \rightarrow \sigma_0} (\sigma - \sigma_0) \tilde{E}(z, \sigma) = \mu(z),$$

gives $\mu(z)$ is invariant under P , and satisfies the equation

$$y^2 \Delta u + \sigma_0(1-\sigma_0)u = 0,$$

$u(z)$ has a fourier expansion

$$u(z) = c_0 y^{1-\sigma_0} + \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x} \frac{1}{\sqrt{y}} \int_0^{\infty} t^{\sigma_0 - \frac{1}{2} - \pi i n y (t + \frac{1}{t})} \frac{dt}{t},$$

assume $c_0 > 0$, and finally

$$\iint_D |u|^2 \frac{dx dy}{y^2} < \infty, \text{ more precisely one}$$

can show

$$\iint_D |u|^2 \frac{dx dy}{y^2} = c_0.$$

Finally one establishes from (7.42) by letting $\sigma \rightarrow \frac{1}{2}$, $\lambda \rightarrow 0$, that $E(z, \sigma)$ is regular at $\sigma = \frac{1}{2}$. If $\varphi(\frac{1}{2}) = -1$, $E(z, \frac{1}{2}) = 0$ for all z , if $\varphi(\frac{1}{2}) = 1$, $E(z, \frac{1}{2})$ does not vanish identically.

In the general case we can proceed in a similar way. We choose a positive number A so large that D contains the \mathcal{H}_i domains $|x_i| \leq \frac{1}{2}$, $y_i \geq A$, for $1 \leq i \leq \mathcal{H}_1$, and such that no two of these domains have a point in common. We denote the domain $|x_i| \leq \frac{1}{2}$, $y_i \geq A$ by $S_A^{(i)}$ and the part that remains of D after all $S_A^{(i)}$ are removed by D_A . Then we define

for $1 \leq i \leq \kappa_1$,

$$(7.43) \quad \tilde{E}_i(z, \lambda, \chi) = \begin{cases} E_i(z, \lambda, \chi) & \text{for } z \text{ in } \mathcal{D}_A \\ E_i(z, \lambda, \chi) - \delta_{ij} \gamma_j^\lambda - \varphi_{ij}(\lambda, \chi) \gamma_j^{1-\lambda}, & \text{for } z \text{ in } S_A^{(j)}, 1 \leq j \leq \kappa_1. \end{cases}$$

Further we write $\tilde{E}(z, \lambda, \chi)$ for the column vector with the κ_1 components $\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{\kappa_1}$. Then for $\sigma \neq \frac{1}{2}$, $\lambda \neq 0$, we can prove the matrix equation

$$(7.44) \quad \iint_{\mathcal{D}} \tilde{E}(z, \sigma + i\lambda, \chi) \overline{\tilde{E}(z, \sigma + i\lambda, \chi)}' \frac{dx dy}{y^2} =$$

$$= \frac{1}{2\sigma - 1} \left\{ A^{2\sigma - 1} E - \phi(\sigma + i\lambda, \chi) \overline{\phi(\sigma + i\lambda, \chi)}' A^{1 - 2\sigma} \right\}$$

$$+ \frac{\overline{\phi(\sigma + i\lambda, \chi)}' A^{2i\lambda} - \phi(\sigma + i\lambda, \chi) A^{-2i\lambda}}{2i\lambda},$$

where E is the $\kappa_1 \times \kappa_1$ identity matrix and the dash ' means transposition. From (7.44) we can now proceed in a similar way as before from (7.42). Denoting by $\varphi(\lambda, \chi)$ the determinant of the matrix $\phi(\lambda, \chi)$ we can formulate the following

Theorem 7.3. The function vector $E(z, s, \chi)$
is in \mathcal{S} a meromorphic function in the
whole complex plane, which can be written
as a quotient between two integral functions
of order ≤ 4 , and such that the denominator
does not depend on z . Furthermore $E(z, s, \chi)$
satisfies the functional equation

