write \( \vec{E}(z, s, x) \) for the column vector with
the \( x_i \) components \( \vec{E}(z, s, x), 1 \leq i \leq x \); and
and similarly we write \( \vec{E}(z, 0, x) \) for the column vector with components
\( \phi(s; x) \) for the matrix
\[
\phi(s; x) = (\phi_{i,j}(s; x)) = \sqrt{\pi} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} L_0(s, x).
\]
And take further for simplicity
\[
A_1 = A_2 = \ldots = A_{2x}, = A_i (\text{actually one can show that they may all be chosen equal to } 1, \text{ but at any rate if } A \text{ is sufficiently large, one can clearly make the } A_i = A \text{ for } 1 \leq i \leq 2x.) \text{ Then (7.48) may be written as a matrix equation}
\]
\[
(7.43) \int_0^1 \vec{E}(z, s + i\tau, x) \vec{E}^T(z, s + i\tau, x) \, dx \, dy_{\tau} = \frac{1}{2^{s-1}} \left\{ A^{2s-1} - \phi(s+i\tau, x) \phi(s+i\tau, x)' A^{s-2s} \right\}
\]
\[
+ \phi(s+i\tau, x) A^{2s} - \phi(s+i\tau, x) A^{-2s}
\]
where \( E \) is the \( x, xx \), identity matrix
and the dash ' means transpose.
From (7.43) one can now proceed in a similar way as from (7.42) before.
Denoting by \( \Phi(s, x) \) the value of the determinant \( |\Phi(s, x)| \), we now get the following results.

\[ E(z, s, x) \] is regular in the region \( \sigma > \frac{1}{2} \), except possibly for a finite number of simple poles on the real axis \( \frac{1}{2} < \sigma \leq 1 \).

If \( \frac{1}{2} < \sigma \leq 1 \) is a pole of \( E(z, s, x) \), then the components of the residue of \( E(z, s, x) \) at \( \sigma = \sigma_0 \) give rise to \( \nu_0 \) linearly independent square integrable eigenfunctions of our problem, where \( \nu_0 \equiv \nu_0(x) \) is the order of the pole of \( \Phi(s, x) \) at \( s = \sigma_0 \). These eigenfunctions satisfy \( y^2 \Delta u + \sigma_0(1-\sigma_0) u = 0 \), and \( u(z^2) = 2^{1/2} \), and are square integrable over the fundamental domain \( D \). The poles at the points \( \sigma_0 \) are the only poles of \( \Phi(s, x) \) in the region \( \sigma > \frac{1}{2} \).

\( \text{§8.} \)

Preliminaries to the proof of the trace formula in the case of a singular \( \chi \).

We shall see for use in the next paragraph are we shall need to develop.
certain estimations for \( E(z, p, x) \) in the region \( \frac{1}{2} \leq \sigma \leq \frac{3}{2} \), and also certain properties of the function \( \varphi(0, x) \).

We first develop some inequalities for the function

\[
\int_0^\infty t^{s-\frac{1}{2}} e^{-y(t+\frac{1}{2})} \frac{dt}{t}
\]

Let \( \frac{1}{2} \leq \sigma \leq \frac{3}{2} \), and \( \delta = \sigma + \imath \rho \), and suppose at first that \( \rho > 0 \); turning the line of integration an angle \( \frac{\pi}{2} - \delta \), \( 0 < \delta \leq \frac{\pi}{2} \) in the positive direction, writing \( t e^{(\frac{\pi}{2}-\delta)i} \) instead of \( t \), we get

\[
\int_0^\infty t^{s-\frac{1}{2}} e^{-y(t+\frac{1}{2})} \frac{dt}{t} = e^{\left(\frac{\pi}{2}-\delta\right)\rho} (\frac{\pi}{2}-\delta)\rho^{-1} \int \frac{t^{s-\frac{1}{2}} e^{-y(t+\frac{1}{2})}}{t} \sin \delta \imath y \left(t e^{(\frac{\pi}{2}-\delta)i}\right) dt
\]

or taking absolute values

\[
(8.1) \quad \left| \int_0^\infty t^{s-\frac{1}{2}} e^{-y(t+\frac{1}{2})} \frac{dt}{t} \right| \leq e^{\left(\frac{\pi}{2}-\delta\right)\rho} \int_0^\infty t^{s-\frac{1}{2}} e^{-y \sin \delta \left(t + \frac{1}{2}\right)} \frac{dt}{t}
\]

\[
\leq 2 e^{\left(\frac{\pi}{2}-\delta\right)\rho} \int_0^\infty t^{s-\frac{1}{2}} e^{-y \sin \delta \left(t + \frac{1}{2}\right)} \frac{dt}{t}
\]

\[
\leq 2 e^{\left(\frac{\pi}{2}-\delta\right)\rho} \int e^{-y \sin \delta t} \frac{dt}{y \sin \delta}
\]

\[
= 2 e^{\left(\frac{\pi}{2}-\delta\right)\rho} \frac{1}{y \sin \delta}
\]
\[ \text{[NB]} \]

or since \( \sin \delta \geq \frac{2}{\pi} \), \( \delta \geq \frac{\delta}{2} \),

\[ (8.1) \quad \left| \int_0^\infty t^{s-\frac{1}{2}} e^{-\gamma(t+\frac{t}{2})} \frac{dt}{t} \right| < 4 \frac{e^{-\frac{(1 - \delta)}{2} \pi}}{\delta \gamma} \]

In the form (8.1) the inequality is obviously true also for \( \gamma < 0 \).

Now consider for \( \frac{1}{2} \leq \sigma \leq \frac{3}{2} \), and \( \gamma > 0 \), the expression

\[ \int_{\gamma}^{\infty} (y^2 - y^2) \left( \frac{y}{2} - \sigma \right) - \infty \int_{0}^{\infty} t^{-\frac{s}{2} + \infty} e^{-\gamma(t+\frac{t}{2})} \frac{dt}{t} \]

\[ = \frac{1}{2} \int_{\gamma}^{\infty} (y^2 - y^2)^{\frac{1}{2} - \infty} - \infty \int_{0}^{\infty} t^{-\frac{s}{2} + \infty} e^{-\gamma^2 t - \frac{t}{2}} \frac{dt}{t} \]

\[ = \frac{1}{2} \int_{\gamma}^{\infty} t^{\frac{s}{2} - \infty} e^{-\gamma^2 t - \frac{t}{2}} dt \int_{0}^{\infty} (y^2 - y^2)^{\frac{1}{2} - \infty} e^{-\gamma^2 y^2 - \frac{y^2}{2}} \frac{dy}{y^2} \]

\[ = \frac{1}{2} \text{P}(1+in) \int_{\gamma}^{\infty} t^{\frac{s}{2} - \infty} e^{-\gamma^2 t - \frac{t}{2}} dt \]

\[ = \frac{1}{2} \text{P}(1+in) \int_{\gamma - \frac{s}{2}}^{\infty} t^{\frac{s}{2} - \infty} e^{-\gamma^2 t + \frac{t}{2}} dt \]

From this we get
\[ \int_{\gamma} y^{\frac{3}{2} - \sigma} \, dy \leq \int_{0}^{\infty} t^{-\frac{3}{2} + \sigma} e^{-\gamma(t + \frac{\sigma}{2})} \, dt \]

\[ \leq \frac{1}{2} \left[ P(1 + \sigma) \right] \int_{0}^{\infty} t^{-\frac{3}{2} + \sigma} e^{-\gamma(t + \frac{\sigma}{2})} \, dt \]

\[ = \frac{1}{2} e^{-\frac{\pi}{2} |\gamma|^{3/2} - \sigma} \int_{0}^{\infty} e^{-\gamma(t + \frac{\sigma}{2})} \, dt \]

\[ > e^{-\frac{\pi}{2} |\gamma|} \frac{\gamma^{\frac{3}{2} - \sigma}}{\gamma^{\frac{3}{2} - \sigma}} \int_{0}^{\infty} e^{-(2\gamma + 1) t} \, dt \]

\[ > e^{-\frac{\pi}{2} |\gamma| - 2\gamma - 1} \frac{\gamma^{\frac{3}{2} - \sigma}}{2\gamma + 1} \]

or since the left-hand side obviously increases with decreasing \( \gamma \), one get easily

\[ \int_{\gamma} y^{\frac{3}{2} - \sigma} \, dy \leq \int_{0}^{\infty} t^{-\frac{3}{2} + \sigma} e^{-\gamma(t + \frac{\sigma}{2})} \, dt \, dy \]

\[ > e^{-\frac{\pi}{2} |\gamma| - 3\gamma - 1} \]

where \( c \) is a sufficiently large positive constant.

