§ 6.

The non-compact case, preparations, the trace formula in ~~one~~ the non-singular case. ~~The Eisenstein-series~~

We have earlier treated the case of the compact fundamental domain and proved the trace-formula for that case. In considering groups with non-compact fundamental domain, we will have to retain the basic restriction that the area of the fundamental domain in the invariant metric be finite.

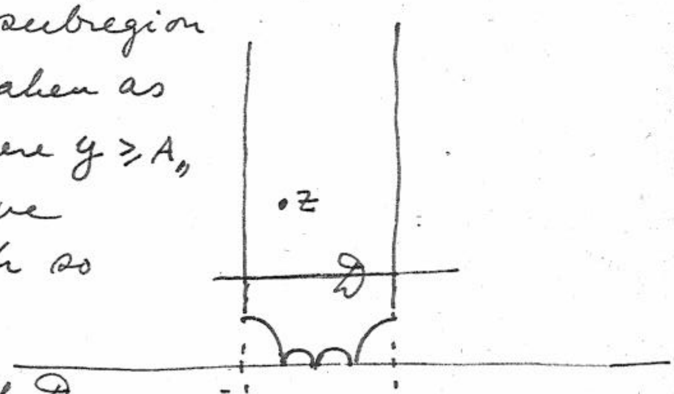
It is then easy to show that non-compactness arises only when there are parabolic transformations in the group Γ, and "cusps", that is vertices where the angle is zero, in the fundamental polygon D. The cusps are the fixpoints of the parabolic transformations, and two cusps are considered equivalent if one of them is carried into the other by some transformation of Γ. It is easily seen that the number of inequivalent cusps is finite, and the same is the case with the number of inequivalent "primitive" parabolic transformations.

Here by "primitive" we mean that the parabolic transformation S should not be a power with exponent > 1 of any other transformation in Γ. It is easily seen that when we look at equivalence with respect to G (the group of all motions of the hyperbolic plane), every parabolic transformation S is either equivalent to $z \to z+1$ or to $z \to z-1$. We will in the following restrict the notion of "primitive" parabolic transformation further, to apply only to those that in addition to the previous condition also is equivalent within G to $z \to z+1$. Thus when S is primitive S^{-1} will not be so.

Our first object will be to study the behaviour of our kernel $K(z, z')$

$$= \sum_{M \subset \Gamma} \chi(M)\, k(z, Mz')$$

under the new conditions. It is easily seen that if the points z and z' are restricted to a compact subregion of the fundamental domain $\mathcal{D}$, our previous arguments for the compact case are still valid, as we merely used the fact that the images of $\mathcal{D}$ under Γ do not overlap, and not that they exhaust the hyperbolic plane completely. Therefore if $k(r)$ satisfies our previous conditions, $K(z, z')$ is still uniformly bounded and continuous, as long as z and z' are restricted to a compact subregion of $\mathcal{D}$. It remains therefore to study the behaviour of $K(z, z')$ if at least one or possibly both points z and z' tend towards the cusps. In the following we will assume that we have $\varkappa$ inequivalent cusps (and so $\varkappa$ inequivalent primitive parabolic transformations $S_1, \dots S_\varkappa$) in Γ. Without loss of generality we may assume that one of the points, say z, is confined to some non-compact subregion of $\mathfrak{D}$ that contains only one cusp, and that this cusp is at ∞ and the corresponding S is $Sz = z+1$. The other point z' is permitted to range freely over $\mathfrak{D}$. The subregion containing z may be taken as the subregion of $\mathfrak{D}$, where $y \geq A$, where A is some positive constant, large enough so that the curve $y = A$ cuts only the sides of $\mathfrak{D}$ that pass through the cusp ($A = 1$, will actually be sufficient always under our assumption about S). Writing now



$$(6.1) \quad K(z,z') = \sum_{M \subset \Gamma} \chi(M)\, k(z, Mz') =$$

$$= \sum_{n=-\infty}^{\infty} \chi^n(S)\, k(z, z'+n) + \sum_{M \subset \Gamma}{}' \chi(M)\, k(z, Mz'),$$

we shall first investigate the second term on the right-hand side, and show that this still remains uniformly bounded under our previous assumptions about $h(r)$ or $k(t)$ $\supset$: $k(t) \leq \frac{A_2}{(2+t)^{1+\varepsilon}}$ for some positive constants A_2, and ε. We have, ~~rewriting~~

$$(6.2) \quad \sum_{\Pi \subset \Gamma}{}' |k(z, Mz')| = \sum_{M \,(\mathrm{mod}\, S)_{\text{left}}}{}' \sum_{n=-\infty}^{\infty} |k(z, S^n M z')|$$

where by $M \,(\mathrm{mod}\, S)_{\text{left}}$ we mean that we sum over a complete system of transformations M that do not differ by a power of S on the left side, and the dash Σ' still indicates that M is not equal to a power of S. Then denoting $Mz' = x'_M + i y_M$ we have

$$|k(z, S^n M z')| = |k(z, x'_M + n + i y_M)|$$

$$= \left| k\left(\frac{y}{y'_M} + \frac{y'_M}{y} - 2 + \frac{(x - x'_M - n)^2}{y'_M y} \right) \right|$$

$$< \frac{A_2 \, {y'_M}^{1+\varepsilon}}{\left(y + \frac{(x - x'_M - n)^2}{y} \right)^{1+\varepsilon}} .$$

Thus

$$\sum_{n=-\infty}^{\infty} |k(z, S^n M z')| < A_2 \, {y'_M}^{1+\varepsilon} \sum_{n=-\infty}^{\infty} \frac{1}{\left(y + \frac{(x - x'_M - n)^2}{y} \right)^{1+\varepsilon}}$$

$$< 2 A_2 \, {y'_M}^{1+\varepsilon} \sum_{n=0}^{\infty} \frac{1}{\left(y + \frac{n^2}{y} \right)^{1+\varepsilon}} .$$

Here because $y > A_1$

$$\sum_{n=0}^{\infty} \frac{1}{\left(y + \frac{n^2}{y} \right)^{1+\varepsilon}} < y^{-1-\varepsilon} + \int_0^{\infty} \frac{du}{\left(y + \frac{u^2}{y} \right)^{1+\varepsilon}} < A_3 \, y^{-\varepsilon},$$

so that

$$\sum_{n=-\infty}^{\infty} |k(z, S^n M z')| < A_4\, y^{-\varepsilon}\, y_M'^{\,1+\varepsilon},$$

(6.2) now gives

$$\sum_{M \subset \Gamma}' |k(z, Mz')| < A_4\, y^{-\varepsilon} \sum_{M \,(\mathrm{mod}\, S)_{\mathrm{left}}}' y_M'^{\,1+\varepsilon},$$

Using the result that will be proved in the next paragraph (§7 p.21), that

$$(6.2a) \quad \sum_{M \,(\mathrm{mod}\, S)_{\mathrm{left}}}' y_M'^{\,1+\varepsilon}$$

is uniformly bounded ~~=~~, (and even tends to zero if z' approaches a cusp), we obtain ~~that~~ that

$$(6.3) \quad \sum_{M \subset \Gamma}' |k(z, Mz')| = O(1).$$

~~Hence~~

It remains to look at the first term on the right-hand side of (6.1). We will now introduce a stronger assumption on $k(t)$ namely that $k(t)$ is of bounded total variation or $\int_0^\infty |d\,k(t)| < \infty$. This will be so if $h(r)$ instead of the previous condition $O\left(\frac{1}{|r|^{2+\varepsilon}}\right)$ as $|r| \to \infty$, satisfies the stronger condition

$$(6.4) \quad h(r) = O\left(\frac{1}{|r|^{3+\varepsilon}}\right) \quad \text{as } |r| \to \infty \text{ in}$$

so in the strip $|J(r)| < \frac{1}{2} + \varepsilon$. We distinguish two cases

(a) $\chi(S) \neq 1$, then we have ~~$\chi(S) = e^{i\alpha}$ where $\frac{\alpha}{2\pi}$ is not integral, and so~~

$$\sum_{n=-\infty}^{\infty} \chi^n(S)\, k(z, z'+n) = \sum_{n=-\infty}^{\infty} \chi(S)^n\, k\left(\frac{(y-y')^2+(x-x'-n)^2}{yy'}\right)$$

or

$$\left|\left(1-\chi(S)\right)\sum_{n=-\infty}^{\infty} \chi^n(S)\, k(z, z'+n)\right| =$$

$$\left|\sum_{n-\infty}^{\infty} \chi(S)^n \left(k\left(\frac{(y-y')^2+(x-x'-n)^2}{yy'}\right) - k\left(\frac{(y-y')^2+(x-x'-n+1)^2}{yy'}\right)\right)\right|$$

$$\leq 2 \int_0^\infty |d\, k(t)|\,,$$

So that in this case $K(z,z')$ is uniformly bounded.

(b) $\chi(S) = 1$, then we have

$$\sum_{n=-\infty}^{\infty} \chi^n(S)\, k(z,z'+n) = \sum_{n=-\infty}^{\infty} \left(k(z,z'+n) - \int_n^{n+1} k(z,z'+u)\,du\right)$$

$$+ \int_{-\infty}^{\infty} k(z,z'+u)\,du\,,$$

here again the first term on the right hand side is seen to be uniformly bounded because k is of bounded total variation, for the second term we get

$$\int_{-\infty}^{\infty} k(z, z'+u)\, du = \int_{-\infty}^{\infty} k\left(\frac{(y-y')^2}{yy'} + \frac{v^2}{yy'}\right) dv$$

$$= \sqrt{yy'} \int_{-\infty}^{\infty} k\left(\frac{(y-y')^2}{yy'} + x^2\right) dx = \sqrt{yy'}\; Q\left(\frac{(y-y')^2}{yy'}\right)$$

$$= \sqrt{yy'}\; g\left(\log \frac{y}{y'}\right).$$

Thus in this case $K(z,z') - \sqrt{yy'}\, g(\log \frac{y}{y'})$ is uniformly bounded. But it is seen that $K(z,z')$ itself is not, as $\sqrt{yy'}\, g(\log \frac{y}{y'})$ tends to infinity as $\sqrt{yy'}$ if z and z' tend towards the cusp ∞ at about the same rate (keeping $|\log \frac{y}{y'}|$ bounded). However since $g(u) = O(e^{-(\frac{1}{2}+\varepsilon)|u|})$ we see that $K(z,z')$ remain ~~is~~ bounded except if z' ~~also~~ and z both tends toward the cusp at ∞.