$$E(z, 1-s, \chi) = \phi(1-s, \chi) E(z, s, \chi).$$

$E(z, s, \chi)$ has poles only where $\phi(s, \chi)$ has
poles and vice versa. There are no poles
in the region $\sigma \geq \frac{1}{2}$, except possibly for
a finite number of simple poles on the
interval $\frac{1}{2} < s \leq 1$ of the real line. If
 $\frac{1}{2} < \sigma_0 \leq 1$ is a pole of $E(z, s, \chi)$, then its residue
at σ_0 gives rise to v_{σ_0} linearly independent
functions $\mu_k(z)$, $1 \leq k \leq v_{\sigma_0}$, with the properties
that $\mu_k(Mz) = \chi(M) \mu_k(z)$ for $M \in P$, and
 $(y^2 \Delta + \sigma_0(1-\sigma_0)) \mu_k = 0$, and $\mu_k(z)$ square-integrable
over \mathbb{D} , where $v_{\sigma_0} \leq \kappa$ is the order of the
pole of $\phi(s, \chi)$ at $s = \sigma_0$. These functions
 $\mu_k(z)$ are not cusp forms. (by a cusp form
we shall in the following mean a

function $f(z)$, which satisfies the conditions $f(Mz) = \chi(M)f(z)$ for M in Γ , $f(z)$ square-integrable over \mathcal{D} , and finally that in the \mathcal{H}_1 Fourier expansions after x_i (for $1 \leq i \leq 2l_1$) the "constant" terms (that is the one free from x_i) shall vanish identically). Possibly

§ 8.

Further preliminaries to the proof of the trace formula in the case of a singular χ .

For the next chapter we shall need certain estimations for $E(z, s, \chi)$ in the region $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$, and also certain properties of the function $\varphi(s, \chi) = |\phi(s, \chi)|$.

Lemma 8.1. For $\frac{1}{2} \leq \sigma \leq 1$, we have

$$\left| \int_0^\infty t^{s-\frac{1}{2}} e^{-y(t+\frac{1}{t})} \frac{dt}{t} \right| < 8 \frac{e^{-2y}}{\sqrt{y}}.$$

We have

$$\begin{aligned} \left| \int_0^\infty t^{s-\frac{1}{2}} e^{-y(t+\frac{1}{t})} \frac{dt}{t} \right| &\leq \int_0^\infty t^{\sigma-\frac{1}{2}} e^{-y(t+\frac{1}{t})} \frac{dt}{t} \\ &< 2 \int_1^\infty e^{-y(t+\frac{1}{t})} \frac{dt}{\sqrt{t}} < 2e^{-2y} \int_1^\infty e^{-y(\sqrt{t}-1)^2} \frac{dt}{\sqrt{t}} = \\ &= \frac{4}{\sqrt{y}} e^{-2y} \int_0^\infty e^{-u^2} du < 8 \frac{e^{-2y}}{\sqrt{y}}. \end{aligned}$$

Lemma 8.2. For $\frac{1}{2} \leq \sigma \leq 1$, $\eta > 0$, we have

$$\int_{\eta}^{\infty} \left| \int_0^{\infty} t^{\sigma-\frac{1}{2}+in} e^{-\eta(t+\frac{1}{t})} \frac{dt}{t} \right|^2 \frac{dy}{y} > e^{-6|2|-8\eta-13}.$$

We consider the expression

$$\int_{\eta}^{\infty} (\eta^2 - y^2)^{in} y^{\frac{3}{2}-\sigma-in} \left(\int_0^{\infty} t^{\sigma-\frac{1}{2}+in} e^{-\eta(t+\frac{1}{t})} \frac{dt}{t} \right) dy =$$

$$= \frac{1}{2} \int_0^{\infty} t^{\sigma-\frac{1}{2}+in} e^{-\eta^2 t - \frac{1}{t}} \left(\int_{\eta}^{\infty} (\eta^2 - y^2)^{in} e^{-(\eta^2 - y^2)t} \frac{d(\eta^2 - y^2)}{d(\eta^2 - y^2)} \right) \frac{dt}{t}$$

$$= \frac{1}{2} P(1+in) \int_0^{\infty} t^{\sigma-\frac{3}{2}} e^{-\eta^2 t - \frac{1}{t}} \frac{dt}{t} =$$

$$= \frac{1}{2} \frac{P(1+in)}{\eta^{\sigma-\frac{3}{2}}} \int_0^{\infty} t^{\sigma-\frac{3}{2}} e^{-\eta(t+\frac{1}{t})} \frac{dt}{t}.$$

This gives

$$\int_{\eta}^{\infty} y^{\frac{3}{2}-\sigma} \left| \int_0^{\infty} t^{\sigma-\frac{1}{2}+in} e^{-\eta(t+\frac{1}{t})} \frac{dt}{t} \right| dy$$

$$\geq \frac{1}{2} \frac{|P(1+in)|}{\eta^{\sigma-\frac{3}{2}}} \int_0^{\infty} e^{-\eta(t+\frac{1}{t})} \frac{dt}{t} >$$

