$$\sum_{1\le i\le \varkappa_1} \varphi_{ij}(\tfrac{1}{2}+ir,\chi)\, E_i^*(z',\tfrac{1}{2}+ir,\bar\chi) = E_j^*(z',\tfrac{1}{2}+ir,\bar\chi),$$

so that the second of the two above integrals reduces to the first by writing $-r$ instead of r. Looking now at the first integral, and moving the line of integration upwards by writing $i(\beta-\frac{1}{2})+r$ instead of r, we obtain, taking absolute values and using (9.6), (9.1) and (8.11) that this integral is

$$= O\left(y_j^{1-\beta} e^{-3y_j'}\right) = O\left((y_j\, y_j')^{1-\beta}\right),$$

and so are the other three integrals.

There remain the terms that do not contain any E^*, of these there are four. We first consider the two integrals

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\, \varphi_{jj}(\tfrac{1}{2}-ir,\bar\chi)\,(y_j\, y_j')^{\frac{1}{2}+ir}\,dr$$

and

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\, \varphi_{jj}(\tfrac{1}{2}+ir,\chi)\,(y_j\, y_j')^{\frac{1}{2}+ir}\,dr.$$

Since $\varphi_{jj}(s,\chi) = \varphi_{jj}(s,\bar\chi)$, the second integral reduces to the first if we write $-r$ instead of r. Shifting again the line of integration upwards by writing $i(\beta-\frac{1}{2})+r$ instead of r, and using (9.1), and the fact that $\varphi_{ij}(s,x)$ is uniformly bounded for $\frac{1}{2}\le\sigma\le\beta$, we obtain that this integral is

$$= O((y_j\, y'_j)^{1-\beta}),$$

and so is the second.

Finally we consider the terms

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\, y_j^{\frac{1}{2}+ir}\, y_j'^{\frac{1}{2}-ir}\, dr,$$

and

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\, y_j^{\frac{1}{2}-ir}\, y_j'^{\frac{1}{2}+ir} \sum_{1\le i\le \varkappa_1} |\varphi_{ij}(\tfrac{1}{2}+ir, x)|^2\, dr.$$

Since $\Phi(\frac{1}{2}+ir, x)$ is a unitary matrix,

$$\sum_{1\le i\le \varkappa_1} |\varphi_{ij}(\tfrac{1}{2}+ir, x)|^2 = 1,$$

and so by writing $-r$ instead of r the second integral reduces to the first. The two taken together give

$$\frac{\sqrt{y_j\, y'_j}}{2\pi}\int_{-\infty}^{\infty} h(r)\, e^{ir\log\frac{y_j}{y'_j}}\, dr =$$

$$= \sqrt{y_j y_j'} \quad g\left(\log \frac{y_j}{y_j'}\right).$$

Combining these results we obtain

$$(9.7) \quad H(z, z', \varkappa) = \sqrt{y_j' y_j} \quad g\left(\log \frac{y_j}{y_j'}\right) + O\left((y_j y_j')^{1-\beta}\right),$$

for z and z' in $S_{4\eta}^{(j)}$, $1 \le j \le \varkappa_1$.

In the same way we can show that if z and z' both are in $S_{4\eta}^{(j)}$ with $\varkappa_1 < j \le \varkappa$,

$$(9.7') \quad H(z, z', \varkappa) = O(1) = O\left((y_j y_j')^{1-\beta}\right).$$

Finally for $i \neq j$, $1 \le i, j \le \varkappa$ and z in $S_{4\eta}^{(i)}$, z' in $S_{4\eta}^{(j)}$, we get

$$(9.7'') \quad H(z, z', \varkappa) = O\left((y_i y_j')^{1-\beta}\right).$$

Now let $K(z, z', \varkappa)$ given by (6.1) be derived from the same function $h(r)$ as in (9.2), we shall consider the integral operator

$$(9.8) \quad \iint_{D} \left\{K(z, z', \varkappa) - H(z, z', \varkappa)\right\} f(z') \frac{dx' dy'}{y'^2}.$$

Combining (9.7), (9.7'), (9.7'') with Theorem 6.1 we see that for z and z' in $\mathcal{D}$,

(9.9) $$K(z,z',\chi) - H(z,z',\chi) = O\Big(\sum_{1\le i,j\le \varkappa} (y_i\, y_j')^{1-\beta}\Big).$$

From this we see, since $\beta > \frac{1}{2}$, that

(9.10) $$\iint_{\mathcal{D}} \iint_{\mathcal{D}} |K(z,z',\chi) - H(z,z',\chi)|^2 \frac{dx\,dy}{y^2}\frac{dx'\,dy'}{y'^2} < \infty.$$