From (8.1) one get, taking \( \delta = \frac{\pi}{2} \),
and \[ T = \pi \ln 1 + 6 \gamma + 2 \zeta + 6 \], that

\[
\begin{align*}
\int_1^\infty & \frac{t^{\gamma - \frac{3}{2}}}{\ln t} \left( \int_0^\infty e^{- \frac{t}{2} \ln t - y(t + \frac{1}{2})} \frac{dt}{t} \right) \, dy \\
& < e^{- \frac{\pi}{2} \ln 1 - 3 \gamma - \zeta - 1},
\end{align*}
\]

Thus we get

\[
\begin{align*}
& \int_1^\infty \frac{t^{\gamma - \frac{3}{2}}}{\ln t} \left( \int_0^\infty e^{- \frac{t}{2} \ln t - y(t + \frac{1}{2})} \frac{dt}{t} \right) \, dy \\
& > e^{- \frac{\pi}{2} \ln 1 - 3 \gamma - \zeta - 1},
\end{align*}
\]

Using Schwartz inequality we get from this:

\[
\begin{align*}
(8.2) & \int_1^\infty \left( \int_0^\infty e^{- \frac{t}{2} \ln t - y(t + \frac{1}{2})} \frac{dt}{t} \right)^2 \, dy \\
& > \left( \int_1^\infty \frac{t^{\gamma - \frac{3}{2}}}{\ln t} \left( \int_0^\infty e^{- \frac{t}{2} \ln t - y(t + \frac{1}{2})} \frac{dt}{t} \right) \, dy \right)^2 \\
& \geq \frac{T}{\zeta - 2 \gamma} \int_1^\infty \frac{t^{\gamma - \frac{3}{2}}}{\ln t} \, dt.
\end{align*}
\]
\[
\begin{align*}
\frac{5-25}{4-2a} \cdot e^{-\frac{\pi}{2T} (x^2 - 6 \eta - 2C - 2)} \\
\frac{1}{4} \cdot e^{-\frac{\pi}{2T} (x^2 - 6 \eta - 2C - 2)} \\
\frac{1}{(1+n+\eta)^4} \cdot e^{-\frac{\pi}{2T} (x^2 - 6 \eta - c')}
\end{align*}
\]

where \( c' \) is an absolute positive constant.

We now turn to the function \( \varphi(s, x) \), from the previous section we know that \( \varphi(s, x) \) is meromorphic, and can be written as a quotient between two integral functions of \( x \) each of order at most 4, further that

\begin{equation}
\varphi(s, x) \varphi(8-s, x) = 1
\end{equation}

and that \( \varphi(s, x) \) is regular for \( s \geq \frac{1}{2} \) except possibly for a finite number of poles in the interval \( \frac{1}{2} \leq s \leq 1 \) of the real axis. Finally

\begin{equation}
\varphi(s, x) = \left\{ \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2})} \cdot \prod_{\rho(\frac{1}{2})} \right\}^{2\pi} \left( \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \right)^{2\pi}
\end{equation}

where \( \Gamma(s) \) is a function of \( s \) with real coefficients, \( \prod_{\rho(s-\frac{1}{2})} \) is the determinant of the matrix \( \rho(s, x) \) which is absolutely convergent for \( s > 1 \).
Now looking at (7.43) with $i = j$, and all $A = A_i = \pi_i$, and observing that the left-hand side is obviously positive, we get

$$\frac{1}{2\sigma-1} \left( A^{2\sigma-1} - A^{-2\sigma} \sum_{k=1}^{\frac{2\sigma-1}{2\sigma-1}} |\varphi_{i+k}(\sigma+\imath x)|^2 \right) + \frac{\varphi_{i\mid}(\sigma+\imath x) A^{-\imath x}}{2\imath n} \varphi_{i\mid}(\sigma+\imath x) A^{-\imath x} > 0.$$

For $|\lambda| > 1$, this gives

$$\sum_{k=1}^{\frac{2\sigma-1}{2\sigma-1}} |\varphi_{i+k}(\sigma+\imath x)|^2 \leq A^{\frac{2\sigma-2}{2\sigma-1}} + \frac{2\sigma-1}{|\lambda|} A^{\frac{2\sigma-1}{2\sigma-1}} |\varphi_{i\mid}(\sigma+\imath x)|$$

for $1 \leq i \leq n$, assuming also $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$, we obtain that all $\varphi_{i+k}(\sigma+\imath x)$ are uniformly bounded for $|\lambda| > 1$, $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$, thus also $\varphi(\lambda, x)$ is uniformly bounded for $s = \sigma + \imath x$, $|\lambda| > 1$, $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$.

Now denote by

(8.5) $\lambda_0(x) = \sum_{k=0}^{\infty} \frac{A_{\lambda_0}}{C_0^{\lambda_0}} x_{\lambda_0} C_0 < C_1 < C_2 \ldots$

and denote by $\sigma_i, 1 \leq i \leq N$ the poles of $\varphi(s, x)$ in the interval $\frac{1}{2} < \sigma \leq 1$, each counted with its proper multiplicity. Consider now the function

(8.6) $\varphi^* (\lambda, x) = C_0^{\lambda-1} \left( \prod_{i=1}^{N} \frac{\lambda - \sigma_i}{\lambda - \sigma_i + \sigma_i} \right) \varphi(\lambda, x)$
This function is new from (8.4) and our result about the boundedness of \( \varphi(s, x) \) for \( \lambda \geq 1 \), \( \frac{1}{2} \leq s \leq \frac{3}{2} \), that \( \varphi^*(s, x) \) is uniformly bounded in the half-plane \( s = \frac{1}{2} \), and since

\[(8.7) \quad |\varphi^*(\frac{1}{2} + it, x)| = |\varphi(\frac{1}{2} + it, x)| = 1, \quad \text{by (8.6) and (8.3)}, \quad \text{we get}
\]

\[(8.8) \quad |\varphi^*(s + x)| \leq 1, \quad \text{for } s \geq \frac{1}{2}.
\]

Denoting by \( p = \beta + it \), \( \beta < \frac{1}{2} \) the poles of \( \varphi(s, x) \) in the half-plane \( s < \frac{1}{2} \), which will also be the poles of \( \varphi^*(s, x) \) in this region. We can prove from (8.4), using well known methods for estimating the number of zeros in a rectangle, that

\[(8.9) \quad \sum_{1 \leq |\lambda| \leq T} \left( \frac{1}{2} - \beta \right) = O(T \log T),
\]

(Actually one can even prove

\[
\sum_{1 \leq |\lambda| \leq T} \left( \frac{1}{2} - \beta \right) = \alpha \frac{T \log T}{T} \log \log T + \alpha_2 T + O(\log T)
\]

where \( \alpha_1 > 0 \), and \( \alpha_2 \) are constants, but this will not be needed.)
From this one shows easily that the product

\[
\prod_{p} \frac{s-1+p}{s-p}
\]

absolutely converges if we combine the term containing \( p \) with the one containing \( \overline{p} \) for the complex \( p \)'s. Therefore, since \( \phi^* \) is at most of new form

\[
\phi^*(s, \chi) = \pm \alpha_3 (s-\frac{1}{2}) + \alpha_4 (s-\frac{1}{2})^3 \prod_{p} \frac{p-1+p}{s-p}.
\]

since \( \phi^*(s, \chi) \phi^*(1-s, \chi) = 1 \). An investigation of the behaviour of \(|\phi^*(s, \chi)|\) as \( s \to \infty \), from

(8.6) and (8.4), shows that \( \alpha_3 = \alpha_4 = 0 \), so that

\[
(8.10) \quad \phi^*(s, \chi) = \pm \prod_{p} \frac{p-1+p}{s-p},
\]

and hence

\[
(8.10') \quad \phi(s, \chi) = \pm c_0 1 - 2s \prod_{\epsilon = 1}^{N} \frac{p-1+\epsilon i}{s-\epsilon^2} \prod_{p} \frac{s-1+p}{s-p}.
\]

\[
* \quad \text{We could here also avoid any combination of terms by putting in the factor} \quad e^{\frac{i}{2}s-1} \quad \text{or by taking the factor} \quad \frac{s-\frac{1}{2}}{\overline{p} - \frac{1}{2}} \quad \text{instead of} \quad \frac{s-\frac{1}{2}}{s-p}.
\]
Forming from (8.10)

\[
(8.11) - \frac{\varphi^{*1}}{\varphi^{*}}(\zeta + i\eta, \chi) = \sum_{\rho} \frac{1 - \frac{1}{2} \beta}{(\beta - \frac{1}{2})^2 + (\rho - \chi)^2},
\]

we observe that this expression is positive and defining

\[
(8.12) \quad \omega(\eta) = 1 - \frac{\varphi^{*1}}{\varphi^{*}}(\zeta + i\eta, \chi),
\]

we get that \(\omega(\eta)\) is positive and further that

\[
\int_{-R}^{R} \omega(\eta) \, d\eta = 2R + \sum_{\rho} \int_{-R}^{R} \frac{1 - \frac{1}{2} \beta}{(\beta - \frac{1}{2})^2 + (\rho - \chi)^2} \, d\eta.
\]

\[
\leq 2R + \sum_{1\chi \leq 2R} \int_{-\infty}^{\infty} \frac{1 - \frac{1}{2} \beta}{(\beta - \frac{1}{2})^2 + (\rho - \chi)^2} \, d\rho + 8R \sum_{1\chi > 2R} \frac{1 - \frac{1}{2} \beta}{1 \chi^2} \leq 2R + 2 \frac{1}{\beta \chi} \sum_{1\chi \leq 2R} 1 + 8R \sum_{1\chi > 2R} \frac{1 - \frac{1}{2} \beta}{1 \chi^2},
\]

here the last term on the right-hand side is \(O(R^2)\) and the second term is \(O(R^5)\). Since \(\varphi(s, \chi)\) and \(\varphi^{*1}(s, \chi)\) is a ratio of integral
functions each of order at most \( q \). Thus we have

\[
(8.13) \quad \int_\mathbb{R} \omega(x) \, dx = O\left( R^{-\frac{5}{2}} \right).
\]