>

$$\geq e^{-\frac{\pi}{2}|n|} \eta^{\frac{3}{2}-\sigma} \int_1^{\infty} e^{-(2\eta+1)t} dt$$

$$= \frac{\eta^{\frac{3}{2}-\sigma}}{2\eta+1} e^{-\frac{\pi}{2}|n|-2\eta-1}.$$

Since the left-hand side of this inequality is clearly increasing when η decreases, we easily get from this that

$$\int_{\eta}^{\infty} \eta^{\frac{3}{2}-\sigma} \left| \int_0^{\infty} t^{\sigma-\frac{1}{2}+in} e^{-\eta(t+\frac{1}{t})} \frac{dt}{t} \right| dy >$$

$$> e^{-\frac{\pi}{2}|n|-3\eta-3}$$

Lemma 8.1 gives taking $T = 2|n| + 2\eta + 4$,

$$\int_T^{\infty} \eta^{\frac{3}{2}-\sigma} \left| \int_0^{\infty} t^{\sigma-\frac{1}{2}+in} e^{-\eta(t+\frac{1}{t})} \frac{dt}{t} \right| dy$$

$$< 8 e^{-\frac{3}{2}T} < \frac{1}{2} e^{-\frac{\pi}{2}|n|-3\eta-3},$$

Thus

$$\int_{\eta}^T \eta^{\frac{3}{2}-\sigma} \left| \int_0^{\infty} t^{\sigma-\frac{1}{2}+in} e^{-\eta(t+\frac{1}{t})} \frac{dt}{t} \right| dy >$$

$$> e^{-\frac{\pi}{2}|n|-3\eta-4}.$$

Applying Schwartz' inequality, we obtain from this

$$\int_{\eta}^{\infty} \left| \int_0^{\infty} t^{\sigma-\frac{1}{2}+in} e^{-\gamma(t+\frac{1}{t})} \frac{dt}{t} \right|^2 \frac{d\gamma}{\gamma} >$$

$$> \frac{\left(\int_{\eta}^T \gamma^{\frac{3}{2}-\sigma} \left| \int_0^{\infty} t^{\sigma-\frac{1}{2}+in} e^{-\gamma(t+\frac{1}{t})} \frac{dt}{t} \right| d\gamma \right)^2}{\int_{\eta}^T \gamma^{4-2\sigma} d\gamma}$$

$$> 3T^{-3} e^{-\pi|n| - 6\eta - 8} > e^{-6|n| - 8\eta - 13}.$$

Turning our attention now to the function $\varphi(s, \chi)$, we know that this is a meromorphic function of s of order ≤ 4 , we have that

$$(8.1) \quad \varphi(s, \chi) \varphi(1-s, \chi) = 1,$$

$$(8.2) \quad |\varphi(\frac{1}{2}+in, \chi)| = 1,$$

$$(8.3) \quad \varphi(s, \chi) = \left\{ \sqrt{\pi} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \right\}^{\chi_1} \ell(s, \chi),$$

where $\ell(s, \chi)$ is a dirichlet series

$$(8.4) \quad \ell(s, \chi) = \sum_{n=1}^{\infty} \frac{a_n}{b_n^{2s}}, \quad a_1 \neq 0, b_1 < b_2 < b_3 < \dots,$$

with real coefficients ($\ell(s, \chi)$ is the determinant of the $\mathbb{Q}_p^{\times} \times L_{\bullet}^{i,j}(s, \chi)$ given by (7.9)), and absolutely convergent for $\sigma > 1$. Furthermore we have $\varphi(s, \chi)$ regular for $\sigma \geq \frac{1}{2}$, except

possibly for a finite number of poles, each with multiplicity $\leq \kappa$, in the interval $\frac{1}{2} < \sigma \leq 1$ of the real line. We shall denote these by $\sigma_1, \sigma_2, \dots, \sigma_N$ each occurring with its multiplicity. Taking the trace of the matrix equation (7.44) and observing that the left-hand side is obviously positive, we get

$$\frac{1}{2\sigma-1} \left\{ \kappa_1 A^{2\sigma-1} - A^{1-2\sigma} \sum_{1 \leq i, j \leq \kappa_1} |\varphi_{ij}(\sigma+in, \kappa)|^2 \right\} + \sum_{1 \leq i \leq \kappa_1} \frac{\overline{\varphi_{ii}(\sigma+in, \kappa)} A^{in} - \varphi_{ii}(\sigma+in, \kappa) A^{-2in}}{2in} > 0.$$