Let now $u(z)$ be a square integrable (over $\mathcal{D}$) solution of

(9.11) $$y^2 \Delta u + \left(\tfrac{1}{4} + r^2\right) u = 0,$$

satisfying the relations

(9.12) $$u(Mz) = \chi(M)\, u(z).$$

One easily sees (from the Fourier expansion of u) that for z in $\mathcal{D}$, $1 \le j \le \varkappa$

$$u(z) = O(y_i^{\alpha})$$

with $\alpha < \frac{1}{2}$, as $y_i \to \infty$. Therefore the integral

(9.13) $$\iint_{\mathcal{D}} u(z)\, \overline{E(z, \tfrac{1}{2} + ir', \chi)}\; \frac{dx\,dy}{y^2},$$

exists for every real r', and the value is

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seen to be zero, by first assuming $r' \neq \pm r$, and considering the expression.

$$\iint_D \left(u(z)\, \Delta\, \overline{E(z, \tfrac{1}{2}+ir', \chi)}' - \overline{E(z, \tfrac{1}{2}+ir', \chi)}'\, \Delta u(z) \right) \frac{dx\,dy}{y^2},$$

which is zero by Greens theorem, since the integrals over the boundary cancel out. Thus (9.13) vanishes for $r' \neq \pm r$, making if r is real r' tend to r and $-r$, we obtain that (9.13) vanishes for all real r'. Hence we ~~also~~ easily get

$$\iint_D H(z, z', \chi)\, u(z') \frac{dx'dy'}{y'^2} = 0,$$

On the other hand Lemma 7.2 gives

$$\iint_D K(z, z', \chi)\, u(z') \frac{dx'dy'}{y'^2} = h(r)\, u(z).$$

Thus

$$(9.14) \quad \iint_D \left(K(z,z',\chi) - H(z,z',\chi) \right) u(z') \frac{dx'dy'}{y'^2} = h(r)\, u(z).$$

Next, we have that the operator $y^2\Delta$ commutes with the operator (9.8), since we have previously shown that it commutes with the integral operator with the kernel $K(z, z'; \chi)$, and to show that it commutes with the operator with kernel $H(z, z'; \chi)$ one observes that (9.2) easily gives that (denoting by subscript the variable on which the differential operator acts),

$$y^2 \Delta_z H(z, z'; \chi) = y'^2 \Delta_{z'} H(z, z'; \chi),$$

an application of Greens theorem then gives the desired conclusion.

From (9.10) we see that the eigenfunctions of the operator (9.8) that do not belong to the eigenvalue ~~zero~~ 0, in the equation

$$(9.15) \quad \iint_D \{K(z, z'; \chi) - H(z, z'; \chi)\} f(z') \frac{dx'dy'}{y'^2} = \lambda f(z),$$

are square integrable over D, and also have the property $f(Mz) = \chi(M) f(z)$ for M in Γ. Furthermore the spectrum is discrete (that is the eigenvalues λ have only zero as point of accummulation); and to each eigenvalue $\lambda \neq 0$, corresponds only a finite number of linearly independent eigenfunctions. Since $y^2\Delta$ commutes with the operator (9.8) one sees that we can choose a complete set of eigen-functions with $\lambda \neq 0$, so that each of them is also an eigenfunction of the operator $y^2\Delta$.

Thus the square integrable solutions of (9.11) and (9.12) give us all eigenfunctions of (9.15) with $\lambda \neq 0$, (and since we have according to (9.14) $\lambda = h(r)$, possibly a finite or infinite number of eigenfunctions with $\lambda = 0$ if (9.11) and (9.12) have solutions for r's which make $h(r)$ vanish).

From this we conclude that the whole class of operators (9.8) commutes (this could have been proved more directly) and that the product of two such operators, derived respectively from the functions $h_1(r)$ and $h_2(r)$, is the operator derived from $h_1(r)h_2(r)$.

We can now proceed to ~~pro~~ compute twice the trace of the operator (9.8), on one hand from the expression

(9.16) $\quad 2 \iint_{\mathfrak{D}} (K(z,\bar{z},\chi) - H(z,\bar{z},\chi)) \dfrac{dx\,dy}{y^2}$, and on the other hand as

(9.17) $\sum h(r_i)$

where we denote the eigenvalues r_i (counting both $+r_i$ and $-r_i$, and $r=0$ with double multiplicity if it occurs) for which ~~of~~ (9.11), (9.12) have square integrable solutions. At first we know that these two expressions are equal if $h(r)$ can be written as a product of two functions $h_1(r)$ and $h_2(r)$ which both satisfy the conditions on $h(r)$ stated in the beginning of this chapter, or what is the same, if $h(r)$ satisfies the condition

(9.18) $h(r) = O(e^{-14|r|})$.