Furthermore since

\[
\sum_{1 \leq i \neq j \leq \infty} \left| \Phi_{ij}(\sigma, \phi) \right|^2
\]

is the trace of the positive definite matrix

\[
\phi(\sigma, \phi) \phi(\sigma, \phi)' \quad \text{and} \quad \left| \Phi(\sigma, \phi) \right|^2 \quad \text{its determinant,}
\]

we have

\[
\sum_{i,j} \left| \Phi_{ij}(\sigma, \phi) \right|^2 \geq \lambda_1 \left( \sum_{i,j} \left| \Phi_{ij}(\sigma, \phi) \right| \right)^2,
\]

and so

\[
\lambda_1 - \sum_{i,j} \left| \Phi_{ij}(\sigma, \phi) \right|^2 \leq \lambda_1 \left( 1 - \left| \Phi(\sigma, \phi) \right|^2 \right)^{\frac{2}{\lambda_1}}.
\]

For \( |\sigma| \geq 1 \), \( \Phi \leq \sigma \leq \frac{3}{2} \), \( \phi = \sigma + \ii \phi \), we get from this

\[
\lambda_1 - \sum_{i,j} \left| \Phi_{ij}(\sigma, \phi) \right|^2 \leq \lambda_1 \left( 1 - \left| \Phi^*(\sigma, \phi) \right|^2 \right)^{\frac{2}{\lambda_1}}
\]

\[+ \lambda_1 \left( 1 - \left| \Phi^*(\sigma, \phi) \right|^2 \right)^{\frac{2}{\lambda_1}} \left( \left| \Phi(\sigma, \phi) \right|^2 \right)^{\frac{2}{\lambda_1}}.
\]

\[
\leq 1 - \left| \Phi^*(\sigma, \phi) \right|^2 + \lambda_1 \left( 1 - c_0 \frac{\phi - \sigma}{\lambda_1} \right).
\]
\[
\text{Since } |\varphi^*(\rho, x)| \leq 1 \text{ and } |\varphi(i_1, x)| \leq C_0 - 2\sigma |\varphi^*(\rho, x)| \text{ by (8.6). Thus,}
\]
\[
(8.14) \quad x, - \sum_{i,j} |\varphi_{ij}(\rho, x)|^2 \leq O(\sigma - \frac{1}{2}) + 1 - |\varphi^*(\rho, x)|^2.
\]

Now, by (8.10)
\[
1 - |\varphi^*(\rho, x)|^2 = 1 - \frac{1 - 2\beta}{\sigma - \rho^2 + (\sigma - \rho^2)} (1 - (2\sigma - 1) \frac{1 - 2\beta}{(\sigma - \rho^2 + (\sigma - \rho^2)}
\]
\[
\leq (2\sigma - 1) \sum_{i,j} \frac{1 - 2\beta}{(\sigma - \rho^2 + (\sigma - \rho^2)} \leq (2\sigma - 1) \sum_{i,j} \frac{1 - 2\beta}{(\sigma - \rho^2 + (\sigma - \rho^2)}
\]
\[
= -(2\sigma - 1) \frac{\varphi^*}{\varphi^*} (\frac{1}{2} + i, x).
\]

We now get from (8.14) and (8.12)
\[
(8.15) \quad x, - \sum_{i,j} |\varphi_{ij}(\rho, x)|^2 \leq O(\sigma - \frac{1}{2}) (1 - \frac{\varphi^*}{\varphi^*} (\frac{1}{2} + i, x))
\]
\[
= O(\sigma - \frac{1}{2}) W(x).
\]

In order to estimate the components of 
\[E(\rho, \beta, x), \text{ we first estimate } \varphi_{ij} \text{ for the fourier coefficients } \varphi_{ij}(\rho, x) \text{ in the expansions (7.5), (7.11), (7.14) for } 1 \leq i \leq \infty, \quad 1 \leq j \leq \infty. \text{ For this purpose we shall go back to formula (7.43). First consider the regions } \gamma_j \geq \gamma_j, |\gamma_j| \leq \frac{1}{2}, \]

for \( 1 \leq j \leq x \), in the notations of the previous paragraph, and take \( \eta > 0 \), so small that the regions together cover \( D \) completely. Next consider the region \( \tilde{D}^{(j)} \) defined by:

\[
\sqrt{j} \geq \eta, \quad |x_j| \leq \frac{1}{2}, \quad \text{for a fixed } j, 1 \leq j \leq x,
\]

and this region in general will not be contained in \( D \), but by adding to \( D \) a finite number of images of \( D \) by suitable transformations of \( G \), one can obtain a domain \( D^{(j)} \) which contains it. Therefore, extending the notation \( \tilde{E}(z, s, x) \), to the images of \( D \) by the relation \( \tilde{E}(x, \xi, x) = \Phi(\xi) \tilde{E}(z, s, x) \), we get from (7.43), taking the trace,

\[
\begin{aligned}
\mathcal{F} \left\{ \tilde{E}(z, s + i\eta, x) \right\}_{D^{(j)}} &= \frac{1}{y^2} \int_{D^{(j)}} \tilde{E}(z, s + i\eta, x) \ dx \ dy \\
&= \mathcal{O}\left( \frac{1}{2^{\sigma-1}} \sum_{1 \leq \xi \leq x} \left( \int_{1 \leq s \leq x} \left( \Phi(D(s, x)) \right) A^{(s, x)} \ dx \right)^2 \right) \\
&\quad + \sum_{1 \leq \xi \leq x} \phi_i(s + i\eta, x) \left( \Phi^{(i)}(s + i\eta, x) A^{(s + i\eta, x)} \right) \\
&\quad \text{or assuming } \left| \Phi_{ij}(s + i\eta, x) \right| \leq \frac{3}{2},
\end{aligned}
\]

(8.16) \[
\begin{aligned}
\mathcal{F} \left\{ \tilde{E}(z, s + i\eta, x) \right\}_{D^{(j)}}& \tilde{E}(z, s + i\eta, x) \ dx \ dy = \\
&= \frac{1}{y^2} \int_{D^{(j)}} \tilde{E}(z, s + i\eta, x) \ dx \ dy.
\end{aligned}
\]
\[
= \mathcal{O} \left( \frac{1}{2\sigma - 1} \kappa_1 (A^{2\sigma - 1} - A^{1-2\sigma}) \right) + \\
+ \mathcal{O} \left( \frac{1}{2\sigma - 1} A^{1-2\sigma} (\kappa_1 - \sum_{1 \leq i, k \leq \kappa_1} |\varphi_{i,j}(\sigma + in, \chi)|^2) \right) \\
+ \mathcal{O}(1)
\]

\[
= \mathcal{O}(1) + \mathcal{O}(\omega(n)) + \mathcal{O}(1) = \mathcal{O}(\omega(n)),
\]
by (8.15), and since \( \varphi_{i,j} \) is uniformly bounded in the region for \( s \) being considered. We now take our constant \( A \) as large that in (8.16)

\[\hat{E}(x, \sigma, x) \text{ differs from } E(x, \sigma, x) \text{ only for } y_j \geq A, \] (8.16) gives since the integrand is non-negative.

\[
(8.17) \sum_{S(\sigma)} \int_{S(\sigma)} \int_{S(\sigma)} \hat{E}(x, \sigma + in, x) E(x, \sigma + in, x) \frac{dx dy}{y^2} = \mathcal{O}(\omega(n))
\]

or for \( 1 \leq i \leq \kappa_1 \),

\[
(8.17') \sum_{S(\sigma)} \int_{S(\sigma)} \int_{S(\sigma)} |\hat{E}_{i,j}(x, \sigma + in, x)|^2 \frac{dx dy}{y^2} = \mathcal{O}(\omega(n))
\]

Suppose first that \( 1 \leq j \leq \kappa_1 \), then from (7.5) and (7.11) we get for \( m \neq 0 \),
\[(8.18) \quad \int_{\eta}^{\infty} \left| \chi_{ij}(s_j, \sigma+in, x) \right|^2 \frac{dy}{q^2} = O(w(r)) \]

Writing now

(N.B.) \( \chi_{ij}(s_j, \sigma, x) = \beta_{ij}(s_j, \sigma, x) \sqrt{\eta} \int_{0}^{\infty} t^{\sigma - \frac{j}{2}} e^{-\frac{1}{2} \left(1 + \eta^2 t^2 \right)} \frac{dt}{t} \)

We get from (8.18)

\[|\beta_{ij}(\sigma + in, x)|^2 \int_{\eta}^{\infty} \int_{0}^{\infty} t^{\sigma - \frac{j}{2}} e^{-\frac{1}{2} \left(1 + \eta^2 t^2 \right)} \frac{dt}{t} \frac{dy}{\eta} = O(w(r)). \]

From (8.2), we now obtain

\[\int_{\eta}^{\infty} \int_{0}^{\infty} t^{\sigma - \frac{j}{2}} e^{-\frac{1}{2} \left(1 + \eta^2 t^2 \right)} \frac{dt}{t} \frac{dy}{\eta} = \int_{\eta}^{\infty} \int_{0}^{\infty} t^{\sigma - \frac{j}{2}} e^{-t \left(1 + \eta^2 \right)} \frac{dt}{t} \frac{dy}{\eta} \]

\[> \frac{1}{(1 + 1/n + n^2 \eta^2)^2} e^{-\pi (21 - 6 \pi ln \eta)} \eta - c. \]