~~Thus we have~~

~~Lemma 6.1. If $h(r)$ is regular for $|\Im(r)| \le \frac{1}{2}+\varepsilon$ and $h(r) = O(\frac{1}{|r|^{3+\varepsilon}})$ in this strip, then the kernel~~

$$\cancel{K(z,z') = \sum_{M \in \Gamma} \chi(M)\, k(z, Mz')}$$

~~remains uniformly bounded, except when z and z' both approach the same ~~the same~~ cusp of the fundamental domain and ~~that~~ the primitive parabolic transformations leaving~~

If we have $\varkappa_1$ cusps $\xi_1, \dots, \xi_{\varkappa_1}$ of the fundamental domain D for which the ~~corresponding~~ primitive parabolic S_i leaving ξ_i fixed has $\chi(S_i) = 1$, for $i = 1, \dots \varkappa_1$ whereas for ~~the~~ other cusps $\chi(S) \neq 1$. Then denoting by z_i the transformation of z (of the form $\frac{-\lambda_i}{z - \xi_i}$) that brings the cusp ξ_i to ∞ and so that $z \to Sz$ corresponds to $z_i \to z_i + 1$) then we have, writing $z_i = x_i + i y_i$ and $z'_i = x'_i + i y'_i$, that

$$(6.5) \quad K(z, z') - \sum_{i=1}^{\varkappa} \sqrt{y_i y'_i}\; g\left(\log \frac{y_i}{y'_i}\right)$$

<u>is uniformly bounded for z and z' in our fundamental domain D.</u>

More generally if χ is a representation of Γ by $n \times n$ unitary matrices, and $|\chi(S_i) - E| = 0$ for $i = 1, \dots \varkappa_1$, whereas $|\chi(S) - E| \neq 0$ for the other cusps of D, ~~we have~~ and writing of 1 is an eigenvalue of $\chi(S_i)$ with multiplicity $0 < \nu_i \le n$, so that

$$\chi(S_i) = U_i \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \ddots 1 & & & \\ & & & \varepsilon_{\nu_i+1} & & \\ & & & & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} \bar{U}_i^{-1}$$

where U_i is a unitary matrix, and $\varepsilon_{\nu_i+1}, \ldots \varepsilon_n$ are the other eigenvalues of $\chi(S_i)$; and denoting by E_ν the diagonal matrix whose first ν diagonal elements are 1 while the rest are zero, we have that

$$(6.5') \quad K(z, z') - \sum_{i=1}^{\kappa} U_i \, E_{\nu_i} \, \bar{U}_i' \sqrt{y_i y_i'} \; g\left(\log \frac{y_i}{y_i'}\right)$$

is uniformly bounded for z and z' in $\mathcal{D}$.

~~For the case when~~

~~We shall in the future call a cusp for which $\chi(S) = 1$ (or in the general case~~ $\cancel{\chi(S) \neq E}$ ~~$|\chi(S) - E| = 0$) for "singular with respect to χ". Thus we have in particular that if all cusps are non-singular with respect to χ~~

We shall in the future say that χ is singular with respect to a cusp if $\chi(S) = 1$ (or in the general case $|\chi(S) - E| = 0$) for the primitive parabolic S that leaves the cusp fixed. In particular if χ is non singular with respect to all cusps, we call χ <u>non singular</u>.

From the preceding it is now easy to extend the trace formula to the non compact case if χ is non singular. We get as before that

$$(6.6) \quad \sum h(r) = 2 \iint_{\mathcal{D}} \sigma(K(z, z)) \frac{dx\,dy}{y^2}$$

under the preliminary assumption that $h(r)$ is a product of two other functions satisfying (6.4) or what is the same

$$(6.7) \quad h(r) = O\left(\frac{1}{|r|^{6+\varepsilon}}\right),$$

and regular in some strip $|\mathcal{I}(r)| < \frac{1}{2}+\varepsilon$.

In the computation of the right-hand side of (6.6), we can not any more split up completely into single terms, namely we must for each primitive parabolic transformation S, keep the terms

$$\sum_{n=-\infty}^{\infty}{}' \chi(S)^n k(z, S^n z) \qquad \left(\textstyle\sum' \text{ means } n \neq 0\right)$$

together, whereas the other terms (identity transformation, elliptic, hyperbolic) are treated ~~ass~~ as before. Taking for simplicity the case of onedimensional χ, we get combining the terms where S is equivalent to say, $z+1$ within Γ, that these contribute, writing $\chi(S) = e^{i\alpha}$, an amount

$$(6.8) \quad 2\iint_{\substack{|x| \le \frac{1}{2} \\ y > 0}} \sum_{-\infty}^{\infty}{}' \cancel{\chi(S)}^{\,in\alpha} k(z, z+n)\, \frac{dx\,dy}{y^2} =$$

~~We approx compute this integral by taking first, for~~

$$\cancel{2\iint_{\substack{|x| \le \frac{1}{2} \\ 0 < y < \frac{1}{\varepsilon}}} \sum_{-\infty}^{\infty}{}' e^{in\alpha} k(z, z+n)\, \frac{dx\,dy}{y^2}}$$

$$= 2 \int_0^\infty \sum_{-\infty}^{\infty}{}' e^{in\alpha} k\left(\frac{n^2}{y^2}\right) \frac{dy}{y^2} =$$

$$= 4 \int_0^\infty \sum_{n=1}^{\infty} \cos n\alpha \; k(n^2 u^2)\, du ,$$

We compute this by first taking the integral

$$\int_\varepsilon^\infty \sum_{n=1}^{\infty} \cos n\alpha \; k(n^2u^2)\, du =$$

$$= \sum_{n=1}^{\infty} \cos n\alpha \int_\varepsilon^\infty k(n^2u^2)\, du =$$

$$= \sum_{n=1}^{\infty} \frac{\cos n\alpha}{n} \int_{n\varepsilon}^\infty k(u^2)\, du =$$

$$= \int_\varepsilon^\infty k(u^2) \left(\sum_{1 \le n \le \frac{u}{\varepsilon}} \frac{\cos n\alpha}{n} \right) du$$

Now for fixed α, $e^{i\alpha} \neq 1$ we have

$$\sum_{1 \le n \le \frac{u}{\varepsilon}} \frac{\cos n\alpha}{n} = \log \frac{1}{|1-e^{i\alpha}|} + O\left(\sqrt{\frac{\varepsilon}{u}}\right)$$

Thus we get

$$\int_\varepsilon^\infty \sum_{n=1}^{\infty} \cos n\alpha \; k(n^2u^2)\, du = \log\frac{1}{|1-e^{i\alpha}|} \int_\varepsilon^\infty k(u^2)\, du$$

$$+ O\left(\sqrt{\varepsilon} \int_0^\infty |k(u^2)| \frac{du}{\sqrt{u}} \right).$$

making $\varepsilon \to 0$, we get

$$\int_0^\infty \sum_{n=1}^\infty \cos n\alpha \, k(n^2u^2)\,du = \log\frac{1}{|1-e^{i\alpha}|} \int_0^\infty k(u^2)\,du,$$

but

$$\int_0^\infty k(u^2)\,du = \tfrac{1}{2}Q(0) = \tfrac{1}{2}g(0)$$

So that from (6.8) the contribution $\underbrace{}_{\text{on the righthand side of the trace formula}}$ of the parabolic transformations in Γ that are equivalent to powers of a particular primitive parabolic transformation S_i, becomes

$$(6.9) \qquad 2\log\frac{1}{|1-\chi(S_i)|}\; g(0),$$

or for representations by $n\times n$ unitary matrices one gets in the same way

$$(6.9') \qquad 2\log\frac{1}{\|E-\chi(S_i)\|}\; g(0)$$

The trace formula $\underbrace{}_{\text{for the nonsingular } \chi}$ can now be written out by adding the new terms to the previous ones, and as before one easily gets rid of the restriction (6.7) introduced and obtains that the formula is valid under the weaker conditions: $h(r)$ regular for $|\Im(r)| < \frac{1}{2}+\varepsilon$ and

$$h(r) = O\left(\frac{1}{|r|^{2+\varepsilon}}\right)$$

for some positive ε.

The consequences about the $Z_\rho(s, \chi)$ can be obtained as in the compact case, the new terms only causing a slight modification of the functional equation, but not influencing the multiplicities of the "trivial" zeros (as the terms coming from elliptic transformation classes did).

§ 7.

Preparations for the case of a singular χ, The Eisenstein series.

In the case that remains, when χ is singular with respect to one or more of the cusps, it is easily seen that we cannot have a trace formula quite like the one we had in the former cases, since the trace does not exist due to the singular behaviour of the kernel $K(z, z')$ when z and z' both tend towards one of the cusps with respect to which χ is singular. We shall however in the next paragraph see that the "singular part" of the kernel can be removed, since it is due to the existence of a continuous spectrum of eigenfunctions for our problem, that are not square integrable over the fundamental domain. Our task is therefore first to study ~~these~~ these eigenfunctions belonging to the continuous spectrum, and from this we will be able to modify $K(z,z')$ in such a way that the continuous spectrum drops out, and only the discrete spectrum remains. The computation of the trace for this modified kernel can then be made, and this will yield what we will call the traceformula in the singular case.