~~instead of (9.7)~~

For simplicity we shall carry out the computation of (9.16) in details only in the case that D has only one cusp, placed at $i\infty$, ~~and~~ that the primitive parabolic transformation which leaves the cusp fixed is $Sz = z+1$, and that $\chi(n)$ is identically equal to 1.

We split the kernel $K-H$ up into

three parts as follows

$$(9.19)\quad K(z,\bar{z}) - H(z,z) = \Big\{ \sum_{m=\infty}^{\infty}{}' k(z+m, z) - H(z,z)\Big\}$$

$$+ \sum_{N(S)_{left}}{}' \sum_{m=-\infty}^{\infty}{}' k(z, N^{-1}S^{m}Nz) + \sum_{M}{}'' k(z, Mz).$$

where the dash $\sum_{m=-\infty}^{\infty}{}'$ denotes that $m=0$ is omitted, $\sum_{N(S)_{left}}{}'$ that N runs over a complete set of transformations that do not differ on the left by a power of S, except that the case N equals a power of S is omitted, finally $\sum_{M}{}''$ is extended over all non-parabolic transformations of Γ (identity, elliptic and hyperbolic transformations). From our previous results it follows that (9.16) can be split accordingly, each of the three resulting integrals existing.

The contribution from the last term on the right hand side of (9.19) is treated precisely as in the compact case, and gives the same contribution from the identity-, elliptic and hyperbolic transformations as found before.

Denoting by D_A the part of $\mathcal{D}$ that lies below the line $y=A$ for large positive A, we have

$$(9.20)\quad \iint_{\mathcal{D}}\Big(\sum_{n=-\infty}^{\infty}{}' k(z,z+n) - H(z,z)\Big)\frac{dx\,dy}{y^2} =$$

$$= \lim_{A\to\infty}\Big\{\iint_{D_A}\sum_{n=-\infty}^{\infty}{}' k(z,z+n)\frac{dx\,dy}{y^2} - \iint_{D_A} H(z,z)\frac{dx\,dy}{y^2}\Big\}.$$

Furthermore, denoting by R the domain remaining of the half strip $y>0$, $|x|\le\frac{1}{2}$, if we remove $\mathcal{D}$, we get that

$$(9.21)\quad \iint_{\mathcal{D}}\sum_{N(S)_{\text{left}}}{}'\ \sum_{n=-\infty}^{\infty}{}' k(z,N^{-1}S^{n}Nz)\frac{dx\,dy}{y^2} =$$

$$= \iint_{R}\sum_{n=-\infty}^{\infty}{}' k(z,z+n)\frac{dx\,dy}{y^2}.$$

From (9.20) and (9.21) we now obtain that the contribution of the two first terms on the right hand side of (9.19) to the expression (9.16) becomes

$$(9.22)\quad \lim_{A\to\infty}\Big\{2\iint_{\substack{|x|\le\frac{1}{2}\\ 0<y\le A}}\sum_{n=-\infty}^{\infty}{}' k(z,z+n)\frac{dx\,dy}{y^2}$$

$$-2\iint_{D_A} H(z,z)\frac{dx\,dy}{y^2}\Big\}.$$

We consider first

$$(9.23)\quad 2\iint_{D_A} H(z,z)\frac{dx\,dy}{y^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\iint_{D_A}\left|E(z,\tfrac12+ir)\right|^2\frac{dx\,dy}{y^2}\,dr.$$

Using (7.42') and the fact that for $y \geq A$,

$$\tilde{E}(z,\tfrac12+ir) = E^*(z,\tfrac12+ir) = O(\sqrt{\omega(r)}\,e^{3|r|-3y}),$$

according to (9.6), we get

$$(9.24)\quad \iint_{D_A}\left|E(z,\tfrac12+ir)\right|^2\frac{dx\,dy}{y^2} = 2\log A - \frac{\varphi'}{\varphi}(\tfrac12+ir)$$

$$+\ \frac{\varphi(\tfrac12-ir)A^{2ir}-\varphi(\tfrac12+ir)A^{-2ir}}{2ir} + O(\omega(r)e^{6|r|-6A}).$$

Now from (8.11) and (9.18),

$$\int_{-\infty}^{\infty}|h(r)|\,\omega(r)\,e^{6|r|-6A}\,dr = O(e^{-6A}).$$

Also

$$(9.25)\quad \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\varphi(\tfrac12-ir)A^{2ir}-\varphi(\tfrac12+ir)A^{-2ir}}{2ir}\,h(r)\,dr$$

$$= \frac{1}{4\pi i}\int_{-\infty}^{\infty}\frac{\varphi(\tfrac12-ir)A^{2ir}-\varphi(\tfrac12)}{r}\,h(r)\,dr$$

$$+\ \frac{1}{4\pi i}\int_{\infty}^{-\infty}\frac{\varphi(\tfrac12+ir)A^{-2ir}-\varphi(\tfrac12)}{r}\,h(r)\,dr.$$