For $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$, $|\kappa| \geq 1$, this gives

$$\sum_{1 \leq i, j \leq \kappa_1} |\varphi_{ij}(\sigma+in, \kappa)|^2 < \kappa_1 A^{4\sigma-2} + \frac{2\sigma-1}{|\kappa|} A^{2\sigma-1} \sum_{1 \leq i \leq \kappa_1} |\varphi_{ii}(\sigma+in, \kappa)|.$$

From this we see that the $\varphi_{ij}(\sigma, \kappa)$ are uniformly bounded for $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$, $|\kappa| \geq 1$. Thus also $\varphi(\sigma, \kappa)$ is uniformly bounded in this region, in addition, from (8.3) and (8.4) we see that $\kappa_1^{-2\sigma} \varphi(\sigma, \kappa)$ is uniformly bounded for $\sigma \geq \frac{3}{2}$, and even tends to zero if $|\kappa|$ tends to infinity within this region.

We now form the function

$$(8.5) \quad \varphi^*(s, \chi) = b_1^{2s-1} \prod_{i=1}^N \frac{s - \sigma_i}{s - 1 + \sigma_i} \varphi(s, \chi).$$

$\varphi^*(s, \chi)$ is regular for $\sigma \geq \frac{1}{2}$, is uniformly bounded for $\sigma \geq \frac{1}{2}$, and tends uniformly to zero as $\sigma \rightarrow \infty$. Also φ^* satisfies the same two relations (8.1) and (8.2) as φ .

Therefore we have for $\sigma \geq \frac{1}{2}$,

$$(8.6) \quad |\varphi^*(s, \chi)| \leq 1.$$

Now denote by $\rho = \beta + i\gamma$, $\beta < \frac{1}{2}$, the poles of $\varphi(s, \chi)$ in the halfplane $\sigma < \frac{1}{2}$, these are also the poles of $\varphi^*(s, \chi)$ in this region, and so the points $1 - \rho$ or, what is the same since the points lie symmetric to the real line the points $1 - \bar{\rho} = 1 - \beta + i\gamma$, are the zeros of $\varphi^*(s, \chi)$ in the halfplane $\sigma \geq \frac{1}{2}$.

From a well known results about analytic functions bounded in a halfplane, we conclude that the series

$$(8.7) \quad \sum_{\rho} \frac{\frac{1}{2} - \beta}{|\rho|^2} < \infty,$$

or since the β 's have a finite lower bound (since $\varphi(s, \chi) \neq 0$ for $\sigma > \alpha$, $\alpha > \frac{1}{2}$ sufficiently large), we have

$$(8.7) \sum_p \frac{\frac{1}{2} - \beta}{1 + |\gamma|^2} < \infty.$$

This implies that the product

$$\prod_p \frac{s-1+\bar{p}}{s-p}, \text{ converges absolutely}$$

if we combine the terms with p and \bar{p} for complex p 's. Therefore since φ^* is a meromorphic function of order ≤ 4

$$\varphi^*(s, x) = \pm e^{\alpha_1(s-\frac{1}{2}) + \alpha_2(s-\frac{1}{2})^3} \prod_p \frac{s-1+\bar{p}}{s-p},$$

where the form of the exponential factor is determined by the fact that φ^* satisfies (2.1).

Because of $|\varphi^*(\frac{1}{2} + it, x)| = 1$, α_1 and α_2 have to be real. A simple investigation of the behaviour of $|\varphi^*(s, x)|$ as $s \rightarrow \infty$, using (8.5) and (8.3) on one hand, and the above expression and (8.7) on the other, shows that $\alpha_1 = \alpha_2 = 0$. Going back to $\varphi(s, x)$ we therefore have

$$(8.8) \quad \varphi(s, x) = \pm b_1^{1-2s} \prod_{i=1}^N \frac{s-1+\sigma_i}{s-\sigma_i} \prod_p \frac{s-1+\bar{p}}{s-p}.$$

*) We could here avoid the combination of terms by putting in the factors $e^{i \frac{2\beta-1}{x}}$, for $x \neq 0$.