Inserting this in the previous inequality we get
\[
\beta^{(i,j)}_{n \lambda}(s \pm i \eta, X)^2 = O\left( \omega(n) e^{\frac{\pi n \lambda_1 + 6 \pi m \eta}{(1 + n \lambda_1 + 10 m \eta)^4}} \right)
\]

\[
= O\left( \omega(n) (1 + n \lambda_1)^4 e^{\pi n \lambda_1 + 20 \pi m \eta} \right).
\]

Thus for \(1 \leq i, j \leq \lambda\),

\[(8.19) \quad \beta^{(i,j)}_{n \lambda}(s \pm i \eta, X) = O\left( \frac{\omega(n)}{(1 + n \lambda_1)^2 e^{\frac{\pi}{2} n \lambda_1 + 10 m \eta}} \right)
\]

Similarly if one for \(\lambda \leq i \leq \lambda\), write

\[(NB) \quad \alpha^{(i,j)}_{n \lambda}(s \pm i \eta, X) = \beta^{(i,j)}_{n \lambda}(s, X) \sqrt{g} \int_0^\infty e^{-\pi m + \alpha_1 t y_j(t + \frac{1}{n \lambda_1})} dt,
\]

we get for all \(m\).

\[(8.19') \quad \beta^{(i,j)}_{n \lambda}(s, X) = O\left( \frac{\omega(n)}{(1 + n \lambda_1)^2 e^{\frac{\pi}{2} n \lambda_1 + 10 m \eta + \alpha_1 \eta}} \right).
\]

From (8.1), we now get for \(\lambda \geq 10 \eta\), assuming first \(1 \leq j \leq \lambda\),

\[(8.20) \quad \alpha^{(i,j)}_{n \lambda}(s \pm i \eta, X) = O\left( \omega(n) e^{2 n \lambda_1 - 2 m \eta y_j} \right),
\]

for \(m \neq 0\),

and similarly for \(\lambda, 1 < j \leq \lambda\), and all \(m\).

\[(8.20') \quad \alpha^{(i,j)}_{m \lambda}(s \pm i \eta, X) = O\left( \omega(n) e^{2 n \lambda_1 - 2 m \eta + \alpha_1 \eta y_j} \right).
\]
Inverting this in the Fourier expansion for \( E_i(\Re, \theta, \sigma, \chi) \), we obtain first for \( 1 \leq j \leq 2 \Re \), and \( \Re_j > 10 \Re \),

\[
(8.21') \quad E_i(\Re, \theta, \sigma, \chi) = \delta_{ij} g_j + \varphi_{ij}(\Re, \sigma, \chi) \Re_j^1 - \Re + O\left( \sqrt{\Re(a)} e^{2(\Re_1 - 2 \Re_j)} \right),
\]

where \( \delta_{ij} = 1 \) if \( i = j \) and 0 otherwise. Similarly for \( \Re_1 < j \leq 2 \Re \), \( \Re_j > 10 \Re \), we get

\[
(8.21') \quad E_i(\Re, \theta, \sigma, \chi) = O\left( \sqrt{\Re(a)} e^{2(\Re_1 - 2 \Re_j)} \right)
\]

if \( \Re_j > 0 \), in \( e^{2 \Re_1 \Re_j} = \chi(S_j) \) is chosen so that \( 12 \Re_j^1 \leq \frac{1}{2} \). Since the regions \( \Re_j > 10 \Re \), \( 1 \leq j \leq 2 \Re \), together cover \( \Re \) completely (8.21) and (8.21') give us estimations for the \( E_i \) and as for \( E(\Re, \theta, \sigma, \chi) \) everywhere in \( \Re \). It is clear that the estimations (8.21) and (8.21') still hold if we instead of the region

\[
\frac{1}{2} \leq \sigma \leq \frac{3}{2}, \quad |\Re_1| > 1,
\]

consider any region obtained from \( \frac{1}{2} \leq \sigma \leq \frac{3}{2}, \) by removing small circular regions with the poles \( \Re = \sigma, \) \( 1 \leq k \leq N \).
centers. In particular this is true for the region $\frac{1}{2} < \sigma < 0$ if $\theta > \frac{1}{2}$ is small enough (note $\theta < 0$, for $1 \leq i \leq N$).

§ 9.

The trace formula in the case of a singular $\chi$.

We shall have to consider in addition to the kernels $K(z, z', x) = \sum_{\eta \in \Gamma} \chi(\eta) k(\eta^{-1} z, \eta^{-1} z')$, another set

\begin{equation}
(9.1) \quad H(z, z', x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} E(z', \frac{1}{2} + i \tau, x) E(z, \frac{1}{2} + i \tau, x) h(\tau) d\tau
\end{equation}

where $h(\tau)$ as before is an even function, analytic in some strip $|Y(\tau)| \leq \frac{1}{2} + \varepsilon$, and for the present we will assume that

\begin{equation}
(9.2) \quad h(\tau) = O(e^{-51|\tau|})
\end{equation}

in this strip.

First, the existence of the integral on the right-hand side of (9.1), follows from the estimations (8.21) and (8.21') combined with (8.13). We shall next
look at the behaviour of \( H(z, z', x) \) as \( z \) or \( z' \) or both tend towards the cusps of the fundamental domain \( \mathcal{D} \). Let us first consider the case that both \( z \) and \( z' \) lie in the region \( \Re s > 10 \), \( |x_j| \leq \frac{1}{2} \) for some \( j \), \( 1 \leq j \leq n \), writing

\[
(9.3) \quad E_i(z, z', x) = \delta_{ij} y_j^s + \gamma_{ij} (0, x) y_j^{1-s} + E_i^*(z, z', x),
\]

we have according to (8.21)

\[
(9.4) \quad E_i^*(z, z', x) = O \left( \sqrt{W(n)} e^{2 |\Re s|} |z|^2 \right),
\]

uniformly for \( \frac{1}{2} \leq \Re s \leq \theta, \quad \theta > \frac{1}{2} \). Inserting the expressions (9.3) in

\[
H(z, z', x) = \frac{1}{4\pi} \int_{\Re(s) = \frac{1}{2}} \sum_{1 \leq i \leq n} E_i(z, \frac{i}{2} + s, x) E_i^*(z', \frac{i}{2} + s, x) \, ds,
\]

we get the left-hand side breaks up into 9 terms, which we shall consider separately. First there is the term that contains

\[
\frac{1}{4\pi} \int_{\Re(s) = \frac{1}{2}} \sum_{1 \leq i \leq n} E_i^*(z, \frac{i}{2} + s, x) E_i^*(z', \frac{i}{2} + s, x) \, ds,
\]

From (9.4) and (8.13), we get immediately
\[ O(e^{-2(g_1 + g'_1)}) = O(1). \]

Then there are 4 terms that contain only one \( E^* \) factor, namely
\[
\frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} f_0(x) y_j \left( \frac{z}{i + \lambda} \right) E^*_j(z', \frac{z}{i + \lambda}, x) \, dz,
\]

\[
\frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} h(x) y_j \sum_{1 \leq i \leq n_1} \varphi_{ij} \left( \frac{z}{i + \lambda}, x \right) E^*_c(z', \frac{z}{i + \lambda}, x) \, dz
\]

and the remaining two are obtained by interchanging \( z \) and \( z' \) and taking \( \overline{\lambda} \) instead of \( \lambda \). Now since
\[
E_j(z', \frac{z}{i + \lambda}, \overline{\lambda}) = \sum_{1 \leq i \leq n_1} \varphi_{ij} \left( \frac{z}{i + \lambda}, \overline{\lambda} \right) E_c(z', \frac{z}{i + \lambda}, \overline{\lambda})
\]
and
\[
\varphi_{ij} \left( \frac{z}{i + \lambda}, \overline{\lambda} \right) = \varphi_{ij} \left( \frac{z}{i + \lambda}, \lambda \right),
\]
we have
\[
\sum_{1 \leq i \leq n_1} \varphi_{ij} \left( \frac{z}{i + \lambda}, \lambda \right) E^*_c(z', \frac{z}{i + \lambda}, \lambda) = \overline{E^*_j(z', \frac{z}{i + \lambda}, \lambda)}
\]

so that the second of the above two terms integrates over into the first by writing \( -i \) instead of \( i \). We therefore look only at the first integral, and moving the line of integration
slightly

slightly upwards by writing \( i (\theta - \frac{1}{2}) \) + \( r \) instead of \( r \), we obtain from (9.4) and (8.13), by
taking absolute values that the integral

\[ O \left( \left( \frac{1}{2} + i \theta \right) \right) = O \left( \left( \frac{1}{2} + i \theta \right)^{1-\delta} \right) \]

and so are the three other terms.

There remain the terms that do not
contain any \( E \), of these there are 4.

We first consider the two terms

\[
\frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{\Phi_{j, j} \left( \frac{1}{2} + i \theta, x \right)}{(y_j - y_{j'} \left( \frac{1}{2} + i \theta \right) \text{ dr}}
\]

and

\[
\frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \Phi_{j, j} \left( \frac{1}{2} + i \theta, x \right) (y_j - y_{j'} \left( \frac{1}{2} + i \theta \right) \text{ dr},
\]

since \( \Phi_{j, j} \left( \frac{1}{2} + i \theta, x \right) = \Phi_{j, j} \left( \frac{1}{2} - i \theta, \bar{x} \right) = \Phi_{j, j} \left( \frac{1}{2} - i \theta, x \right) \)

the second integral goes over into the first

if we replace \( r \) with \( -r \) instead of \( r \).