We will again for simplicity suppose that we have a one-dimensional representation of Γ by $\chi(M)$, and that the fundamental domain $\mathfrak{D}$ has ~~the~~ $\varkappa$ inequivalent cusps at

$\xi_1, \xi_2 \dots, \xi_\varkappa$ with corresponding $S_1,$

$\dots \quad S_\varkappa$, primitive parabolic transformations,

and that $\chi(S_1) = \dots = \chi(S_{\varkappa_1}) = 1$, while $\chi(S_i) \neq 1$ for $\varkappa_1 < i \leq \varkappa$. Let us for any

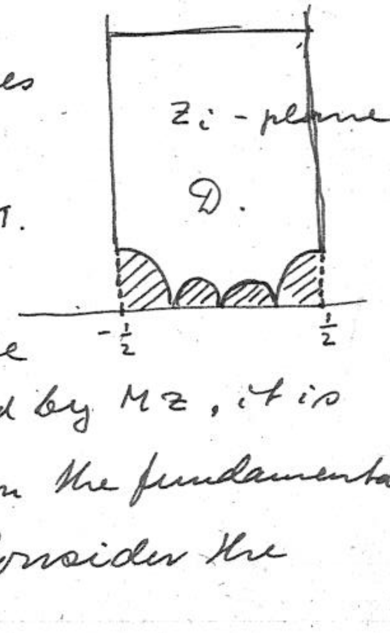
ξ_i, S_i, write $z_i = \dfrac{-\lambda_i}{z - \xi_i}$; λ_i being so chosen that $z \to S_i z$ corresponds to $z_i \to z_i + 1$, furthermore let $M_\xi^{(i)} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ be the transformation $z_i \to M_\xi^{(i)} z_i$ that corresponds to $z \to Mz$. We now form the series for $s = \sigma + it$; $\sigma > 1$, for $i = 1, 2, \ldots \varkappa_1$

$$(7.1)\quad E_i(z, s; \chi) = \sum_{M \,(\mathrm{mod}\, S_i)_{\mathrm{left}}} \overline{\chi}(M) \left\{ \mathcal{Y}\left(-\frac{\lambda_i}{Mz - \xi_i} \right) \right\}^s$$

$$= \sum_{M_z^{(i)} (\mathrm{mod} \left(\begin{smallmatrix}1 & 1 \\ 0 & 1\end{smallmatrix}\right))_{\mathrm{left}}} \overline{\chi}(M_{z}) \frac{y_i^s}{|c_i z_i + d_i|^{2s}}$$

$$= \sum_{c_i, d_i} \overline{\chi}_{c_i, d_i} \frac{y_i^s}{|c_i z_i + d_i|^{2s}}.$$

We shall first have to prove the absolute convergence of this series, and uniform convergence in any compact domain. This fact is simply deduced as follows; taking absolute values we get the series



$$(7.2)\quad \sum_{c_i, d_i} \frac{y_i^\sigma}{|c_i z_i + d_i|^{2\sigma}}, \text{ with } \sigma > 1.$$

Now because of the invariance of this series when z is replaced by Mz, it is enough to study the behaviour in the fundamental domain in the z_i-plane. Consider the integral

$$\iint y_i^\sigma \frac{dx_i dy_i}{y_i^2}$$

over the shaded region on the figure, namely ~~the points~~ the part of the strip $-\frac{1}{2} \le x_i \le \frac{1}{2}$, $y_i \ge 0$, that remains after the fundamental domain has been taken out. The integral is obviously finite since $\sigma > 1$. Mapping everything back into $\mathfrak{D}$, ~~by~~ ~~using~~ we get, that the integral equals

$$\iint_{\mathfrak{D}} \left(\sum_{c_i \neq 0, d_i} \frac{y_i^{\sigma}}{|c_i z_i + d_i|^{2\sigma}} \right) \frac{dx_i\, dy_i}{y_i^2}$$

since the terms are positive, this proves that

$$\sum_{\substack{c_i, d_i}} \frac{y_i^{\sigma}}{|c_i z_i + d_i|^{2\sigma}}$$ $\underbrace{\text{converges}}_{\text{almost everywhere}}$ to an integrable

function ~~in $\mathfrak{D}$~~. The convergence in every point then follows because each term is an eigenfunction of our operators, namely consider a point $z_i^{(0)}$ and denote by C_ρ a geodesic circle around it with radius ρ (in the hyperbolic metric), Then we have

$$\frac{y_i^{(0)\,\sigma}}{|c_i z_i^{(0)} + d_i|^{2\sigma}} = \mu(\rho, \sigma) \iint_{C_\rho} \frac{y_i^{\sigma}}{|c_i z_i + d_i|^{2\sigma}} \frac{dx_i\, dy_i}{y_i^2}$$

where $\mu(\rho,\sigma)$ is a positive constant depending only on ρ and σ. From this the convergence at each point, and also the uniform convergence in any compact domain becomes evident.

We now want to study the behaviour of $E_i(z, s; \chi)$; first of all $E_i(z, s; \chi)$ is an eigenfunction of $y^2\Delta$ our fundamental operator, since each term is annihilated by $y^2\Delta + s(1-s)$ and so

$$(7.3) \qquad \left(y^2\Delta + s(1-s)\right) E_i(z, s; \chi) = 0$$

Furthermore we have evidently

$$(7.4) \qquad E_i(Mz, s; \chi) = \bar{\chi}(M)\, E_i(z, s; \chi)$$

We need to know the ~~behaviour~~ behaviour of $E_i(z, s)$ when z approaches the cusps of the fundamental domain. This is obtained by first finding the fourier expansions of $E_i(z, s; \chi)$ in terms of x_j $(z_j = x_j + i y_j)$ for $1 \le j \le n$. Let us first look at the expansion of $E_i(z, s; \chi)$ in terms of x_i, since $E_i(z, s) = E_i(S_i z, s; \chi)$, we have

$$(7.5) \qquad E_i(z, s; \chi) = \sum_{m=-\infty}^{\infty}{}' \, \alpha_m^{(i,i)}(\overset{s}{\underset{\vee}{y}}_i; \chi)\, e^{2\pi i m x_i}$$

where

$$(7.6) \qquad \alpha_m^{(i,i)}(\overset{s}{\underset{\vee}{y}}_i; \chi) = \int_0^1 E_i(z, s; \chi)\, e^{-2\pi i m x_i}\, dx_i =$$

$=$ [illegible]

We now write from (7.1)

$$E_i(z,s;\chi) = y_i^{s} + \sum_{c_i \neq 0} \sum_{0 \le d_i < |c_i|} \overline{\chi}_{c_i,d_i} \sum_{n=-\infty}^{\infty} \frac{y_i^{s}}{|c_i(z_i+n)+d_i|^{2s}}$$

$$= y_i^{s} + \sum_{c_i \neq 0} \sum_{0 \le d_i < |c_i|} \frac{\overline{\chi}_{c_i,d_i}}{|c_i|^{2s}} \sum_{n=-\infty}^{\infty} \frac{y_i^{s}}{\left|z_i+n+\frac{d_i}{c_i}\right|^{2s}}.$$

(7.6) now gives taking first $n=0$,

$$\alpha_0^{(i,i)}(y_i;\chi) = y_i^{s} + \sum_{c_i \neq 0} \sum_{0 \le d_i < |c_i|} \frac{\overline{\chi}_{c_i,d_i}}{|c_i|^{2s}} \int_{-\infty}^{\infty} \frac{y_i^{s}\,dx_i}{|z_i|^{2s}}$$

Here

$$\int_{-\infty}^{\infty} \frac{y_i^{s}\,dx_i}{|z_i|^{2s}} = \int_{-\infty}^{\infty} \frac{y_i^{s}}{(x_i^2+y_i^2)^{s}}\,dx_i =$$

$$= y_i^{1-s} \int_{-\infty}^{\infty} \frac{du}{(u^2+1)^{s}} = \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, y_i^{1-s},$$

So that

$$(7.7)\quad \alpha_0^{(i,i)}(y_i;s;\chi) = y_i^{s} + \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, L_0^{(i,i)}(s;\chi)\, y_i^{1-s},$$

where for integral n,

$$(7.8)\quad L_n^{(i,i)}(s;\chi) = \sum_{c_i \neq 0} \frac{1}{|c_i|^{2s}} \sum_{0 \le d_i \le |c_i|} \overline{\chi}_{c_i,d_i}\, e^{2\pi n i \frac{d_i}{c_i}}.$$

Next take $n \neq 0$; then the above expression for $E_i(z,s;\chi)$ gives

$$(7.9)\quad \alpha_m^{(i,i)}(y_i;\chi) = \sum_{c_i \neq 0} \frac{1}{|c_i|^{2s}} \sum_{0 \leq d_i \leq |c_i|} \bar{\chi}_{c_i, d_i}\, e^{2\pi i m \frac{d_i}{c_i}} \,.$$

$$\cdot \int_{-\infty}^{\infty} \frac{y_i^{\,s}\, e^{-2\pi m i x_i}}{|z_i|^{2s}}\, dx_i \,.$$

Here

$$\int_{-\infty}^{\infty} \frac{y_i^{\,s}}{|z_i|^{2s}}\, e^{-2\pi m i x_i}\, dx = y_i^{1-s} \int_{-\infty}^{\infty} \frac{e^{-2\pi m i y_i u}}{(u^2+1)^s}\, du$$

$$= \frac{y_i^{1-s}}{\Gamma(s)} \int_0^{\infty} dt \int_{-\infty}^{\infty} t^{s-1} e^{-t(u^2+1) - 2\pi m i y_i u}\, du =$$

~~inverting here order of integrations we get further~~

$$= \frac{y_i^{1-s}}{\Gamma(s)} \sqrt{\pi} \int_0^{\infty} t^{s-\frac{3}{2}}\, e^{-t - \frac{\pi^2 m^2 y_i^2}{t}}\, dt =$$

$$= \frac{\sqrt{y_i}\; \sqrt{\pi}^{\,s}\, |m|^{s-\frac{1}{2}}}{\Gamma(s)} \int_0^{\infty} t^{s-\frac{1}{2}}\, e^{-\pi |m| y_i (t + \frac{1}{t})}\, \frac{dt}{t}$$

Thus from (7.9) we get

$$(7.10)\quad \alpha_m^{(i,i)}(y_i; \overset{s}{\wedge} \chi) = \frac{\pi^s |m|^{s-\frac{1}{2}}}{\Gamma(s)}\, L_m^{(i,i)}(s, \chi) \sqrt{y_i} \int_0^{\infty} t^{s-\frac{1}{2}} e^{-\pi |m| y_i (t + \frac{1}{t})}\, \frac{dt}{t}$$

The expansion (7.5) is thus determined.