Since

$$(8.9) \quad -\frac{\varphi^*}{\varphi} \left(\frac{1}{2} + ir, \chi \right) = \sum_p \frac{1-2\beta}{(\beta - \frac{1}{2})^2 + (r-\gamma)^2},$$

the left-hand side is positive. We define

$$(8.10) \quad w(r) = 1 - \frac{\varphi^*}{\varphi} \left(\frac{1}{2} + ir, \chi \right),$$

then $w(r) > 1$, for all r , and further

$$\int_{-R}^R w(r) dr = 2R + \sum_p \int_{-R}^R \frac{1-2\beta}{(\beta - \frac{1}{2})^2 + (r-\gamma)^2} dr \leq$$

$$\leq 2R + \sum_{|\gamma| \leq 2R} \int_{-\infty}^{\infty} \frac{1-2\beta}{(\beta - \frac{1}{2})^2 + (r-\gamma)^2} dr +$$

$$+ 8R \sum_{|\gamma| > 2R} \frac{1-2\beta}{|\gamma|^2} = 2R + 2\pi \sum_{|\gamma| \leq 2R} 1 +$$

$$+ 8R \sum_{|\gamma| > 2R} \frac{1-2\beta}{|\gamma|^2}.$$

Here the last term on the right-hand side is $o(R)$ because of (8.7) and the second term is $O(R^5)$, since φ^* is of order ≤ 4 . Thus

$$(8.11) \quad \int_{-R}^R w(r) dr = O(R^5).$$

Since

$$\sum_{1 \leq i, j \leq \kappa_1} |\varphi_{ij}(\lambda, \chi)|^2$$

is the trace of the matrix $\Phi(\lambda, \chi) \overline{\Phi(\lambda, \chi)}'$, and $|\varphi(\lambda, \chi)|^2$ its determinant, we have

$$\kappa_1 - \sum_{1 \leq i, j \leq \kappa_1} |\varphi_{ij}(\lambda, \chi)|^2 \leq \kappa_1 (1 - |\varphi(\lambda, \chi)|^{\frac{2}{\kappa_1}}).$$

For $\frac{1}{2} \leq \sigma \leq 1$, $|\chi| \geq 1$, we get from this

$$\kappa_1 - \sum_{1 \leq i, j \leq \kappa_1} |\varphi_{ij}(\lambda, \chi)|^2 \leq \kappa_1 (1 - |\varphi^*(\lambda, \chi)|^{\frac{2}{\kappa_1}})$$

$$+ \kappa_1 (|\varphi^*(\lambda, \chi)|^{\frac{2}{\kappa_1}} - |\varphi(\lambda, \chi)|^{\frac{2}{\kappa_1}})$$

$$\leq 1 - |\varphi^*(\lambda, \chi)|^2 + \kappa_1 (1 - |\chi|^{\frac{2-4\sigma}{\kappa_1}}),$$

since $|\varphi^*(\lambda, \chi)| \leq 1$ and by (8.5)

$$|\varphi(\lambda, \chi)| \geq |\chi|^{1-2\sigma} |\varphi^*(\lambda, \chi)| \text{ in this region.}$$

Furthermore we have

$$1 - |\varphi^*(\lambda, \chi)|^2 = 1 - \prod_p \left(1 - \frac{(2\sigma-1)(1-2\beta)}{(\sigma-\beta)^2 + (\kappa-\gamma)^2} \right) \leq$$

$$\leq (2\sigma-1) \sum_p \frac{1-2\beta}{(\sigma-\beta)^2 + (\kappa-\gamma)^2} \leq (2\sigma-1) \sum_p \frac{1-2\beta}{(\beta-\frac{1}{2})^2 + (\kappa-\gamma)^2}$$

$$= -(2\sigma-1) \frac{\varphi^*'}{\varphi^*} \left(\frac{1}{2} + i\kappa, \chi \right).$$

Thus we get for $\frac{1}{2} \leq \sigma \leq 1$, $|n| \geq 1$

$$(8.12) \quad u_1 = \sum_{1 \leq i, j \leq n_1} |\varphi_{ij}(s, \chi)|^2 \leq -(2\sigma-1) \frac{\varphi^*}{\varphi^*}(\frac{1}{2} + in, \chi)$$

$$+ O(\sigma - \frac{1}{2}) = O((\sigma - \frac{1}{2}) \omega(n)),$$

using (8.10).