Moving the line of integration slightly

slightly upwards by writing \( i (\theta - \frac{1}{2}) \) + \( r \) instead of \( r \),

and observing that \( \Phi_{j, j} (x, \bar{x}) \) is uniformly

bounded for \( \frac{1}{2} \leq \theta \leq \theta \), we obtain
again that these terms are $O((y_j, y_j')^{-\alpha})$.

Finally we consider the terms

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} h(r) \, y_j^{\frac{1}{2} + \Im r} y_j^{\frac{1}{2} - \Im r} \, dr$$

and

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} h(r) \, y_j^{\frac{1}{2} - \Im r} y_j^{\frac{1}{2} + \Im r} \sum_{1 \leq i \leq \infty} |\psi_{ij}(\frac{1}{2} + \Im r, x)|^2 \, dr$$

Observing that

$$\sum_{1 \leq i \leq \infty} |\psi_{ij}(\frac{1}{2} + \Im r, x)|^2 = 1,$$

since $\phi(\frac{1}{2} + \Im r, x)$ is unitary, we get by writing $-\Im r$ instead of $\Im r$ in the last integral, that the two terms together give

$$\sqrt{\frac{y_j}{y_j'}} \int_{-\infty}^{\infty} h(r) \, e^{\Im \log \frac{y_j}{y_j'}} \, dr$$

$$= \sqrt{y_j y_j'} \, g(\log \frac{y_j}{y_j'}) \, .$$

Combining these results we obtain

(9.5) $H(2, 2', x) = \sqrt{y_j y_j'} \, g(\log \frac{y_j}{y_j'}) + O((y_j, y_j')^{-\alpha})$.
with $|\theta| > \frac{1}{2}$, for $\theta$ and $\theta'$. In the region $y_j \geq 10\gamma_j$; $|x_j| \leq \frac{1}{2}$, for some $j$, $1 \leq j \leq n$. 

In the same way one shows that for $x_k < j \leq n$, we have if $z$ and $z'$ both lie in the region $y_j \geq 10\gamma_j$; $|x_j| \leq \frac{1}{2}$, that

\[(9.5) \quad H(z, z', x) = O(1) = O((\gamma_j \gamma_{j'})^{-\delta}).\]

Finally, if $i \neq j$, $1 \leq i, j \leq n$, and $z$ lies in the region $y_i \geq 10\gamma_j$; $|x_j| \leq \frac{1}{2}$, and $z'$ in the region $y_j \geq 10\gamma_j$; $|x_j| \leq \frac{1}{2}$, we obtain in a similar way

\[(9.5') \quad H(z, z', x) = O((\gamma_j \gamma_{j'})^{-\delta}).\]

We now consider the integral operator

\[(9.6) \quad \int_{\mathbb{R}^n} (K(z, z', x) - H(z, z', x)) f(x') \, dx' dy'\]

where $K$ and $H$ are derived from the same function $f(x)$. Combining our previous results about the behavior of $K(z, z', x)$ with the ones just obtained for $H(z, z', x)$,
we see that for \( z \) and \( z' \) both in \( D \),

\[
K(z, z', x) - H(z, z', x) = O \left( \frac{\alpha}{y_1^{1-\alpha}} y_2^{1-\alpha} \right). 
\]

This implies that the integrals

\[
\iint_D \left| K(z, z', x) - H(z, z', x) \right|^2 \frac{dxdy}{y_1^2} \frac{dxd'y_1}{y_1'^2} < \infty.
\]

**Theorem.**

Now consider that \( u(z) \) be a square-integrable solution of

\[
y^2 \Delta u + \left( \frac{r}{2} + r^2 \right) u = 0
\]

and such that \( u(1/z) = N(1)u(z) \),

then one easily sees (from the Fourier expansions of \( u \) ) that \( u(z) = O \left( \frac{1}{y_1^\alpha} \right) \),

with \( \alpha < \frac{1}{2} \) as \( y_1 \to \infty \).

Therefore the integral

\[
\iint_D u(z) E(z, 1/z + i\nu, x) \frac{dxdy}{y_1^2}
\]

exists, and the value is easily seen to be zero, by first assuming \( r' + i\nu \),
and considering the expression

\[ \iint_D (u(z) \Delta E(2, \frac{x+i'y}{y}, x) - E(2, \frac{x+i'y}{y}, x) \Delta u(z)) \, dx \, dy \]

which is zero by Green's theorem, making \( \mu \rightarrow \infty \) we obtain the result

\[ \iint_D u(z) E(2, \frac{x+i'y}{y}, x) \, dx \, dy = 0 \]

for all real \( n' \). Therefore we also get

\[ \iint_D H(2, 2', n) u(z) \, dx \, dy = 0, \]

furthermore since *

\[ \iint_D K(2, 2', x) u(z) \, dx \, dy = \frac{h(n) u(z)}{y'^2} \]

we obtain

\[ (9.10) \quad \iint_D \left( K(2, 2', x) - H(2, 2', x) \right) u(z) \, dx \, dy = \frac{h(n) u(z)}{y'^2}. \]

*) Earlier we have actually only shown that if a function \( f(\gamma) \) satisfies (9.9) and also is an eigenfunction of \( \iint_D k(2, 2', x) f(\gamma) \, dx \, dy \)

then the latter expression has the value \( h(n) f(\gamma) \). However, that \( u(z) \) is actually an eigenfunction of \( \iint_D k(2, 2', x) u(z) \, dx \, dy \)

\[ \frac{\iint D k(2, 2', x) u(z) \, dx \, dy}{y'^2} \]

can be seen for instance from its Fourier expansion, where each term is an eigenfunction of the last operator with eigenvalue \( h(n) \).
On the other hand, since the operator $y^2 \Delta$ commutes with the operator
\[
S(E(K(\mathbf{x}, \mathbf{r}, \mathbf{x}')) - H(\mathbf{x}_{\mathbf{r}}, \mathbf{r}', \mathbf{x}')) f(\mathbf{x}_{\mathbf{r}}') \frac{d \mathbf{x}_{\mathbf{r}}'}{y^2}
\]
becomes (denoting by subscript to $\Delta$ the variable upon which the differential operator acts)
\[
y^2 \Delta_\mathbf{r} (E(K(\mathbf{x}, \mathbf{r}, \mathbf{x}')) - H(\mathbf{x}_{\mathbf{r}}, \mathbf{r}', \mathbf{x}'))
\]
\[
= y^2 \Delta_\mathbf{r}' (E(K(\mathbf{x}, \mathbf{r}, \mathbf{x}')) - H(\mathbf{x}_{\mathbf{r}}, \mathbf{r}', \mathbf{x}'))
\]
and because of Green's theorem. Therefore, for $\mathbf{r} = \mathbf{r}'$, the set of eigenfunctions of (9.6) which do not belong to the eigenvalue zero are square integrable over $\mathbf{D}$, and the set of eigenvalues has zero as its only point of accumulation, and to each eigenvalue $\neq 0$, belongs only a finite number of linearly independent eigenfunctions. Therefore, since the operator (9.6) commutes with $y^2 \Delta$, it follows easily that the orthonormal system of eigenfunctions can be chosen so that they are all also eigenfunctions of $y^2 \Delta$. Thus, the square integrable solutions of (9.9) give us all eigenfunctions of (9.6) that do not belong to the eigenvalue zero, and possibly a finite or infinite number that belong to the eigenvalue zero, if (9.9) has solutions for
values of \( r \) that make \( h_r(x) = 0 \).

From this we also conclude that the whole class of operators (9.6) commute, and that to the product of two such operators, derived respectively from the functions \( h_1(x) \) and \( h_2(x) \), is the operator derived from \( h_1(x) h_2(x) \).

We now proceed to compute the trace of the operator (9.6),

\[
(9.11) \quad \int_S (K(z, z, x) - H(z, z, x)) \, dx dy \quad \text{with}
\]

This expression on one hand is equal to the sum \( \sum \text{tr} h_r(x) \) extended over the

\( r \)'s for which (9.9) has square integrable solutions on the arc assuming for this reason assuming that \( h_r(x) \) can be written as a product of two functions of \( r \) that each satisfy the conditions assumed fulfilled by \( h_r(x) \) at the beginning of this paragraph, or what is the same, if we assume that replace the condition (9.2) by

\[
(9.12) \quad h_r(x) = 0 \left( e^{-10\pi^2 r} \right),
\]

and leave the other conditions unchanged,
For simplicity we shall carry out first in detail the computation of (9.11) in the case that we have only one cusp, which is placed at \( \infty \), so that the primitive parabolic transformation leaving the cusp fixed is \( S^2 = z + 1 \), and that
\[ K(1) = 1 \]
for all \( m \). We split the expression
\[ K(2, 2) - H(2, 2) \]
up into three parts.