In the first integral on the right-hand side we shift the line of integration upwards by writing $i(\beta-\frac{1}{2})+r$ instead of r, this is permitted because of the fact that $\varphi(s)$ uniformly bounded in the domain $\frac{1}{2}\le\sigma\le\beta$, we then get that the first integral becomes

$$-\frac{\varphi(\frac{1}{2})}{4\pi i}\int_{i(\beta-\frac{1}{2})-\infty}^{i(\beta-\frac{1}{2})+\infty}\frac{h(r)}{r}\,dr+O(A^{1-2\beta}).$$

For the second integral we get in the same way shifting the line of integration downwards that it is

$$-\frac{\varphi(\frac{1}{2})}{4\pi i}\int_{-i(\beta-\frac{1}{2})+\infty}^{-i(\beta-\frac{1}{2})-\infty}\frac{h(r)}{r}\,dr+O(A^{1-2\beta}).$$

Combining we find that the left hand side of (9.25) equals

$$\frac{1}{2}h(0)\varphi(\tfrac{1}{2})+O(A^{1-2\beta}).$$

Finally

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\,dr=g(0).$$

Inserting now (9.24) in (9.23) and using the above results, we obtain

$$(9.26)\quad 2\iint_{\mathcal{D}_A} H(z,z)\,\frac{dx\,dy}{y^2} = 2g(0)\log A - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\,\frac{\varphi'}{\varphi}\left(\tfrac{1}{2}+ir\right)dr$$

$$+ \tfrac{1}{2}h(0)\,\varphi\left(\tfrac{1}{2}\right) + O\left(A^{1-2\beta}\right),$$

as $A \to \infty$.

Next we have

$$(9.27)\quad 2\iint_{\substack{|x|\le\frac{1}{2}\\ 0<y\le A}} \sum_{m=-\infty}^{\infty}{}' k(z,z+m)\,\frac{dx\,dy}{y^2} =$$

$$= 4\int_0^A \sum_{m=1}^{\infty} k\left(\frac{m^2}{y^2}\right)\frac{dy}{y^2} =$$

$$= 4\sum_{m=1}^{\infty}\frac{1}{m}\int_{\frac{m}{A}}^{\infty} k(u^2)\,du =$$

$$= 4\int_0^{\infty} k(u^2)\sum_{m\le Au}\frac{1}{m}\,du.$$

For $u>0$, we have uniformly

$$\sum_{m\le Au}\frac{1}{m} = \log(Au)+c + O\left(\frac{1}{\sqrt{Au}}\right),$$

where c is Eulers constant. Thus

$$(9.28)\quad 2\iint_{\substack{|x|\le\frac{1}{2}\\ 0<y\le A}} \sum_{m=-\infty}^{\infty}{}' k(z,z+m)\,\frac{dx\,dy}{y^2} =$$

$$= 4(\log A + C)\int_0^\infty k(u^2)\,du + 4\int_0^\infty \log u\, k(u^2)\,du$$

$$+ O\left(\frac{1}{\sqrt{A}}\right).$$

Here

$$\int_0^\infty k(u^2)\,du = \frac{1}{2}\int_0^\infty \frac{k(t)}{\sqrt{t}}\,dt = \frac{1}{2}g(0).$$

Further

$$4\int_0^\infty \log u\, k(u^2)\,du = \int_0^\infty k(t)\,\frac{\log t}{\sqrt{t}}\,dt$$

Now

$$k(t) = -\frac{1}{\pi}\int_t^\infty \frac{dQ(w)}{\sqrt{w-t}}, \quad Q(e^u+e^{-u}-2) = g(u).$$

Hence

$$4\int_0^\infty \log u\, k(u^2)\,du = -\frac{1}{\pi}\int_0^\infty dQ(w)\cdot\left\{\int_0^w \frac{\log t}{\sqrt{t(w-t)}}\,dt\right\}.$$

One easily shows that

$$\int_0^w \frac{\log t}{\sqrt{t(w-t)}}\,dt = \pi(\log w - 2\log 2),$$

so that

$$4\int_0^\infty \log u\, k(u^2)\,du = -\int_0^\infty (\log w - 2\log 2)\,dQ(w)$$

$$= -\int_0^\infty \log(e^u+e^{-u}-2)\,dg(u) - 2\log 2\, g(0)$$

$$= -2\int_0^\infty \log(1-e^{-u})\,dg(u) - \int_0^\infty u\,dg(u) - 2\log 2\, g(0) =$$

$$= -2\int_0^\infty \log(1-e^{-u})\,dg(u) + \int_0^\infty g(u)\,du - 2\log 2\, g(0)$$

$$= -2\int_0^\infty \log(1-e^{-u})\,dg(u) + \tfrac{1}{2}h(0) - 2\log 2\; g(0).$$

Now

$$g'(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ir\,h(r)\,e^{iru}\,dr,$$