We now denote for $1 \leq j \leq n$, the domain defined by $|x_j| \leq \frac{1}{2}$, $y_j \geq \eta$ by $S_{\eta}^{(j)}$. Clearly we can choose $\eta > 0$ so small that the x -regions

$S_{\frac{1}{4}\eta}^{(j)}$ cover D completely. The region

$S_{\eta}^{(j)}$ will in general not be contained in D , but it can clearly be completely covered by a finite number of images of D by suitable transformations of P .

Extending the notation \tilde{E} used in the previous chapter outside of D , by the relation

$\tilde{E}(Mz, s, \chi) = \chi(M) \tilde{E}(z, s, \chi)$ for $M \in P$, we get therefore, using (7.44) for each fundamental domain, and taking the trace, that for $1 \leq i \leq n$, $1 \leq j \leq n$,

$$\iint_{S_{\eta}^{(j)}} |\tilde{E}_i(z, \sigma + in, \chi)|^2 \frac{dx dy}{y^2} =$$

(62)

$$= O\left(\frac{1}{2\sigma-1} \left\{ A^{2\sigma-1} \kappa_1 - A^{1-2\sigma} \sum_{1 \leq i, j \leq \kappa_1} |\varphi_{ij}(\sigma+ir, \chi)|^2 \right\} \right. \\ \left. + \sum_{1 \leq i \leq \kappa_1} \frac{\overline{\varphi_{ii}(\sigma+ir, \chi)} A^{2ir} - \varphi_{ii}(\sigma+ir, \chi) A^{-2ir}}{2ir} \right),$$

or assuming $|r| \geq 1$; $\frac{1}{2} \leq \sigma \leq 1$,

$$(8.13) \quad \iint_{S_\eta^{(i)}} |\tilde{E}_i(z, \rho, \chi)|^2 \frac{dx dy}{y^2} = O\left(\frac{A^{2\sigma-1} - A^{1-2\sigma}}{2\sigma-1}\right)$$

$$+ O\left(\frac{1}{2\sigma-1} A^{1-2\sigma} \left(\kappa_1 - \sum_{1 \leq i, j \leq \kappa_1} |\varphi_{ij}(\sigma+ir, \chi)|^2\right)\right)$$

$$+ O(1) = O(\omega(r))$$

by (8.12), and since the φ 's are uniformly bounded in the region considered. If we denote by σ_1 the smallest of the numbers σ_i , $1 \leq i \leq N$, then clearly (8.13) holds ~~even~~ uniformly even in the strip $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + \frac{\sigma_1 - \frac{1}{2}}{2} = \beta$, where $\beta > \frac{1}{2}$. #

Now we choose the positive number A so large that in $S_\eta^{(i)}$, $\tilde{E}_i(z, \rho, \chi) = \bar{E}_i(z, \rho, \chi)$ ~~except~~ for $y_j \leq A$. This can clearly be done. ~~since the domain~~ $|x_j| \leq \frac{1}{2}, y_j \leq A$

Inserting now in (8.13) the Fourier expansion for $\tilde{E}_i(z, \rho, \chi)$ in terms of x_j , which is

obtained from that for $E_i(z, s, x)$, and differs (63)
 from it only in the "constant" term (that is
 the one that is free from x_j) in $S_\eta^{(0)}$, we get
 using
 for $1 \leq j \leq x_1$, from (7.5) and (7.10), for $m \neq 0$

$$|\varphi_{ij}^{(*)}(s, x)|^2 \int_{\eta}^{\infty} \left| \int_0^{\infty} t^{\sigma+in-\frac{1}{2}} e^{-\pi |m|y(t+\frac{1}{t})} \frac{dt}{t} \right|^2 \frac{dy}{y}$$

$$= O(\omega(r)),$$

Using here Lemma 8.2 for the integral, we
 get for $m \neq 0$,

$$(8.14) \quad |\varphi_{ij}^{(m)}(s, x)| = O(\sqrt{\omega(r)} e^{+3|r| + 4\pi|m|\eta}).$$

Similarly for $x_1 < j \leq x$, we obtain from
 (7.12) and (7.14) and all integral m ,

$$(8.15) \quad \varphi_{ij}^{(m+\alpha_j)}(s, x) = O(\sqrt{\omega(r)} e^{3|r| + 4\pi|m+\alpha_j|\eta}).$$