More generally we can determine the

fourier expansion of $E_i(z,s)$ after x_j where $j \neq i$, if we first assume $1 \le j \le \varkappa_1$ we get

$$(7.11)\quad E_i(z,s;\chi) = \sum_{-\infty}^{\infty} \alpha_m^{(i,j)}(y_j, \overset{s}{\vee};\chi)e^{2\pi i m x_j}$$

where now

$$(7.12)\quad \alpha_0^{(i,j)}(y_j, \overset{s}{\vee};\chi) = \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)} L_0^{(i,j)}(s;\chi)\, y_j^{1-s}$$

and for $m \neq 0$

$$(7.13)\quad \alpha_m^{(i,j)}(y_j, \overset{s}{\vee};\chi) = \frac{\pi^s |m|^{s-\frac{1}{2}}}{\Gamma(s)} L_m^{(i,j)}(s;\chi)\sqrt{y_j}\int_0^{\infty} t^{s-\frac{1}{2}} e^{-\pi|m|y_j(t+\frac{1}{t})}\frac{dt}{t}$$

where $L_m^{(i,j)}(s,\chi)$ is a series of a similar but slightly more complicated form as $L_m^{(i,i)}(s;\chi)$.

If finally $\varkappa_1 < j \le \varkappa$, we have $\chi(S_j) = e^{2\pi i \alpha_j} \neq 1$ therefore the fourier expansion has the form,

$$(7.14)\quad E_i(z,s;\chi) = \sum_{-\infty}^{\infty} \alpha_m^{(i,j)}(y_j, \overset{s}{\vee},\chi)e^{2\pi i(m+\alpha_j)x_j}$$

where

$$(7.15)\quad \alpha_m^{(i,j)}(y_j, \overset{s}{\vee};\chi) = \frac{\pi^s |m+\alpha_j|^{s-\frac{1}{2}}}{\Gamma(s)} L_{m,\alpha_j}^{(i,j)}(s,\chi)\sqrt{y_j}\int_0^{\infty} t^{s-\frac{1}{2}} e^{-\pi|m+\alpha_j|y_j(t+\frac{1}{t})}\frac{dt}{t}$$

and $L_{m,\alpha_j}^{(i,j)}(s,\chi)$ is a dirichlet-series of a similar but slightly more complicated form than the above $L_m^{(i,j)}(s,\chi)$.

From these fourier expansions the behaviour of $E_i(z, s, \chi)$ at any of the cusps can be readily determined. From the form of the terms (7.10), (7.13) or (7.15) it is seen that the contribution of these in any case goes exponentially to zero ~~if z~~ (that is as $e^{-\beta y_j}$, with some positive β), as z approaches a cusp ξ_j. Therefore we have the results:

a) $$E_i(z, s; \chi) = y_i^{s} + \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)} L_0^{(i,i)}(s,\chi)\, y_i^{1-s} + O(e^{-\beta y_i})$$ as z approaches ξ_i (or $y_i \to \infty$)

b) $$E_i(z, s; \chi) = \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)} L_0^{(i,j)}(s,\chi)\, y_j^{1-s} + O(e^{-\beta y_j})$$ as z approaches ξ_j for $i \neq j$, $1 \leq j \leq \kappa_1$.

c) $$E_i(z, s; \chi) = O(e^{-\beta y_j})$$ as z approaches ξ_j for $\kappa_1 < j \leq \kappa$.

In case of a) and b) one easily sees that β may be taken as $2\pi - \varepsilon$, for any positive ε. In case of c) if $\chi(S_j) = e^{2\pi i \alpha_j}$, $|\alpha_j| \leq \frac{1}{2}$, $\alpha_j \neq 0$ we may take $\beta = 2\pi|\alpha_j| - \varepsilon$ for any positive ε.

In particular as an easy consequence we have the result used in the preceding paragraph that the expression (6.2a) is uniformly bounded.

We now want to prove ~~a few~~ two lemmas that will be usefull later

Lemma 7.1. For $1 \le i, j \le \varkappa_1$, we have

$$L_0^{(i,j)}(s,\chi) = L_0^{(j,i)}(s,\bar{\chi}).$$

~~or the matrix $(L^{(i,j)}(s))$ is a symmetric matrix.~~

This could have been seen directly, if we had obtained the explicit dirichlet series for $L^{(i,j)}(s)$. But it can also be obtained from Green's formula, in this way.

Consider the fundamental domain D and let us cut off the cusps by taking out the $\varkappa$ ~~dom~~ subdomains of D where $y_\nu \ge A_\nu$ for $\nu = 1, 2 \ldots, \varkappa$, where the A_ν are chosen so large that the curve $y_\nu = A_\nu$ cuts only the two sides of D that pass through the cusp ξ_ν, ~~and~~ and also that no two ~~none~~ of the subdomains have a point in common, call the compact domain remaining $\bar{D}$, and consider

$$0 = \iint_{\bar{D}} \left(E_i(z,s,\chi)\Delta E_j(z,s,\bar{\chi}) - E_j(z,s,\bar{\chi})\Delta E_i(z,s,\chi)\right)\frac{dx\,dy}{\mathfrak{Y}}$$

$$= \oint_{\mathfrak{C}} \left(E_i(z,s,\chi)\frac{\partial E_j(z,s,\bar{\chi})}{\partial n} - E_j(z,s,\bar{\chi})\frac{\partial E_i(z,s,\chi)}{\partial n}\right)d\sigma$$

where $\frac{\partial}{\partial n}$ is the normal derivative, and ~~the~~ $d\sigma$ the differential of the arclength on the boundary ℓ of $\overline{\mathfrak{D}}$. The contributions from the boundary of $\mathfrak{D}$ cancel out, because corresponding sides are traversed twice and with opposite sign of the normal derivative, so that only the integrals over the parts of the curves $y_\nu = A_\nu$ ~~inside~~ that belong to the boundary of $\overline{\mathfrak{D}}$. ~~Thus~~ we get using the fourier expansions for each cusp, that the contribution from the curves $y_\nu = A_\nu$ tend to zero as $A_\nu \to \infty$, except for $\nu = i$ or $\nu = j$. First assume that $i = j$, then we get

$$\left(y_i^{\,s} + \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, L_0^{i,i}(s,\chi)\, y_i^{\,1-s}\right) \frac{\partial}{\partial y_i}\left(y_i^{\,s} + \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, L_0^{(i,i)}(s,\bar{\chi})\, y_i^{\,1-s}\right)$$

$$-\left(y_i^{\,s} + \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, L_0^{(i,i)}(s,\bar{\chi})\, y_i^{\,1-s}\right) \frac{\partial}{\partial y_i}\left(y_i^{\,s} + \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, L_0^{(i,i)}(s,\chi)\, y_i^{\,1-s}\right)$$

$= 0$ for $y_i = A_i$, A_i arbitrary but sufficiently large. This implies after some reduction

$$(20\text{-}1)\quad \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\left(L_0^{(i,i)}(s,\chi) - L_0^{(i,i)}(s,\bar{\chi})\right) = 0,$$

so that our lemma is true for $i = j$.

Next let $i \neq j$, then we get

$$\left(y_i^{\,s} + \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, L_0^{(i,i)}(s,\chi)\, y_i^{\,1-s}\right) \frac{\partial}{\partial y_i}\left(\sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, \mathcal{L}_0^{(j,i)}(s,\bar{\chi})\, y_i^{\,1-s}\right)$$

$$-\sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, L_0^{(j,i)}(s,\bar{\chi})\, y_i^{\,1-s} \frac{\partial}{\partial y_i}\left(y_i^{\,s} + \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, L_0^{(i,i)}(s,\chi)\, y_i^{\,1-s}\right) +$$

$$+ \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,L_0^{(i,j)}(s,\bar{\chi})\,y_j^{1-s}\,\frac{\partial}{\partial y_j}\Big(y_j^{s} + \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,L_0^{(j,j)}(s,\bar{\chi})\,y_j^{1-s}\Big)$$

$$- \Big(y_j^{s} + \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,L_0^{(j,j)}(s,\bar{\chi})\,y_j^{1-s}\Big)\frac{\partial}{\partial y_j}\Big(\sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,L_0^{(i,j)}(s,\bar{\chi})\,y_j^{1-s}\Big) =$$

$= 0$; for $y_i = A_i$, $y_j = A_j$, both sufficiently large, but otherwise arbitrary. From this follows in a similar way the statement of the lemma for $i \neq j$.