So that

$$-2\int_0^\infty \log(1-e^{-u})\,dg(u) = -\frac{i}{\pi}\int_{-\infty}^{\infty} h(r)\,r\left\{\int_0^\infty e^{iru}\log(1-e^{-u})\,du\right\}dr$$

Now one can show that

$$-ir\int_0^\infty e^{iru}\log(1-e^{-u})\,du = -\frac{\Gamma'}{\Gamma}(1-ir) - C,$$

where C again is Eulers constant. Inserting this above we get

$$-2\int_0^\infty \log(1-e^{-u})\,dg(u) = -2C\,g(0) - \frac{1}{\pi}\int_{-\infty}^{\infty} h(r)\,\frac{\Gamma'}{\Gamma}(1-ir)\,dr$$

$$= -2C\cdot g(0) - \frac{1}{\pi}\int_{-\infty}^{\infty} h(r)\,\frac{\Gamma'}{\Gamma}(1+ir)\,dr,$$

since $h(r)$ is even.

Inserting our results in (9.28), we finally obtain

$$(9.29)\quad 2\iint\limits_{\substack{|x|\le\frac12\\0<y\le A}} \sum_{n=-\infty}^{\infty}{}' k(z,z+n)\,\frac{dx\,dy}{y^2} = 2\log A\, g(0)$$

$$-2\log 2\, g(0) + \tfrac12 h(0) - \frac{1}{\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)\,dr + O\!\left(\tfrac{1}{\sqrt{A}}\right),$$

as $A\to\infty$.

From (9.26), (9.29) and (9.28) we get since $\beta>\frac12$, that the contribution of the two first terms on the right-hand side of (9.19) to the expression (9.16) is

$$(9.30)\quad \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\,\frac{\varphi'}{\varphi}(\tfrac12+ir)\,dr - \frac{1}{\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)\,dr$$

$$-2\log 2\, g(0) + \tfrac12\left(1-\varphi(\tfrac12)\right)h(0).$$

In the general case we can proceed in a similar way, and obtains then as the contribution to (9.16) of all parabolic transformations (including the ones treated in §6 already where χ of the primitive transformation is $\neq 1$).

$$(9.30')\quad \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\,\frac{\varphi'}{\varphi}(\tfrac12+ir,\chi)\,dr - \frac{\varkappa_1}{\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)\,dr$$

$$-2\varkappa_1 \log 2\, g(0) + 2 \sum_{\varkappa_1 < i \le \varkappa} \log \frac{1}{|1 - \chi(S_i)|}\, g(0)$$

$$+ \tfrac{1}{2}(\varkappa_1 - \sigma(\Phi(\tfrac{1}{2}, \chi)))\, h(0).$$

It should be observed that as the eigenvalues of the $\varkappa_1 \times \varkappa_1$ matrix $\Phi(\frac{1}{2}, \chi)$ are all ± 1, the coefficient of $h(0)$ in (9.30′) is always a non-negative integer $\le \varkappa_1$.

We can now write down the trace formula in the general case of a onedimensional χ, at first on the assumption ~~that~~ (9.18),

$$(9.31)\quad \sum h(r_i) = \mu(D) \int_{-\infty}^{\infty} r \frac{e^{\pi r} - e^{-\pi r}}{e^{\pi r} + e^{-\pi r}}\, d\nu(r)\, dr$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\varphi'}{\varphi}(\tfrac{1}{2} + ir, \chi)\, dr - \frac{\varkappa_1}{\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'}{\Gamma}(1 + ir)\, dr$$

$$-2\Big(\varkappa_1 \log 2 + \sum_{\varkappa_1 < i \le \varkappa} \log|1 - \chi(S_i)|\Big)\, g(0)$$

$$+ \tfrac{1}{2}(\varkappa_1 - \sigma(\Phi(\tfrac{1}{2}, \chi)))\, h(0)$$

$$+ \sum_{\{R\}} \sum_{1 \le \nu \le m-1} \frac{2\chi^{\nu}(R)}{m \sin \frac{\pi\nu}{m}} \int_{-\infty}^{\infty} \frac{h(r)\, e^{-\frac{2\pi\nu}{m} r}}{1 + e^{-2\pi r}}\, dr +$$

$$+2\sum_{\{P\}}\sum_{k=1}^{\infty}\frac{\chi^{k}(P)\log N\{P\}}{(N\{P\})^{\frac{k}{2}}-(N\{P\})^{-\frac{k}{2}}}\,g(k\log N\{P\}).$$

Here as before $\mu(D)=\frac{A(D)}{2\pi}$ where $A(D)$ is the area of $\mathcal{D}$ in the invariant metric. $\{R\}$ runs over the primitive elliptic classes, and $m=m\{R\}$ is the order of the primitive elliptic transformation. Similarly $\{P\}$ runs over the primitive hyperbolic classes and $N\{P\}$ denotes the norm of P.