Inserting the estimations (8.14) in (7.10) and (7.5)
 and using Lemma 8.1, we find for $1 \leq j \leq x$,
 and $\eta_j \geq 4\eta$,

$$(8.26) \quad E_i(z, s, x) = \delta_{i,j} y_j^s + \varphi_{ij}(s, x) y_j^{1-s} + \\ + O(\sqrt{\omega(r)} e^{3|r| - 3\eta_j})$$

and similarly using (8.15), (7.14) and (7.12) and lemma 1, for $n_1 < j \leq n$, and $y_j \geq 4\eta$

$$(8.17) \quad E_j(z, s, x) = O(\sqrt{\omega_n}) e^{3|z| - 3|z_j| y_j},$$

both (8.16) and (8.17) holding uniformly for $\frac{1}{2} \leq \sigma \leq \beta$. Since the regions $S_{4\eta}^{(j)}$ $1 \leq j \leq n$ together cover D completely, at least one of the estimations (8.16) or (8.17) is always valid when z lies in D .

§ 9.

The trace formula in the case of a circular χ .

Let now $h(n)$ be an even function, analytic and regular in some strip $|y(n)| \leq \frac{1}{2} + \varepsilon$, $\varepsilon > 0$ and satisfying ~~for the time being~~ until this restriction is explicitly removed, the condition

$$(9.1) \quad h(n) = O(e^{-7|n|}),$$

in this strip.

Consider now the expression

$$(9.2) \quad H(z, z', x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} h(n) \overline{E(z', \frac{1}{2} + in, x)}' E(z, \frac{1}{2} + in, x) dn.$$

First the existence of the integral follows from (9.1), the estimations (8.16) and (8.17) and (8.11). Furthermore we have

$$(9.3) \quad \overline{H(z, z', \chi)} = H(z', z, \chi),$$

and

$$(9.4) \quad H(Mz, z', \chi) = \chi(M) H(z, z', \chi),$$

for M belonging to Γ .

We shall next look at the behaviour of $H(z, z', \chi)$ ^{with z and z' in \mathcal{D}} as one or both of the points tend towards cusps of \mathcal{D} . Let us first consider the case that both z and z' lie in the domain $S_{4\eta}^{(j)}$ defined in the previous chapter for some j with $1 \leq j \leq \kappa$. Writing

$$(9.5) \quad E_i(z, s, \chi) = \delta_{ij} y_j^s + \varphi_{ij}(s, \chi) y_j^{1-s} + E_i^*(z, s, \chi),$$

we have according to (8.16) for $1 \leq i \leq \kappa$,

$$(9.6) \quad E_i^*(z, s, \chi) = O(\sqrt{w(z)}) e^{3|n|-3y_j},$$

uniformly for $\frac{1}{2} \leq \sigma \leq \beta$, $s = \sigma + i\tau$. Inserting the expressions (9.5) in (9.2), we observe that the ^{right} ~~left~~ hand-side breaks up into 9 terms - which we shall consider

separately. First we take the term

$$\frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \sum_{1 \leq i \leq n_1} E_i^*(z, \frac{1}{2} + ir, \chi) E_i^*(z', \frac{1}{2} - ir, \bar{\chi}) dr,$$

from (9.6) and (9.1) and (8.11), we get immediately that this term is

$$O(e^{-3(y_j + y'_j)}) = O(1).$$

Then we have four terms that contain only one E^* factor, namely

$$\frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) y_j^{\frac{1}{2} + ir} E_j^*(z', \frac{1}{2} - ir, \bar{\chi}) dr,$$

$$\frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) y_j^{\frac{1}{2} - ir} \sum_{1 \leq i \leq n_1} \varphi_{ij}(\frac{1}{2} + ir, \chi) E_i^*(z', \frac{1}{2} - ir, \bar{\chi}) dr,$$

and the remaining two are obtained by interchanging z and z' and taking the conjugate of the integrand apart from $h(r)$ which is left unchanged. Now since

$$\varphi_{ij}(\frac{1}{2} + ir, \chi) = \varphi_{ji}(\frac{1}{2} + ir, \bar{\chi})$$

from Theorem 7.2, and

$$\sum_{1 \leq i \leq n_1} \varphi_{ji}(\frac{1}{2} + ir, \bar{\chi}) E_i(z', \frac{1}{2} - ir, \bar{\chi}) = E_j(z', \frac{1}{2} + ir, \bar{\chi}),$$

because of (7.36), we have also