(9.13) \[ K(2, 2) - H(2, 2) = \left\{ \sum_{n=\infty}^{\infty} k(2, 2+n) - H(2, 2) \right\} + \]
\[ \sum_{N}^{\infty} \sum_{n=\infty}^{\infty} k \left( 2, NS^{-1} \right) + \sum_{M}^{\infty} k \left( 2, M \right), \]
where the dash \( \Sigma' \) denotes that \( m = 0 \) is omitted,
\( \Sigma \) is taken over a complete set of transformations of \( \Gamma \) that do not differ by a power of \( S \) on the right side, and finally \( \Sigma'' \) is extended over all non-parabolic transformations of \( \Gamma \), including the identity transformation. From our previous results it follows that the integral (9.11) can be split correspondingly, each of the three resulting integrals existing. The last term is treated precisely as in the compact case, and gives the same contribution from the identity, the elliptic and hyperbolic transformations.
as before, so we shall only consider the chosen contribution from the first two, writing \( \overline{D} \) for the part of \( D \) that lies below the line \( y = A \) for large positive \( A \), we have

\[
(9.14) \quad \mathcal{S} \mathcal{S} \left\{ \frac{2k}{2 + \pi} (z, z + \pi) - H(z, z) \right\} \frac{dx dy}{y^2} =
\]

\[
= \lim_{A \to \infty} \mathcal{S} \mathcal{S} \frac{1}{2} \sum_{-\infty}^{\infty} k(z, z + \pi) \frac{dx dy}{y^2} - \mathcal{S} \mathcal{S} H(z, z) \frac{dx dy}{y^2}.
\]

Furthermore, we have denoting by \( \bar{D} \) the chip \( y > 0; |x| < \frac{1}{2} \) (which we may assume contains \( D \)) and by \( \bar{S} - \bar{D} \) the part of \( S \) that does not belong to \( D \), that

\[
(9.15) \quad \mathcal{S} \mathcal{S} \sum_{N} \sum_{n=-\infty}^{\infty} k(2, Ns^2 + n^{-2}) \frac{dx dy}{y^2} =
\]

\[
= \mathcal{S} \mathcal{S} \sum_{n=-\infty}^{\infty} k(2, 2 + n) \frac{dx dy}{y^2}.
\]

Therefore writing \( \overline{P} \) for the part of \( P \) that lies below the line \( y = A \), we obtain combining \( (9.14) \) and \( (9.15) \) that the contribution of the two first terms in \( (9.13) \) to the expression \( (9.11) \) is
\[(9.16) \lim_{A \to \infty} \left\{ \frac{2}{\pi} \sum_{h=-\infty}^{\infty} k(2, 2+n) \frac{dxdy}{y^2} - 2 \sum_{h=-\infty}^{\infty} H(2, 2) \frac{dxdy}{y^2} \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(n) \, dr \sum_{h=-\infty}^{\infty} |E(2, \frac{1}{2} + in)|^2 \frac{dxdy}{y^2}.

Here we observe first consider first

\[(9.17) \frac{\sum_{r=0}^{\infty} H(2, 2) \frac{dxdy}{y^2}}{\mathcal{D}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(n) \, dr \sum_{h=-\infty}^{\infty} |E(2, \frac{1}{2} + in)|^2 \frac{dxdy}{y^2}.

Combining (7.42'), with the fact that for \(y > A\)

\[E(2, \frac{1}{2} + in) = E(z, \frac{1}{2} + in) - \frac{g^{2/3}}{\pi} \Phi(\frac{1}{2} + in) y^{1/2 - in} - O(\sqrt{\omega(n)} e^{2in1 - 2ny}),\]

we obtain

\[(9.18) \sum_{r=0}^{\infty} |E(2, \frac{1}{2} + in)|^2 \frac{dxdy}{y^2} = 2 \log A - \frac{\Phi}{\pi} \left( \frac{1}{2} + in \right) + \frac{\Phi(\frac{1}{2} + in) A^{2in}}{\Phi(\frac{1}{2} + in) A^{-2in}} + O(\omega(n) e^{4in1 - 4A}).\]

Now

\[\int_{-\infty}^{\infty} |h(n)| W(n) e^{i\pi n - 4A} \, dr = O(e^{-4A})\]

and

\[\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi(\frac{1}{2} - in) A^{2in} - \Phi(\frac{1}{2} + in) A^{-2in}}{\Phi(\frac{1}{2} - in) + \Phi(\frac{1}{2} + in)} h(n) \, dr = \]
\[
\begin{align*}
&= \frac{1}{2\pi} \int \left( \varphi \left( \frac{1}{2} + i \psi \right) A \sin \frac{\varphi}{2} - \varphi \left( \frac{1}{2} + i \psi \right) A \sin \frac{\varphi}{2} \right) \frac{h(n)}{A} \, dr + o(1) \\
&= \frac{1}{2\pi} \int \frac{h(0) \varphi \left( \frac{1}{2} \right) \sin 2\pi \log A}{\pi} \, dr \\
&\quad + \frac{1}{2\pi} \int \frac{\varphi \left( \frac{1}{2} - i \psi \right) h(n) - \varphi \left( \frac{1}{2} \right) h(0)}{2\pi} A \sin \frac{\varphi}{2} \, dr \\
&\quad + \frac{1}{2\pi} \int \frac{\varphi \left( \frac{1}{2} + i \psi \right) h(n) - \varphi \left( \frac{1}{2} \right) h(0)}{2\pi} A \sin \frac{\varphi}{2} \, dr + o(1) \\
&= \frac{1}{2\pi} h(0) \varphi \left( \frac{1}{2} \right) \int \frac{\sin 2\pi \log A}{\pi} \, dr + o(1) \\
&= \frac{1}{2\pi} h(0) \varphi \left( \frac{1}{2} \right) \int \frac{\sin x}{x} \, dx + o(1) \\
&= \frac{1}{2\pi} h(0) \varphi \left( \frac{1}{2} \right) \int \frac{\sin x}{x} \, dx + o(1) = \frac{1}{2} h(0) \varphi \left( \frac{1}{2} \right) + o(1).
\end{align*}
\]

as \( A \to \infty \).

Also
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} h(n) \, dn = \varphi(0).
\]

Thus from (9.17) and (9.18) we get
\[
(9.19) \quad 2\int_{-\infty}^{\infty} H(2, z) \frac{dx}{y^2} = 2\varphi(0) \log A - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(n) \frac{\varphi'}{\varphi \left( \frac{1}{2} + i \psi \right)} \, dr \\
+ \frac{1}{2} h(0) \varphi \left( \frac{1}{2} \right) + o(1),
\]

as \( A \to \infty \).
Secondly, we have

\[
\begin{align*}
(9.20) \quad 2 \sum_{n=0}^{\infty} \int_{\frac{A}{\sqrt{n}}}^{\infty} k(z, z+n) \, dx \, dy &= \\
&= 4 \sum_{n=1}^{\infty} \int_{A}^{\infty} k \left( \frac{m^2}{y^2} \right) \, dy \\
&= 4 \sum_{n=1}^{\infty} \left[ \frac{1}{m} \int_{A}^{\infty} k(u^2) \, du \right] \\
&= 4 \int_{A}^{\infty} k(u^2) \left( \sum_{m<Au} \frac{1}{m} \right) \, du,
\end{align*}
\]

here uniformly for \( u > 0 \),

\[
\sum_{\frac{A}{\sqrt{n}} < Au} \frac{1}{m} = \log A + \log u + c + O \left( \frac{1}{\sqrt{Am}} \right)
\]

where \( c \) is Euler's constant, thus

\[
(9.21) \quad 2 \sum_{n=0}^{\infty} \int_{\frac{A}{\sqrt{n}}}^{\infty} k(z, z+n) \, dx \, dy = 4 \left( \log A + c \right) \int_{A}^{\infty} k(u^2) \, du +
\]

\[
4 \int_{0}^{\infty} \log u \, k(u^2) \, du + O \left( \frac{1}{\sqrt{A}} \right).
\]

Here

\[
\int_{0}^{\infty} k(u^2) \, du = \frac{1}{2} \int_{0}^{\infty} \frac{K(t)}{\sqrt{t}} \, dt = \frac{1}{2} \vartheta (0).
\]
and further
\[ 4 \int_0^\infty \log u \ k(u^2) \, du = \int_0^\infty \ k(t) \ \frac{\log t}{\sqrt{t}} \, dt \]

Now
\[ k(t) = -\frac{1}{\pi} \int_0^\infty \frac{d \ Q(\omega)}{t} \frac{1}{\sqrt{\omega - t}} \]

where \ Q(\omega) = e^{-\omega^2} - 1\), inserting this we get
\[ 4 \int_0^\infty \log u \ k(u^2) \, du = -\frac{1}{\pi} \int_0^\infty \ d \ Q(\omega) \left\{ \int_0^\infty \frac{\log t}{\sqrt{t} (\omega - t)} \, dt \right\} \]

\[ \log t \]

Now for \ s \ real > -1
\[ \int_0^w \frac{t^s}{\sqrt{\omega - t}} \, dt = w^{s+\frac{1}{2}} \int_0^1 \frac{t^{s+\frac{1}{2}}}{\sqrt{1-t}} \, dt = w^{s+\frac{1}{2}} \frac{\Gamma(s+1) \Gamma\left(\frac{1}{2}\right)}{\Gamma(s+\frac{1}{2})} \]

Differentiating with respect to \ w \ and putting \ s = -\frac{1}{2},
we get easily
\[ \int_0^w \frac{\log t}{\sqrt{t} (\omega - t)} \, dt = \frac{1}{\pi} \left( \log w - 2 \log 2 \right) \]
\[ 4 \int_0^\infty \log u \, k(u^2) \, du = -\int_0^\infty (\log (\pi u - 2 \beta_2)) \, d \varphi(u) \]
\[ = -\int_0^\infty \log (e^u + e^{-u}) \, d \varphi(u) - 2 \beta_2 \varphi(0) = \]
\[ = -2 \int_0^\infty \log (e^{\frac{u}{2}} - e^{-\frac{u}{2}}) \, d \varphi(u) - 2 \beta_2 \varphi(0) \]
\[ = -2 \int_0^\infty \log (1 - e^{-u}) \, d \varphi(u) - 2 \int_0^\infty u \, d \varphi(u) - 2 \beta_2 \varphi(0) \]
\[ = -2 \int_0^\infty \log (1 - e^{-u}) \, d \varphi(u) + \int_0^\infty \varphi(u) \, du - 2 \beta_2 \varphi(0) \]
\[ = -2 \int_0^\infty \log (1 - e^{-u}) \, d \varphi(u) + \frac{1}{2} \varphi(0) - 2 \beta_2 \varphi(0). \]