We shall first show that the range of (9.31) can be extended to the case that $h(r)$ instead of (9.18) only satisfies the usual condition

$$h(r)=O\left(\frac{1}{|r|^{2+\varepsilon}}\right),$$

as $|r|\to\infty$ in ~~the~~ strip $|\Im(r)|\le\frac{1}{2}+\varepsilon$, in which $h(r)$ is regular and even.

For this purpose we observe that

$$-\frac{\varphi'}{\varphi}\left(\tfrac{1}{2}+ir,\chi\right)=1-\frac{\varphi^{*\prime}}{\varphi^{*}}\left(\tfrac{1}{2}+ir,\chi\right)+O(1)$$

$$=\omega(r)+O(1),$$

and we move the term in (9.31)

containing $\omega(r)$, after we have inserted the above expression, over to the left hand side of (9.31), which then becomes

$$\sum h(r_i) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega(r)\, h(r)\, dr ,$$

Inserting now for $h(r)$ the function $e^{-\frac{r^2}{R^2}}$, which clearly is admissible, we easily get

$$(9.32) \quad \sum_{|r_i| \le R} 1 + \frac{1}{2\pi}\int_{-R}^{R} \omega(r)\, dr = O(R^2),$$

as $R \to \infty$. This implies that all series and integrals occurring in (9.31) converge absolutely if $h(r)$ only satisfies the usual conditions mentioned above. Also for a class of functions which satisfies these conditions uniformly, convergence of series and integrals is seen to be uniform. Considering now for a fixed $h(r)$ the class $h(r)e^{-\varepsilon r^2}$ with $0 \le \varepsilon \le 1$, these constitute such a class, we have

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for $\varepsilon > 0$ that (9.12) is satisfied and so (9.31) valid. Making $\varepsilon \to 0$, we obtain

Theorem 9.1. The trace formula (9.31) is valid if $h(r)$ is even and analytic and regular in some strip $|\Im(r)| \leq \frac{1}{2} + \varepsilon$, and $O\left(\frac{1}{|r|^{2+\varepsilon}}\right)$ in this strip with $\varepsilon > 0$.

(9.32) can actually easily be sharpened to

$$(9.33) \quad \sum_{|r_i| \leq R} 1 - \frac{1}{2\pi} \int_{-R}^{R} \frac{\varphi'}{\varphi}\left(\tfrac{1}{2} + ir, \chi\right) dr \sim \mu(\mathcal{D}) R^2,$$

as R tends to infinity (and ~~other~~ much ~~better~~ estimations with a remainder term $O(R/\lg R)$ can be given.

Unfortunately however, we have in the general case no means of ~~independently~~ separately estimating the two terms on the left hand side of (9.33) so that the asymptotic formula for the distribution of the eigenvalues r_i can not be given. Only in some special cases, when the function $\varphi(s, \chi)$

n be expressed in terms of functions that are own from analytic numbertheory can we o this, and in all these special cases the second erm on the lefthand side of (9.33) is $O(R \log R)$ as $R \to \infty$.

In all these special cases $\varphi(s,\chi)$ is a meromorphic function of order 1. From (9.33) or even (9.32) we can easily show that in the general case $\varphi(s,\chi)$ is a meromorphic function of order at most 2, or more precisely that, if $\rho = \beta + i\gamma$; $\beta < \frac{1}{2}$, again denoting the poles of $\varphi(s,\chi)$ in the halfplane $\sigma < \frac{1}{2}$, we have

$$\sum_{|\gamma| \leq R} 1 = O(R^2).$$

We may now, as in the compact case, study the function defined by

$$Z_p(s,\chi) = \prod_{\{P\}} \prod_{\nu=1}^{\infty} \left(1 - \chi\{P\}(N\{P\})^{-s-\nu}\right),$$

for $\sigma > 1$, by inserting in (9.31) the special function

$$h(r) = (s-\tfrac{1}{2})\left\{\frac{1}{(s-\frac{1}{2})^2 + r^2} - \frac{1}{(a-\frac{1}{2})^2 + r^2}\right\},$$

here a is a real constant > 1, and we at first assume $\Re(s) > 1$. The new term occurring in our trace formula produces some important changes in the properties of the function $Z_P(s,\chi)$. First of all, $Z_P(s,\chi)$ is no longer an integral function of s, since the third term on the right-hand side of (9.31) produces poles at the points $s = -n + \frac{1}{2}$, $n = 1, 2, 3, \dots$ of order $\varkappa_1$, in addition the fifth term produces a pole at $s = \frac{1}{2}$ of order $\varkappa_1 - \sigma(\Phi(\frac{1}{2},\chi))$. (This latter pole might however, if some of the $r_i = 0$, be superimposed on a zero, and so if sufficiently many $r_i = 0$, be cancelled out completely).