Lemma 7.2., Any function $f(z)$, which for some s with $\Re s > 1$, satisfies the equation

(1) $(y^2 \Delta + s(1-s))\, f(z) = 0$,

and the condition

(2) $f(Mz) = \chi(M)\, f(z)$

for all $M \subset \Gamma$, and which furthermore in $\mathfrak{D}$ as z tends towards a cusp ξ_j ; $1 \leq j \leq \varkappa$ satisfies the relation

(3) $f(z) = O(e^{\varepsilon y_j})$

for every positive ε, is necessarily a linear combination of the functions

$E_i(z, s, \chi)$ for $1 \leq i \leq \varkappa$, .

This can be proved by looking at the fourier expansion of $f(z)$ at each with respect to the x_j. ~~[illegible]~~

For $\varkappa_1 < j \le \varkappa$, this fourier expansion has the form

$$f(z) = \sum_{m=-\infty}^{\infty} c_m^{(j)}(y_j)\, e^{2\pi i (m+\alpha_j) x_j}$$

here each term has to satisfy the equation (1) and also the relation (3) as $y_j \to \infty$. (1) gives an ordinary linear differential equation of second order for $c_m^{(j)}(y_j)$, and ~~the~~ the only solution that satisfies (3) is

$$c_m^{(j)}(y_j) = c_m^{(j)} \sqrt{y_j} \int_0^\infty t^{s-\frac{1}{2}} e^{-\pi |m+\alpha_j| y_j (t+\frac{1}{t})} \frac{dt}{t}$$

Therefore $f(z)$ will tend to zero as z tends to ξ_j for $\varkappa_1 < j \le \varkappa$. For $1 \le j \le \varkappa$, we get that the fourier-expansion by the same reasoning has the form

$$f(z) = a_j y_j^s + b_j y_j^{1-s} + \sum_{m=-\infty}^{\infty}{}' c_m^{(j)} e^{2\pi i m x_j} \sqrt{y_j} \int_0^\infty t^{s-\frac{1}{2}} e^{-\pi |m| y_j (t+\frac{1}{t})} \frac{dt}{t}$$

where all terms except $a_j y_j^s$ tend to zero as $z \to \xi_j$ (or $y_j \to \infty$). From this it follows that the function

$$\tilde{f}(z) = f(z) - \sum_{j=1}^{\varkappa_1} a_j E_j(z, s, \varkappa)$$

which also satisfies (1) and (2), is square -

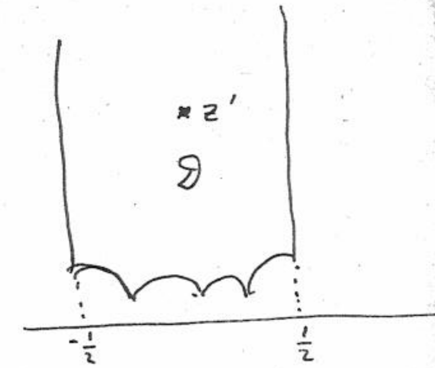
integrable in $\mathcal{D}$ or

$$\iint_{\mathcal{D}} |\tilde{f}|^2 \frac{dx\,dy}{y^2} < \infty .$$

But from the fact that $y^2\Delta$ is an elliptic operator, it is well-known that a square-integrable function satisfying (2) and being an eigenfunction

$(y^2\Delta + \lambda)\tilde{f} = 0$, where λ is not both real and non-negative, has to vanish identically. Since $s(1-s)$ for $R(s) > 1$ cannot be real and non negative, this proves that $\tilde{f}$ vanishes identically.

We have up to now only studied the functions $E_i(z, s; \chi)$ for $R(s) > 1$, where the series by which they were defined converge. It will be necessary for our purposes to show that they ~~can~~ be analytically continued in the whole s-plane except for poles, and so actually are meromorphic functions in s. ~~[illegible]~~ For simplicity we shall carry out this proof ~~only~~ in the case that $\chi(M)$ is identically 1, and the fundamental domain has only one cusp, ~~[illegible]~~ which we place at ∞. It will afterwards be briefly indicated how one proceeds in the general case.



Let z' be a point in $\mathcal{D}$, which we for simplicity at first assume to be in the interior. By means of ~~the~~ dirichlet's principle one can then show that there exist a function $G(z,z')$ which is harmonic in z throughout $\mathcal{D}$, except for a logarithmic singularity at $z = z'$, so that

(7.16) $\quad G(z,z') - \log\frac{1}{|z-z'|}$,

is regular and harmonic at $z = z'$, and with another logarithmic singularity (in the local uniformizing variable $e^{2\pi i z}$) at $z = i\infty$, so that

(7.18) $\quad G(z,z') + 2\pi y$

is regular in $w = e^{2\pi i z}$ around and in $w = 0$, and such that $G(z,z')$ and its normal derivative agree in corresponding points on the sides of $\mathcal{D}$ that correspond to each other. $G(z,z')$ will actually be the real part of an abelian integral of the third kind, with logarithmic singularities at z' and $i\infty$, and with purely imaginary periods.

It is clear that $G(z,z')$ can be continued throughout the interior of the upper complex halfplane, except at the points Mz' for $M \subset \Gamma$, where it has logarithmic singularities, and that

$$(7.18) \qquad G(Mz, z') = G(z, z')$$

Because of the arbitrariness involved in G in that any function depending on z' only, may be added, we norm G in such a way that

$$(7.19) \qquad G(z, z') + 2\pi y \to 0 \quad \text{as } z \to i\infty .$$

~~Hence consider the domain~~

Now let us have two points z' and z'' in D and consider the domain $\mathcal{D}^*$ left by cutting of from $\mathcal{D}$ the part $y > A$, for some large A, so that the cusp is removed, and furthermore removing two small circular discs around the points z' and z''. Consider the integral

$$\iint_{D^*} \left(G(z,z') \Delta G(z,z'') - G(z,z'') \Delta G(z,z') \right) dx\,dy = 0$$

~~and as~~, applying Green's formula to the left-hand side, we find that the integrals over the boundary cancels out, except for the integrals over the two circles and over the line segment $-\frac{1}{2} \leq x \leq \frac{1}{2}$, $y = A$. From (7.19) making A tend to ∞, we find that ~~this part~~ the contribution of the line segment $-\frac{1}{2} \leq x \leq \frac{1}{2}$, $y = A$ also vanishes.

Finally making the radii of the two circles tend to zero, and using (7.16) for $G(z,z')$ and $G(z,z'')$, we obtain

~~(7.20)~~ $$G(z'', z') = G(z', z'')$$

or writing again z for z'',

(7.20) $$G(z,z') = G(z',z).$$

We may then remove the restriction that z' lie in the interior of $\mathcal{D}$, by properly modifying the definition, so that $G(z,z')$ then have a logarithmic singularity at z' and the other points on the boundary of $\mathcal{D}$ that correspond to z'. Finally if z' is at a fixpoint of a primitive elliptic transformation of order m, (7.16) has to be modified, so that we require

(7.21) $$G(z,z') - m \log \frac{1}{|z-z'|}$$

to be regular at $z=z'$. Then all the other statements continue to hold, ~~and we have that~~

Defined in this way $G(z,z')$ is now harmonic in the upper half plane in both variables, except when the two points z and z' are equivalent under Γ, and it is also automorphic in both variables, or

(7.22) $$G(Mz, M'z') = G(z,z')$$

if M and M' belong to Γ.

We shall now obtain certain estimates for $G(z,z')$ as the points z and z' range over the fundamental domain, of most interest is the behaviour when one or both points approach the cusp. Let A be a positive constant so large that the line $y = A$ cuts the boundary of the fundamental domain $\mathfrak{D}$ only in the two sides passing through the cusp.

Suppose first that z and z' both lie below the line $y = A$; so that $y \leq A$, $y' \leq A$; denoting by $M_1, M_2, \ldots, M_\nu$ the transformations of Γ for which the image of the domain $\mathfrak{D}$ under M_i, $i = 1, 2, \ldots, \nu$, $M_i \mathfrak{D}$ has at least one boundary point in common with $\mathfrak{D}$ (not counting the cusp as a boundary point). Then we have obviously that

$$(7.23) \quad G(z,z') = \cancel{\sum_{\nu=1}} \log \frac{1}{|z-z'|} + \sum_{i=1}^{\nu} \log \frac{1}{|z - M_i z'|} + g_1(z,z')$$

where $g(z,z')$ is uniformly bounded for z, z' in $\mathfrak{D}$ and $y \leq A$, $y' \leq A$.