Here
\[ \varphi'(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sin h(u) e^{iu} \, du, \]

inserting this we have
\[ -2 \int_0^\infty \log (1 - e^{-u}) \, d \varphi(u) = -\frac{2\beta_2}{\pi} \int_{-\infty}^{\infty} h(u) \, du \int_{-\infty}^{\infty} e^{iu} \log (1 - e^{-u}) \, du, \]

and
\[ -i \pi \int_{-\infty}^{\infty} e^{iu} \log (1 - e^{-u}) \, du = \sum_{n=1}^{\infty} \frac{i \pi}{\sin(n \pi)} \int_{-\infty}^{\infty} e^{i(n-1)u} \, du \]
\[ = \sum_{n=1}^{\infty} \frac{i \pi}{\sin(n \pi)} \int_{-\infty}^{\infty} \left( \frac{1}{\sin(n \pi \sin u)} - \frac{1}{n \pi} \right) \, du = \]

\[ \sum_{n=1}^{\infty} \frac{i \pi}{\sin(n \pi \sin u)} - \frac{1}{n \pi}. \]
\[ = - \frac{P'}{P} (1 - i \pi) - c, \]

where \( c \) is Euler's constant. Thus

\[ -2 \int_0^\infty \log (1 - e^{-u}) \log(u) \, du = -\frac{1}{\pi} \int_0^\infty h(n) \left( \frac{P'}{P} (1 - i \pi) + c \right) \, dn \]

\[ = -2 \arg(0) - \frac{1}{\pi} \int_0^\infty h(n) \frac{P'}{P} (1 + i \pi) \, dn. \]

Hence

\[ 4 \int_0^\infty \log u \, f(x^2) \, du = -\frac{4}{\pi} \int_0^\infty h(n) \frac{P'}{P} (1 + i \pi) \, dn \]

\[ = 2(\log 2) \arg(0) + \frac{1}{2} h(0). \]

Inserting our results in (9.21) we get

\[ (9.22) \quad 2 \int_0^\infty \sum_{n=-\infty}^{\infty} \frac{f(x^2, x^2 + m)}{y^2} \, dx \, dm = 2 \log A \, \arg(0) \]

\[ - 2 \log 2 \, \arg(0) + \frac{1}{2} h(0) - \frac{4}{\pi} \int_0^\infty h(n) \frac{P'}{P} (1 + i \pi) \, dn + O\left(\frac{1}{A}\right), \]

as \( A \to \infty \). Finally, inserting (9.19) and (9.22) in (9.16) and letting \( A \to \infty \), we find the contribution of the parabolic harmonics to be

\[ (9.23) \quad \frac{1}{2 \pi} \int_0^\infty h(n) \frac{P'}{P} (1 + i \pi) \, dn - \frac{4}{\pi} \int_0^\infty h(n) \frac{P'}{P} (1 + i \pi) \, dn \]
\[ -2 \log g(0) + \frac{1}{2} \left( 1 - \Phi(\frac{1}{2}) \right) h(0). \]

In the general case one may proceed in a similar way and obtains then as the contribution of all parabolic transformations (including the ones treated already in \( \S \), when \( \mathfrak{X} \) of the primitive transformation \( \neq 0 \)),

\[(9.23') \quad \frac{1}{2\pi i} \int_{-\infty}^{\infty} h(n) \Phi'(\frac{1}{2} + i\tau, x) \, d\tau - \frac{2\pi i}{\pi} \sum_{n \neq 0} \int_{-\infty}^{\infty} h(n) \Phi'(\frac{1}{2} + i\tau, x) \, d\tau \]

\[ + \frac{1}{2} \left( \mathfrak{X}, - \Phi(\frac{1}{2}, x) \right) h(0). \]

It should be observed that since the eigenvalues of the \( \mathfrak{X} \) matrix \( \Phi(\frac{1}{2}, x) \) are all \( \pm 1 \), because \( \Phi(\frac{1}{2}, x) \) is unitary and equal to its conjugate transposed, the coefficient of \( h(0) \) in \((9.23')\) is always a non-negative integer \( \leq \mathfrak{X} \).

We can now write down the formula...
\begin{align*}
(9.24) \sum_i h(n_i) &= \frac{N(\theta)}{lim} \int_{-\infty}^{\infty} e^{\frac{un - e^{-un}}{e^{un} + e^{-un}}} h(n) \, dn \\
+ \sum_{\{R^2\}} \sum_{k=1}^{n-1} \frac{\chi_k(R)^2}{\sin \frac{k \omega}{\omega}} \int_{-\infty}^{\infty} e^{-2\pi n \frac{k \omega}{\omega}} h(n) \, dn \\
+ 2 \sum_{\{P^2\}} \sum_{k=1}^{\infty} \frac{\chi_k(P) \log N\xi P^2}{(N\xi P^2)^{\frac{k}{2}} - (N\xi P^2)^{-\frac{k}{2}}} \log \left( \log N\xi P^2 \right) \\
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} h(n) \frac{\phi'(1 + i\xi, \chi)}{\phi(1 + i\xi, \chi)} \, dn - \frac{\chi_i}{\pi} \int_{-\infty}^{\infty} h(n) \frac{P'}{P}(1 + i\xi) \, dn \\
- 2\pi i \log 2 \cdot g(0) + 2 \sum_{\xi, \xi' \leq \xi} \log \left( \frac{1}{1 - \chi(\xi, \xi')} \right) \cdot g(0) \\
+ \frac{1}{2} \left( \chi_1 - \sum \left( \phi(\frac{1}{2}, \chi) \right) \right) \cdot h(0),
\end{align*}

This formula has now been proved valid under the condition (9.12) in addition to the earlier conditions imposed upon \( h(n) \). We now wish to show that (9.24) is valid under the conditions

(i) \( h(n) = h(-n) \); (ii) \( h(n) \) regular for \( |Y(n)| < \frac{1}{2} + \varepsilon \), and instead of (9.12) the condition

\( ki \leq \alpha \).
(9.25) \[ h(r) = O \left( \frac{1}{(1+r^2)^{2+\epsilon}} \right) \]

For this purpose we move the second term on the right-hand side of (9.24) over to the left-hand side, and put in the function

\[ h(r) = e^{-\frac{r^2}{R^2}}, \]

where \( R \) is a large positive number. This function clearly satisfies (9.12) and the other conditions on \( h(r) \). Furthermore, we have from (8.10)

\[-\frac{\phi'}{\phi} (\frac{1}{2} + in, x) = -\frac{\phi_*}{\phi_*} (\frac{1}{2} + in, x) + O(1), \]

where the first term on the right-hand side is non-negative. Also \( \frac{r'}{r} (1 + in) = O \left( \log(2+2ri) \right) \).

Using this we easily get from (9.24)

(9.26) \[ \sum_{|n| \leq R} \frac{1}{2\pi} \int_{-R}^{R} \frac{\phi_*}{\phi} (\frac{1}{2} + in, x) \, dr = O(R^2) \]

or

(9.26') \[ \sum_{|n| \leq R} \frac{1}{2\pi} \int_{-R}^{R} |\frac{\phi'}{\phi} (\frac{1}{2} + in, x)| \, dr = O(R^2) \]

This implies actually that all series
and infinite series occurring in (9.24) converges absolutely if \( h(n) \) satisfies (9.25) and the other standard conditions; also, for a class of functions \( h(n) \) that satisfy these conditions uniformly, we see that the convergence of the series and integrals is uniform. Taking now for a fixed \( h(n) \) the class \( h(n) e^{-\varepsilon R^2} \), \( \varepsilon > 0 \), these constitute another class, and for \( \varepsilon > 0 \), (9.12) is satisfied as that (9.24) is valid for the function \( h(n) e^{-\varepsilon R^2} \). Letting \( \varepsilon \to 0 \), we obtain that the formula (9.24) is valid if \( h(n) \) is even, analytic in some region \( |f(n)| < \frac{1}{2} + \varepsilon \), and \( h(n) = O\left(\frac{1}{(1 + |n|)^{2+\varepsilon}}\right) \).

Before we go. From (9.26) and (8.11), we get

\[
\sum_{n} (1 - 2\beta) \int_{-2R}^{2R} \frac{1}{(\beta - \varepsilon)^2 + (n-\lambda)^2} \, dn = O\left(R^2\right)
\]

Here we have, using (8.9).
\[
\sum_{1/\lambda \leq R} \int_{-2R}^{2R} \frac{1}{(\beta-\frac{1}{2})^2 + (\lambda-x)^2} \, dr = \\
= \sum_{1/\lambda \leq R} \int_{-\infty}^{\infty} \frac{dr}{(\beta-\frac{1}{2})^2 + \lambda^2} + O\left(\sum_{1/\lambda \leq R} \frac{1}{\lambda - \beta}\right) \\
= 2\pi \sum_{1/\lambda \leq R} 1 + O(\log R),
\]

Thus we get

(9.27) \( \sum_{1/\lambda \leq R} 1 = O\left(R^2\right) \)

which shows that \(\varphi(s, x)\) actually is a quotient of integral functions at most of order \(2\), the same result carries over to the \(E_i(z, s, x)\) as functions of \(s\).