As before the first and sixth term produces zeros at the negative integers, with multiplicities that can be determined from $\mu(D)$, the $\chi(R)$'s and the m's. The second term on the right hand side of (9.31) produces zeros at the poles of $\rho = \beta + i\gamma$ of $\varphi(s,\chi)$ in the halfplane $\sigma < \frac{1}{2}$, and poles at the points $1 - \sigma_i$. (the σ_i as before denoting the eventual poles of $\varphi(s,\chi)$ in the interval $\frac{1}{2} < s \leq 1$ of the real line), these poles are however cancelled out by zeros produced by the term on the lefthand side of (9.31) at the points $\frac{1}{2}+ir_i$, since the points $\pm(\sigma_i-\frac{1}{2})i$, belong to the set r_i, ~~with an~~ according to Theorem 7.3.

One can from (9.31) also find a functional equation for $Z_P(s,\chi)$, which we shall not give here. Whereas in the compact case and the case treated in §6, the functional equation was of the form

$$\frac{Z_P(1-s,\chi)}{Z_P(s,\chi)} = \text{Canonical product that can be given explicitly},$$

we have in the case of a singular χ that

$$\frac{Z_P(1-s,\chi)}{Z_P(s,\chi)} = \varphi(s,\chi) \text{ times canonical product that can be given explicitly.}$$

Returning to the eigenfunctions, if we denote by $u_i(z)$ the square integrable eigenfunction corresponding to the eigenvalues $\pm r_i$, normed so that

$$\iint_D |u_i(z)|^2 \frac{dx\,dy}{y^2} = 1,$$

and, in case of multiple eigenvalues, chosen so that they form a set of orthogonal functions one can show that one has a fourier expansion,

$$f(z) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \alpha(r)\, E(z, \tfrac{1}{2}+ir, \chi)\, dr + \sum_i \beta_i\, u_i(z),$$

convergent in the L_2 sense, if $f(z)$ is a function satisfying $f(Mz) = \chi(M) f(z)$ and $f(z)$ square integrable over D. Here

$$\alpha(r) = \iint_D f(z)\, \overline{E(z, \tfrac{1}{2}+ir, \chi)}\, \frac{dx\,dy}{y^2}$$

and

$$\beta_i = \iint_D f(z)\, \overline{u_i(z)}\, \frac{dx\,dy}{y^2}.$$

Also a Plancherel formula

$$\iint_D |f(z)|^2 \frac{dx\,dy}{y^2} = \frac{1}{4\pi}\int_{-\infty}^{\infty} |\alpha(r)|^2\, dr + \sum_i |\beta_i|^2,$$

holds.

~~Finally~~ We should also mention that the square integrable eigen-functions obtained in Theorem 7.3 are the only ones that are not cuspforms.

Finally it is possible to generalize all these results to the case that χ is not one dimensional. The resulting formulas are of course somewhat more complicated. We may also consider groups Γ that contain transformations of the form $\frac{a\bar{z}+b}{c\bar{z}+d}$, where

a, b, c, d are real and $ad-bc=-1$. The new types of transformations (there will really be only two types, represented respectively by $z \to -\rho\bar{z}$ $\rho \neq 1$ (which plays a role similar to the hyperbolic transformation) and $z \to -\bar{z}$, will of course give us new terms in the trace formula. In this case the definition of $Z_P(s, \chi)$ has to be modified.

§10.

Concluding remarks.

A typical example of a case where $\varphi(s, \chi)$ can be expressed in terms of "known" functions, we get by taking Γ to be the full modular group and $\chi(M)$ identically 1. One then easily finds

$$\varphi(s) = \sqrt{\pi}\ \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\ \frac{\zeta(2s-1)}{\zeta(2s)},$$

one sees that the poles of $\varphi(s)$ in this case are $s=1$, and $s = \frac{\rho}{2}$, where ρ runs over the nontrivial zeros of the Riemann zetafunction.

One can show by examples that $\varphi(s, \chi)$ for suitable Γ and χ may have arbitrary many poles in the interval $\frac{1}{2} < s \leq 1$ of the real line. Also that for the poles in the halfplane $\sigma < \frac{1}{2}$,

$= \beta + i\gamma$, $\beta - \frac{1}{2}$ may become arbitrarily large and negative even for a fixed Γ (and variable χ), and that with variable Γ and χ, β may become arbitrarily close to $\frac{1}{2}$, even to the extent that a sequence Γ_i, χ_i can be constructed so that every point on the line $\sigma = \frac{1}{2}$ is a point of accumulation for the ρ's as $i \to \infty$.