Next suppose that at least one point lies above the line $y = A$. ~~First assume that~~ Then we easily see from (7.16), (7.17), (7,19), (7.20) and the fact that $G(z,z')$ is periodic in both x and x' with period 1, that:

$$(7.24)\quad G(z,z') = \log \frac{1}{|e^{-2\pi i z} - e^{-2\pi i z'}|} + g_2(z,z'),$$

where $g_2(z,z')$ is a regular harmonic function of $w = e^{2\pi i z}$, $w' = e^{2\pi i z'}$ as long as z and z' lie in $\mathcal{D}$ (the cusp included), and at least one of the numbers $|w|$ or $|w'|$ is $\le e^{-2\pi A}$. Because of (7.19)

$$g_2(z,z') = 0 \text{ if } w \text{ or } w' \text{ equals zero.}$$

From this follows that when y or $y' \geq A$, we have

$$(7.25)\quad g_2(z,z') = O(|w\,w'|) = O(e^{-2\pi(y+y')}).$$

For $y > y'$, $y \geq A$, (7.24) and (7.25) give

$$(7.26)\quad G(z,z') = -2\pi y + O\left(e^{-2\pi|y-y'|}\log\left(2 + \frac{1}{|y-y'|}\right)\right),$$

similarly because of the symmetry for $y' > y$, $y' \geq A$,

$$(7.26')\quad G(z,z') = -2\pi y' + O\left(e^{-2\pi|y-y'|}\log\left(2+\frac{1}{|y-y'|}\right)\right).$$

Define now for z' in $\mathcal{D}$,

$$(7.27)\quad \tilde{G}(z,z') = G(z,z') + 2\pi y' \quad \text{for } y' \geq A,$$

and

$$(7.27')\quad \tilde{G}(z,z') = G(z,z') + 2\pi A \quad \text{for } y' \leq A.$$

Suppose that $u(z) = y^2 \Delta v(z)$; where $v(z)$ is automorphic under Γ, and has continuous partial derivatives up to second order, that satisfy the condition that they are $O(e^{2\pi y})$ as $y \to \infty$. Consider the domain $\mathcal{D}_\rho$ obtained from $\mathcal{D}$ by removing a small circular disc C_ρ with radius ρ around the point z^*, and consider the integral

$$(7.28) \quad \iint_{\mathcal{D}} \tilde{G}(z,z')\, u(z') \frac{dx'dy'}{y'^2},$$

writing here

$$\iint_{\mathcal{D}} \tilde{G}(z,z')\, u(z') \frac{dx'dy'}{y'^2} =$$

$$= \lim_{\rho \to 0} \iint_{\mathcal{D}_\rho} \left(\tilde{G}(z,z') \Delta v(z') - v(z') \Delta \tilde{G}(z,z') \right) \frac{dx'dy'}{y'^2}$$

and applying Greens formula taking into account the fact that the integrals over the boundary of $\mathcal{D}$ cancel out, and the discontinuity of the normal derivative of $\tilde{G}$ along the line $y' = A$, we obtain *)

$$\iint_{\mathcal{D}} \tilde{G}(z,z')\, u(z') \frac{dx'dy'}{y'^2} = \lim_{\rho \to 0} \int_{C_\rho} \left(\tilde{G}(z,z') \frac{\partial v(z')}{\partial n} - v(z') \frac{\partial \tilde{G}(z,z')}{\partial n} \right) ds$$

$$+ 2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} v(x'+iA)\, dx' = -2\pi v(z^*) + 2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} v(x'+iA)\, dx',$$

using (7.16) when taking the limit as $\rho \to 0$.

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*) We suppose for simplicity here that the line $y' = A$ does not cross the circle C_ρ.

Thus the operation

$$(7.29) \qquad \frac{1}{2\pi} \iint_{D} \tilde{G}(z,z')\, u(z')\, \frac{dx'dy'}{y'^2}$$

inverts the operation $y^2\Delta$, conversely one can show by the standard methods ~~for~~ used in the theory of partial differential equations, in connection with the Greens function, that if $u(z')$ is continuous and automorphic under Γ, and satisfies our condition $u(z') = O(e^{2\pi y'})$ for $y' \to \infty$, then the function

$$v(z) = \frac{1}{2\pi} \iint_{D} \tilde{G}(z,z')\, u(z')\, \frac{dx'dy'}{y'^2},$$

which is obviously automorphic in z, since $\tilde{G}(z,z')$ is, satisfies the relation

$$y^2 \Delta v(z) = u(z), \text{ and } \int_{-\frac{1}{2}}^{\frac{1}{2}} v(x+iA)\,dx = 0;$$

Now consider the integral equation of third kind

$$(7.30) \qquad u(z,\lambda) = 1 + \frac{\lambda}{2\pi} \iint_{D} \tilde{G}(z,z')\, u(z',\lambda)\, \frac{dx'dy'}{y'^2},$$

Writing the equation in the form

$$(7.31) \qquad e^{-\varepsilon y} u(z,\lambda) = e^{-\varepsilon y} + \frac{\lambda}{2\pi} \iint_{D} e^{\varepsilon(y'-y)} \tilde{G}(z,z')\, e^{-\varepsilon y'} u(z',\lambda)\, \frac{dx'dy'}{y'^2}$$

where $\varepsilon < 2\pi$ is a ~~small~~ positive number, Now from (7.27), (7.27') and (7.26) and (7.26') we get, for $y' \geq A$; $y' > y$, that

$$(7.32) \quad e^{\varepsilon(y'-y)} \tilde{G}(z,z') = O\left(e^{-(2\pi-\varepsilon)|y-y'|} \log\left(2+\frac{1}{|y-y'|}\right)\right)$$

and for $y \geq A$; $y > y'$, that

$$(7.32') \quad e^{\varepsilon(y'-y)} \tilde{G}(z,z') = O\left(e^{-\varepsilon|y^*-y'|} \left(|y-y'| + \log\left(2+\frac{1}{|y-y'|}\right)\right)\right).$$

Finally for $y < A$, $y' < A$, we have from (7.23) and (7.27')

$$(7.32'') \quad e^{\varepsilon(y'-y)} \tilde{G}(z,z') = O\left(1 + |\log|z-z'|| + \sum_{i=1}^{\nu} |\log|z-M_i z'||\right).$$

From these estimations we see that the Fredholm solution of our equation (7.31) or (7.30) converges, the kernel being square integrable in the sense that

$$\iint_{D} \iint_{D} |e^{\varepsilon(y'-y)} \tilde{G}(z,z')|^2 \frac{dx'dy'}{y'^2} \frac{dx\,dy}{y^2} < \infty$$

and also the function $e^{-\varepsilon y}$ being square integrable over D.

From this we deduce that the solution $u(z,\lambda)$ of (7.30) ~~exists~~, is a meromorphic function of λ, which actually can be written as $\dfrac{D(z,\lambda)}{D(\lambda)}$, where the $D(z,\lambda)$ and $D(\lambda)$

being respectively the numerator and the denominator of the Fredholm solution, are integral functions of λ of order 2 at most.

Furthermore $u(z,\lambda)$ is automorphic in z, that is

(7.33) $$u(Mz,\lambda) = u(z,\lambda) \text{ for } M \subset \Gamma,$$

From equation (7.31) it can be seen ~~that~~, since ε may be taken arbitrarily close to zero, that for fixed λ (not equal to a zero of $D(\lambda)$), we have

(7.34) $$u(z,\lambda) = O(e^{\varepsilon y}),$$

for z in $\mathcal{D}$, and every $\varepsilon > 0$.

Finally using the fact that the operator (7.29) inverts the operation $y^2\Delta$, we get from (7.30) that

(7.34) $$y^2 \Delta\, u(z,\lambda) = \lambda\, u(z,\lambda)$$

Now put $\lambda = -s(1-s)$, where s is a complex variable and, suppose first that $\Re s > 1$. Lemma 7.2 then gives, because $u(z,-s(1-s))$ satisfies all conditions, that

(7.35) $$u(z,-s(1-s)) = a(s)\, E(z,s).,$$ where

$E(z,s)$ is the single Eisenstein series that occurs for this problem, and $a(s)$ is some function of s. $a(s)$ can be easily determined explicitly from the fact that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} u(x+iA, -s(1-s))\, dx = 1,$$

this gives, using (7.7),

$$a(s)\left(A^s + \sqrt{\pi}\, \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} L_0(s)\, A^{1-s}\right) = 1,$$

but the explicit expression is not necessary. It is enough that we know, that in the Fourier expansion of $u(z, -s(1-s))$ after x, the "constant" term (that is the term independent of x), for each y is a meromorphic function of s, which can be written as the quotient of two integral functions ~~which~~ which are at most of ~~order~~ ~~4~~ in s.

This term is

$$a(s)\left(y^s + \sqrt{\pi}\, \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} L_0(s)\, y^{1-s}\right)$$

Therefore (taking two values for y and eliminating $L_0(s)$), we get that $a(s)$ also is a quotient of two integral functions in s of order at most 4, thus from (7.35), we obtain that $E(z,s)$ can be continued in the whole s-plane except for isolated poles (not depending on z) and that $E(z,s)$ can be written as a quotient, ~~between~~ where the denominator is independent of z and is an integral function of s of order ≤ 4, while the numerator is a function of z and s, which in s is an integral analytic function of order ≤ 4, and in z is automorphic with respect to Γ, and furthermore an eigenfunction of the operator $y^2\Delta$.

Before going further we shall briefly indicate how one proceeds ~~in th~~ with the above arguments in the case when there are more than one cusp of $\mathfrak{D}$, and $\chi(M)$ is not necessarily always equal 1, in order to avoid too much complexity of notations we still suppose χ to be one-dimensional (that is: a number, not a matrix).

Take first the case when there are $\varkappa$ inequivalent cusps, ~~and~~ say at $\xi_1, \dots, \xi_\varkappa$ and $\chi(M) = 1$ always. Bringing the first cusp ξ_1 to ∞, we can again ~~construct~~ show the existence of our ~~function~~ harmonic function

(7.36) $G(z, z')$

with a logaritmic singularity ~~at~~ z at the cusp ξ_1, and at z', which is symmetric in z and z' and automorphich with respect to Γ in both variables. Denoting by

$$g_j(z) \quad \text{for } j = 2, 3, \dots \varkappa,$$

the harmonic function arising from (7.36) if we let z' go into the cusp ξ_j, we and ~~form~~ writing $g_1(z) = 1$, we can form an expression

$$\tilde{G}(z,z') = G(z,z') + \sum_{j=1}^{\varkappa} \omega_j(z')\, g_j(z)$$

where the functions $\omega_j(z')$ are chosen so as to be automorphic and with continuous derivatives up to second order, except for a finite number of curves in the compact part of $\mathcal{D}$, and furthermore so that as z' tends to the cusp ξ_j for $j = 1, \dots, \varkappa$,

$\tilde{G}(z,z')$ vanishes sufficiently strongly.