In certain cases, for instance for the modular group, the formula (9.31) can be considerably generalized. For the modular group and χ identically 1, we may combine our integral operators with the so called Hecke operators T_n $(n = 1, 2, \ldots)$, with which they commute. It is possible to compute the trace of the product in a similar way as before, and for $n = 2, 3, 4, \ldots$ we obtain new formulas, that actually can be looked upon as transcendental analoga of the classical class number relations due to Kronecker. ~~for positive definite forms.~~

One can also generalize the preceding approach to cover also functions that transform in the manner

$$(10.1) \quad f(M z) = \chi(M)\,(cz+d)^{k} f(z)$$

where k is a real number, and $Mz = \frac{az+b}{cz+d}$ belongs to Γ. This can be fitted into the general setup by passing to a certain 3-dimensional space, enlarging also Γ in a suitable way, and associating with the function $f(z)$ a certain function, defined on the 3 dimensional space, which ~~by~~ for a transformation M' belonging to the enlarged group Γ' ~~transforms~~ multiplies only by a factor $\chi'(M')$; where χ' is a unitary representation of Γ' derived from χ and k. The details shall not be given here; but going back to the functions $f(z)$ one sees that it ~~corresponds~~ leads to considering the class of integral operators

$$(10.2)\quad \iint_H k\left(\frac{|z-z'|^2}{yy'}\right) \frac{y'^{k}}{(z-\bar{z}')^{k}} f(z') \frac{dx'dy'}{y'^2},$$

which commute among themselves and also with the ~~transformations~~ operation $f(z) \to (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$.

A new feature is the singular position occupied by the analytic functions. For $k>2$ one may chose $k\left(\frac{|z-z'|^2}{yy'}\right)$ in (10.2) identically 1, so that the kernel becomes analytic in z.

This fact leads to a splitting up of the trace-formula corresponding to the ring of operators (10.2) for $h=2$, into a trancendental part which is much like what we had before, and an algebraic part which turns out to be the expression for the number of linearly independent analytic regular functions (10.1) given by the Riemann-Roch formula, $h=2$ can be treated by a limit procedure.

In case of the modular group, we may consider the analytic modular forms that transform in the way

$$f(Mz) = (cz+d)^h f(z)$$

for h an ~~odd~~ even positive integer. We may again combine our operator

$$\iint_H \frac{y'^k}{(\bar{z}-\bar{z}')^k} f(z') \frac{dx'dy'}{y'^2}$$

(after having reduced the domain of integration to a fundamental domain in a corresponding way as when we passed from $k(z,z')$ to the kernel $K(z,z';\chi)$ previously) with the Hecke operators T_n with which it commutes. The trace formula we obtain in this case gives us the trace of the Hecke operator T_n applied to the $f(z)$ that are cusp-forms, namely for k even > 2,

$$\sigma(T_n) = - \sum_{0 \le m < 2\sqrt{n}}{}' H(4n-m^2)\,\frac{\eta_m^{k-1} - \bar{\eta}_m^{k-1}}{\eta_m - \bar{\eta}_m}$$

$$- \sum_{\substack{d \mid n \\ d \le \sqrt{n}}}{}' d^{k-1} + \frac{k-1}{12}\,\delta(\sqrt{n})\, n^{\frac{k}{2}-1},$$

where $\delta(x) = 1$ or 0 according as x is an integer or not. Σ' means that whenever the summation variable equals one of the limits stated the corresponding term is taken with weight $\frac{1}{2}$ (that is for $m = 0$ in the first sum and for $d = \sqrt{n}$ if n is a square in the second term. Further

$$\eta_m = \frac{m + i\sqrt{4n - m^2}}{2}$$

and $H(d)$ denotes the number of inequivalent classes of quadratic forms with determinant d, counted in the usual way. For $k = 2$ the result modifies ~~into~~ and gives the classical Kronecker class number relation, whereas for $k = 4, 6, 8, 10, 14$ the left hand side is zero since there are no analytic cuspforms. For $k = 12$, the left hand side is the so-called Ramanujan function $\tau(n)$.

Finally, in carrying the program indicated in the introductory lectures out for higher dimensional analogs as we have here done for the hyperbolic plane, there are two stages to overcome. The first is to find the explicit formulation of the trace formula for the ~~non~~-compact case. The difficulties here are essentially not too hard and can be overcome. The second stage, namely the treatment of the case of a noncompact fundamental domain with finite volume, however is far more difficult and can at present only be done in special cases, as for instance the general n-dimensional hyperbolic space (it turns out that the case n odd in some respects is simpler than n even). The cases that we can handle completely have essentially that in common that the non-compact part of D is pointlike, that is, it has ~~no~~ dimension zero.