Then we consider the equation

$$u_j(z,\lambda) = g_j(z) + \frac{\lambda}{2\pi} \iint_{\mathcal{D}} \tilde{G}(z,z')\, u_j(z',\lambda) \frac{dx'\,dy'}{y'^2}$$

These functions will again exist as meromorphic functions of λ, they will be linearly independent because the $g_j(z)$ are, automorphic with respect to Γ and $O(e^{\varepsilon y_j})$ as z tends to ξ_j in $\mathcal{D}$, and satisfy the equations

$$y^2 \Delta u_j(z,\lambda) = \lambda u_j(z,\lambda);$$

writing again $\lambda = -s(1-s)$, we get from Lemma 7.2

again that the $u_j(z,-s(1-s))$ are linear combinations of the $E_j(z,s)$ for $j=1,\dots \varkappa$, or

$$u_i(z,-s(1-s)) = \sum_{j=1}^{\varkappa} a_{ij}(s)\, E_j(z,s) \quad ;$$

for $\mathcal{R}(s)>1$. Since the u_i are linearly independent for $i=1,2\dots \varkappa$; the $E_j(z,s)$ can conversely be expressed by the $u_i(z,-s(1-s))$, or

$$E_j(z,s) = \sum_{i=1}^{\varkappa} b_{ij}(s)\, u_i(z,-s(1-s)),$$

again one can show that the $b_{ij}(s)$ have to be meromorphic functions of s, ~~of the~~ and one gets finally that the $E_j(z,s)$ are meromorphic functions of s in the same way as for $E(z,s)$ above, with the same restrictions on the order.

When finally $\chi(M)\neq 1$ for some M, let us suppose that χ is singular with respect to the cusps $\xi_1,\dots \xi_{\varkappa_1}$ and non-singular with respect to $\xi_{\varkappa_1+1},\dots \xi_{\varkappa}$.

~~We can then show the~~

In this case one can show the existence of a harmonic function $G(z,z';\chi)$ which satisfies the relation

$$G(Mz,z';\chi) = \chi(M)\, G(z,z';\chi)$$

for $M\subset\Gamma$, and now with only one logaritmic singularity in D at the point z', we do not any more need to have another placed at a cusp. ~~Let i~~ It can again be shown that $G(z,z';\chi)$ can be taken so that it ~~is symmetric~~ satisfies the relation

$$\cancel{G(z,z';\chi) = G(z',z;\chi)}$$

$$G(z,z';\chi) = \overline{G(z',z;\chi)}.$$

Letting z' go into the cusp ξ_i for $i=1,2,\ldots \varkappa$, we obtain a harmonic function

$$g_i(z;\chi)$$

which has a logarithmic singularity at the cusp ξ_i (in the local uniformizing variable) and for which

$$g_i(Mz;\chi) = \chi(M)\, g_i(z,\chi).$$

Again we construct

$$\tilde{G}(z,z',\chi) = G(z,z',\chi) + \sum_{i=1}^{\varkappa} \omega_i(z')\, g_i(z;\chi)$$

where the $\omega_i(z')$ are functions satisfying

$$\omega_i(Mz') = \overline{\chi(M)}\, \omega_i(z')$$

and continuous and with continuous derivatives up to second order in D, except for a finite number of curves in the compact part of D where discontinuities may occur, and further chosen so that $\tilde{G}(z,z';\chi)$ vanishes sufficiently strongly as z' tends towards any cusp ξ_j for $1 \le j \le \varkappa$. Consider the equations

$$u_i(z, \chi, \lambda) = g_i(z, \lambda) + \frac{\lambda}{2\pi} \iint_D \tilde{G}(z, z'; \chi) u_i(z', \chi, \lambda) \frac{dx' dy'}{y'^2}$$

for $i = 1, 2, \ldots \varkappa_1$. Then as in the former case one gets the u_i meromorphic in λ, linearly independent, and $u_i(Mz, \chi; \lambda) = \chi(M) u_i(z, \chi, \lambda)$ for $M \subset \Gamma$, and eigenfunctions of the operator $y^2\Delta$. Again for $\mathcal{R}(s) > 1$, one gets for $i = 1, 2 \ldots \varkappa_1$,

$$E_i(z; s; \chi) = \sum_{j=1}^{\varkappa_1} b_{ij}(s) u_j(z, \chi; -s(1-s)),$$

where the $b_{ij}(s)$ are meromorphic functions of s, and one finally gets in this case also that the $E_i(z, s; \chi)$ are meromorphic functions of s, with the same restriction that they can be written as quotients of integral functions of s of order ≤ 4, and where the denominator is independent of z.

Letting $\mathcal{R}s > 1$, and taking s so that $1 - s$ is not a pole for the E_i, we find that the ~~system~~ functions

$$E_i(z, 1-s; \chi) \quad , i = 1, 2, \ldots \varkappa_1,$$

satisfy the conditions of Lemma 7.2. Thus they can be expressed in terms of these, the connection is easily found by looking at the terms independent of x_j in the fourier-expansions of the systems $E_i(z, 1-s; \chi)$ and $E_i(z, s; \chi)$ for $i = 1, 2, \dots \varkappa_1$, where $x_j + i y_j = z_j = -\frac{\lambda_j}{z - \xi_j}$, for $j = 1, 2 \dots \varkappa_1$

Writing $E(z, s, \chi)$ for the column vector

$$\begin{pmatrix} E_1(z,s,\chi) \\ E_2(z,s;\chi) \\ \vdots \\ E_{\varkappa_1}(z,s,\chi) \end{pmatrix}$$

and $L_0(s, \chi)$ for the $\overbrace{\text{matrix}}^{\varkappa_1 \times \varkappa_1 \text{ rowed}}$ $\left(L_0^{(i,j)}(s,\chi) \right)$,*)

we find easily,

$$(7.37) \quad E(z, 1-s; \chi) = \sqrt{\pi}\, \frac{\Gamma(\frac{1}{2} - s)}{\Gamma(1-s)}\, L_0(1-s, \chi)\, E(z, s; \chi).$$

From (7.37) follows when changing s into $1-s$ and comparing

$$(7.38) \quad \sqrt{\pi}\, \frac{\Gamma(\frac{1}{2} - s)}{\Gamma(1-s)}\, L_0(1-s, \chi)\, \sqrt{\pi}\, \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)}\, L_0(s, \chi) = E$$

where E is the $\varkappa_1 \times \varkappa_1$ Identity matrix.

---

*) It is clear that $L_0^{(i,j)}(s;\chi)$ as all ~~other~~ expressions for ~~the various fouri~~ the other coefficients in the various fourierexpansion for the $E_i(z, s; \chi)$ are also meromorphic functions.

Lemma 7.1, implies that

$$L_0'(s,\bar{\chi}) = L_0(s,\chi)$$

where L' denotes transposition of the matrix L, if $s = \frac{1}{2} + ir$, r real, we have

$$L_0(\tfrac{1}{2} - ir, \bar{\chi}) = \overline{L_0(\tfrac{1}{2} + ir, \chi)},$$

Because

$$E(z,\bar{s},\bar{\chi}) = \overline{E(\bar{z}, s, \chi)},$$

from the definition of the Eisenstein series.

Thus

$$L_0'(\tfrac{1}{2} - ir, \chi) = \overline{L_0'(\tfrac{1}{2} + ir, \chi)},$$

comparing this with (7.38) we find that

(7.39) $$\sqrt{\pi}\,\frac{\Gamma(ir)}{\Gamma(\frac{1}{2}+ir)}\,L_0(\tfrac{1}{2} + ir, \chi)$$

is a <u>unitary matrix</u>.

Before we sum up our results, we shall obtain some results about the location of the poles of the $E_i(z,s,\chi)$ or, what is the same, the vector $E(z,s,\chi)$. For simplicity we again consider the case with only one cusp of $\mathfrak{D}$, placed at ∞, and take $\chi(M)$ identically 1, so we can omit the χ in the formulas.

It is at once clear that $E(z,s)$ can not have a pole in the half-plane $\mathcal{R}(s) > \frac{1}{2}$, unless

$\frac{1}{2}<s\leq 1$ | it be located on the part $1 \geq s > \frac{1}{2}$ of the real line. Supposing that there was such a pole s_0 of order $\nu > 0$, then the value of $(s-s_0)^\nu E(z,s)$ at the point $s = s_0$, would be a function of z, with the following properties: It would be automorphic in z under the group Γ, and also an eigenfunction of $y^2\Delta$ corresponding to a complex eigenvalue $s_0(1-s_0)$; finally it would be square integrable over the fundamental domain D, because its fourier expansion after x would be of the form

$$\alpha_0 y^{1-s_0} + \sum_{n=-\infty}^{\infty}{}' \alpha_n e^{2\pi i n x} \sqrt{y} \int_0^\infty t^{s_0 - \frac{1}{2}} e^{-\pi |n| y (t + \frac{1}{t})} \frac{dt}{t},$$

and $\mathcal{R}(1-s_0) < \frac{1}{2}$, which gives a contradiction.

In order to obtain more, we have ~~first~~ to develop a ~~[illegible]~~ formula involving two functions $E(z,s)$ and $E(z,s')$. Taking again the number A so large that D contains the domain $|x| \leq \frac{1}{2}$, $y \geq A$, and writing for brevity

$$\varphi(s) = \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,L_0(s),$$

and defining

$$\tilde{E}(z,s) = E(z,s) \text{ in } \mathcal{D}, \text{ for } y \le A,$$

and

$$\tilde{E}(z,s) = E(z,s) - y^s - \varphi(s)\,y^{1-s}, \text{ for } y \ge A;$$

we have

$$(7.40)\quad (s'-s)(s+s'-1)\iint_{\mathcal{D}} \tilde{E}(z,s)\,\tilde{E}(z,s')\,\frac{dx\,dy}{y^2} =$$

$$= \iint_{\mathcal{D}} \left(\tilde{E}(z,s)\,\Delta\,\tilde{E}(z,s') - \tilde{E}(z,s')\,\Delta\,\tilde{E}(z,s)\right) dx\,dy.$$

Using here Green's formula, taking into account the discontinuity along the line $y = A$, we observe that the integrals over the boundary of $\mathcal{D}$ cancel out, and that the contribution coming from the line-segment $y = A$, $|x| \le \frac{1}{2}$, becomes

$$(y^s + \varphi(s)y^{1-s})\,\frac{d}{dy}\,(y^{s'} + \varphi(s')y^{1-s'}) -$$

$$(y^{s'} + \varphi(s')y^{1-s'})\,\frac{d}{dy}\,(y^{s} + \varphi(s)\,y^{1-s}) =$$

$$= (s'-s)\,(y^{s+s'-1} - \varphi(s)\varphi(s')\,y^{1-s-s'})$$

$$+ (s+s'-1)\,(\varphi(s)\,y^{s'-s} - \varphi(s')\,y^{s-s'}),$$

for $y = A$. (7.40) thus gives

$$(7.41)\quad \iint_D \tilde{E}(z,s)\,\tilde{E}(z,s')\,\frac{dx\,dy}{y^2} =$$

$$= \frac{A^{s+s'-1} - \varphi(s)\varphi(s')A^{1-s-s'}}{s+s'-1} +$$

$$+ \frac{\varphi(s)A^{s'-s} - \varphi(s')A^{s-s'}}{s'-s}$$

Taking first $s=\sigma+ik$, $s'=\sigma-ik$; $k\neq 0$, $\sigma\neq\frac{1}{2}$, we get

$$(7.42)\quad \iint_D |\tilde{E}(z,\sigma+ik)|^2\,\frac{dx\,dy}{y^2}$$

$$= \frac{A^{2\sigma-1} - |\varphi(\sigma+ik)|^2 A^{1-2\sigma}}{2\sigma-1} + \frac{\overline{\varphi(\sigma+ik)}\,A^{2ik} - \varphi(\sigma+ik)A^{-2ik}}{2ik}.$$

Making here $\sigma\to\frac{1}{2}$, we obtain, observing that by (7.39) $\varphi(\frac{1}{2}+ik)$ is necessarily regular at the point $s=\frac{1}{2}+ik$, and $|\varphi(\frac{1}{2}+ir)|=1$, that for r real and $\neq 0$,

$$(7.42')\quad \iint_D |\tilde{E}(z,\tfrac{1}{2}+ir)|^2\,\frac{dx\,dy}{y^2} =$$

$$= 2\log A - \mathcal{R}\left\{\frac{\varphi'(\frac{1}{2}+ir)}{\varphi(\frac{1}{2}+ir)}\right\} + \frac{\overline{\varphi(\frac{1}{2}+ir)}\,A^{2ir} - \varphi(\frac{1}{2}+ir)A^{-2ir}}{2ir}.$$

From this follows that <u>$E(z,s)$ is regular at the point $s=\frac{1}{2}+ir$</u>, since otherwise if say $E(z,s)$ had a pole there of order $\nu>0$, we would get that $\lim_{s \to \frac{1}{2}+ir} (s-\frac{1}{2}-ir)^{\nu} E(z,s) = u(z)$,

would be a solution of

$$\left(y^2\Delta + \tfrac{1}{4} + r^2\right) u = 0,$$

which is automorphic under Γ, and furthermore, since the right hand side of (7,42') is always finite for $r \neq 0$, we would get

$$\iint_{\mathcal{D}} |u|^2 \frac{dx\,dy}{y^2} = 0,$$

so that u vanishes identically, against the assumption. Letting next $\sigma > \frac{1}{2}$ and making $r \to 0$ in (7.42), we obtain

$$(7.42'') \quad \iint_{\mathcal{D}} \left|\tilde{E}(z,\sigma)\right|^2 \frac{dx\,dy}{y^2} = \frac{A^{2\sigma-1} - |\varphi(\sigma)|^2 A^{1-2\sigma}}{2\sigma-1} + 2\varphi(\sigma)\log A - \varphi'(\sigma).$$

From this, we can not quite exclude the existence of poles on the segment $1 \geqq \sigma \geqq \frac{1}{2}$; but observing that the right hand side of (7.42)'' must by necessity be positive, we see that $|\varphi(\sigma)|$ cannot grow large without $-\varphi'(\sigma)$ becomming positive and large of the order of $|\varphi(\sigma)|^2$. Therefore if poles of $\varphi(s)$ occur on this stretch, they must be poles of first order only, and the residue at the pole must also be positive. If $\varphi(s)$ has such a pole at $s = \sigma_0$, $1 \geq \sigma_0 > \frac{1}{2}$, we find that $E(z,s)$ has a pole of first order there ~~too~~, and $\lim\limits_{s\to\sigma_0}(s-\sigma_0)\,E(z,s)$, will in this case give us a function $u(z)$ which is ~~an eigen~~ a solution of the equation

$$\left(y^2\Delta + \sigma_0(1-\sigma_0)\right)u = 0$$

and automorphic in z with respect to the group Γ, with a fourier expansion after x of the form

$$u(z) = c\,y^{1-\sigma_0} + \sum_{n=-\infty}^{\infty}{}' \alpha_n e^{2\pi i n x}\sqrt{y}\int_0^\infty t^{\sigma_0-\frac{1}{2}} e^{-\pi|n|y(t+\frac{1}{t})}\frac{dt}{t},$$

where $c>0$, finally $u(z)$ is square integrable over the fundamental domain $\mathcal{D}$ or

$$\iint_{\mathcal{D}}|u|^2\frac{dx\,dy}{y^2} < \infty, \text{ more precisely } \iint_{\mathcal{D}}|u|^2\frac{dx\,dy}{y^2} = c.$$

At the points σ that are not poles for $\varphi(s)$, we find that $E(z,s)$ is regular.

Finally we may investigate the point $s=\frac{1}{2}$, by letting in (7.42) $\sigma\to\frac{1}{2}$ and $r\to 0$, one finds that $E(z,s)$ is regular at $s=\frac{1}{2}$. If $\varphi(\frac{1}{2})=-1$, $E(z,\frac{1}{2})=0$ for all z, if $\varphi(\frac{1}{2})=1$, $E(z,\frac{1}{2})$ does not vanish identically:

Considering now the more general case where there are $\varkappa$ inequivalent cusps and χ is not necessarily identically 1. Let the fundamental domain $\mathcal{D}$ as before have the cusps at $\xi_1, \dots, \xi_\varkappa$, and let χ be singular with respect to the $\varkappa_1$ first cusps and non-singular with respect to the rest. And let again $z_i = -\frac{\lambda_i}{z-\xi_i}$ for $i=1,2\dots\varkappa$ where the λ_i are so chosen that the primitive parabolic transformation leaving ξ_i fixed takes the form $z_i \to z_i+1$. Writing for brevity

$$\varphi_{i,j}(s,\chi) = \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,L_0^{(i,j)}(s,\chi),$$

and taking the positive numbers $A_1, A_2, \dots, A_{\varkappa_1}$ so large that the lines $y_i = A_i$ cuts the boundary of ~~domain~~ $\mathfrak{D}$ only in the two sides of $\mathfrak{D}$ that pass through ξ_i, and furthermore that the ~~domains~~ regions $y_i \geq A_i$ for $i=1,2\dots\varkappa_1$ have no point in common, we now define for $i=1,2,\dots\varkappa_1$ and z in $\mathfrak{D}$

(a) $\tilde{E}_i(z,s;\chi) = E_i(z,s;\chi)$

~~for~~ $y_j \leq A_j$ for all j with $1 \leq j \leq \varkappa_1$,

for $y_i \geq A_i$ we put

(b) $\tilde{E}_i(z,s;\chi) = E_i(z,s;\chi) - y_i^s - \varphi_{ii}(s,\chi)y_i^{1-s}$,

and for $y_j \geq A_j$ where $i \neq j$, we put

(c) $$\tilde{E}_i(z,s;\chi) = E_i(z,s;\chi) - \varphi_{ij}(s,\chi)y^{1-s}.$$

Instead of (7.40) we now consider, for $1 \le i,j \le \varkappa_1$,

$$(s'-s)(s+s'-1)\iint_{\mathcal{D}} \tilde{E}_i(z,s;\chi)\,\tilde{E}_j(z,s',\bar{\chi})\,\frac{dx\,dy}{y^2}.$$

Proceeding in a similar way as before, and putting $s=\sigma+ir$, $s'=\sigma-ir$, and observing that $\tilde{E}_j(z,\bar{s},\bar{\chi}) = \overline{\tilde{E}_i(z,s,\chi)}$, we obtain for $\sigma \neq \frac{1}{2}$, $r \neq 0$, $1 \le i,j \le \varkappa_1$,

$$(7.43)\quad \iint_{\mathcal{D}} \tilde{E}_i(z,\sigma+ir,\chi)\,\overline{\tilde{E}_j(z,\sigma+ir,\chi)}\,\frac{dx\,dy}{y^2}$$

$$= \frac{1}{2\sigma-1}\left(\delta_{ij}A_i^{2\sigma-1} - \sum_{k=1}^{\varkappa_1}\varphi_{ik}(\sigma+ir,\chi)\overline{\varphi_{jk}(\sigma+ir,\chi)}\,A_k^{1-2\sigma}\right)$$

$$+ \frac{\overline{\varphi_{ji}(\sigma+ir,\chi)}\,A_i^{2ir} - \varphi_{ij}(\sigma+ir,\chi)\,A_j^{-2ir}}{2ir},$$

where $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$.
A simpler and more ~~[illegible]~~ transparent form of these equations, we get by introducing vector- and matrix notations again,