More precisely one can prove from (9.24) that

(9.28) \( \sum_{1/\lambda \leq R} 1 + \sum_{1/\lambda \leq R} 1 = \mu(\beta) R^2 + O(R \log R) \)

Unfortunately, however, we have no means of estimating the two terms on the left-hand side separately, except in some special cases, when the function \(\varphi(s, x)\) can be expressed
in terms of functions that are known from
analytic number theory. Such a case will
be mentioned later in connexion with
the modular group. It should be mentioned
that in all these cases where there are
those have
\[ \sum_{1 \leq \| \alpha \| \leq R} 1 = O(R \log R), \]
so that
\[ \sum_{1 \leq \| \alpha \| \leq R} 1 = \mu(\mathbb{D}) R^2 + O(R \log R), \]
in these cases. It is an open question
whether this is true in the general case.

We may now as in the previous cases
study the function
\[ Z_p(\alpha, x) = \prod_{\| \alpha \| \leq R} \prod_{v=0}^{\infty} (1 - x \xi_p \left( N\xi_p \right)^{-\alpha - \nu}) \]
by inserting in (9.24) the special
function
\[ h(R) = \left( R^{-\frac{1}{2}} \right) \frac{1}{\left( \alpha - \frac{1}{2} \right)^2 + R^2} - \frac{1}{(a - \frac{1}{2})^2 + R^2} \]
where \( a \) is a constant > 1, and \( R \) is > 1. The new terms occurring in
The trace formula produces some important changes in the function properties of the function. First of all, \( Z_p(0, x) \) is no longer an integral function, the first term on the right-hand side of (9.24) produces poles at the points \( s = \frac{1}{2}, m = 1, 2, 3, \ldots \) of order \( k_i \), in addition the fifth term produces a pole at \( s = \frac{1}{2} \) of order

\[ k_i - 1 + \phi(\frac{1}{2}, x) \]

which however may be superimposed on some zero if some of the \( k_i = 0 \). As before, the first and ninth term produces zeros at 0 and the negative integers. The first term produces poles of \( Z_p(0, x) \) at the zeros \( p = \beta + i\gamma \) of \( \psi(0, x) \) in the half-plane \( \Re(\beta) < \frac{1}{2} \), accordingly the poles and zeros at the points \( 1 - 5x \), these poles however are cancelled out by the zeros produced by the term on the left-hand side of (9.24) at the points \( \frac{1}{2} + i\pi \), since
the points \( \pm (5i - \frac{1}{2}) \), belong to the set \( \mathbb{R} \).

According to our results mentioned at the end of §7.

One can also from (9.24) find a functional equation for \( \Xi(r, x) \), which we shall not give here, it differs from the previous case in that it has the form

\[
\frac{\Xi(r - 1, x)}{\Xi(r, x)} = \Psi(r, x) \text{ times a simpler function}
\]

that can be given explicitly as a canonical product.

Let us finally consider as an explicit case the modular group and \( \Xi(m) = 1 \) for all \( m \). The formulas in §8 then easily give that

\[
\Psi(r) = \sqrt{\pi} \frac{\Gamma(r - \frac{1}{2})}{\Gamma(r)} \frac{\zeta(2r)}{\zeta(2r - 1)}
\]

\[
= \sqrt{\pi} \frac{\Gamma(r - \frac{1}{2})}{\Gamma(r)} \frac{\zeta(2r - 1)}{\zeta(2r)}
\]

where \( \Psi(m) \) is Euler's function \( \Psi(m) = m \prod_{n|m} (1 - \frac{1}{n}) \), and \( \zeta \) the Riemann zeta function. From this expression we see that the only poles
of \( \varphi(s) \) in this case are the points \( s = \frac{p}{2} \), where \( p \) runs over the nontrivial zeros of the Riemann zeta function, and \( s = 1 \), also in this case \( \varphi(\frac{1}{2}) = -1 \).

In this case (9.24) can be generalized considerably.

In that the operator (9.6) can be combined with one of the so-called "Hecke operators" \( T \) (with which it commutes), and the trace of the product computed in a similar way.

Also one can in the general case show that if we have a \( \varphi(t) \) which is square-integrable over \( \mathbb{T} \) and satisfies \( \varphi(t) = \chi(t) \varphi(t) \), then \( \varphi(t) \) has an expression convergent in the \( L^2 \) sense,

\[
f(t) = \frac{1}{4\pi} \int \alpha(\xi) E(2, t + i\alpha, \xi) \, d\alpha + \sum \beta(\xi) \zeta(\xi)
\]

where \( \zeta(\xi) \) is the square-integrable eigenfunction corresponding to the eigenvalue \( \xi \), also a Parseval formula can be proved. Here
\[ \alpha(n) = \frac{SS \int_{\mathcal{D}} f(z) \Phi(z, \xi, \chi) \, dx \, dy}{\eta} \]

and
\[ \beta(n) = SS \int_{\mathcal{D}} f(z) \mu(z) \, dx \, dy \]

also a Plancherel formula

\[ SS |f(z)|^2 \, dx \, dy = \frac{1}{4\pi} \int_{-\infty}^{\infty} |\alpha(n)|^2 \, dn + \sum |\beta(n)|^2 \]

is valid.

\section{Concluding remarks}

Finally it should be remarked that these results can all be extended to the case that \( X \) is a representation by \( r \times r \) dimensional unitary matrices where \( r \geq 1 \), the resulting formulas are of course somewhat more complicated.

\section{Conclusion}

We can generalize the previous theory to cover also the case not only the automorphic functions, but also automorphic forms, that is functions transforming in
The manner
\[ f(M_2) = \chi(n)(CZ+d)^k f(z) \]
where \( k \) is a real number, and
\[ M_2 = \frac{aZ+b}{cZ+d}. \]
This is easiest to fit into the general set-up given in the introduction by considering a three-dimensional space with points \((Z, \Phi)\), \(Z = x + iy\), \(y > 0\) and \( \Phi \) real \(-\infty < \Phi < \infty\). Our group \( G \) consists of
\[ (Z, \Phi) \rightarrow (Z', \Phi' = \text{arg}(CZ+d) + \alpha) \]
where \( M_2' = \frac{aZ+b}{cZ+d} \), \( ad - bc = 1 \), and \( \alpha \) is a real constant, in addition we define the transformation \( \mu \) as
\[ (Z, \Phi) \rightarrow (-Z, -\Phi). \]
The two differential forms
\[ \frac{dx^2 + dy^2}{y^2} \quad \text{and} \quad dy + \frac{dx}{2y} \]
are invariant under \( G \) and the first also under \( \mu \), while the second changes sign under \( \mu \). Therefore any combination
\[ a \frac{\partial^2 x^2 + \partial y^2}{\gamma^2} + b \left( \frac{\partial x + \partial y}{\gamma} \right)^2 \]

with positive \( a \) and \( b \), gives us an invariant metric. The point-pair invariants under \( g \)

This space and group \( G \) satisfy all the necessary requirements, the point-pair invariants are of the form \( K \left( \frac{12 - z^2}{4y^2}, \varphi - \varphi' - \arg(z - \bar{z}) \right) \).

The invariant operators have as a basis the differential operator \( \frac{\partial}{\partial \varphi} \) and the Laplace operator corresponding to the above metric.

We now form the group \( P \) in the upper half plane and construct a discontinuous group in the three-dimensional space by associating with the transformation

\[ Mz = \frac{az + b}{cz + d} \]

the transformations \( Mz \) given by

\[ (z, \varphi) \Rightarrow (Mz, \varphi - \arg(cz + d) + k\pi) \]

for \( k = 0, \pm 1, \pm 2, \ldots \)

The volume of the fundamental domain of the new group in the three-dimensional space is finite if that of \( P \) in \( H \) was. We may then ask for functions \( f(z, \varphi) \) that satisfy the relations

\[ f(Mz, \varphi - \arg(cz + d) + k\pi) = x(Mz) f(z, \varphi) \]
in particular
\[ F(z, \varphi + \pi) = e^{i \beta} F(z, \varphi) \]
with some real \( \beta \), and in particular are eigenfunctions of our two fundamental operators, in particular \( \frac{\partial}{\partial \varphi} \), therefore they must have the form
\[ F(z, \varphi) = f(z) e^{ik \varphi} \]
with \( k = 2\hbar \varphi + \beta \); \( k \) an integer.

From this we get
\[ F(z, \varphi) = f(z) e^{-i k \varphi} = f(z) e^{i k \varphi} \]

or writing
\[ e^{-\frac{i k}{\hbar} \varphi} f(z) = f(z), \]
we have easily
\[ \psi(z, \varphi) = \chi(k)(z) (Cz + d)^{-k} f(z). \]

If we prefer we may translate everything into characteristic functions, as can be done by fixing \( k \) considering the operators
\[ \sum_{k} \kappa \left( \frac{12-21^2}{8y} \right) \left( \frac{4}{2-21} \right)^{k} \int (2') \frac{dx'dy'}{y12}, \]

the integral being extended over the upper half plane \( \mathcal{H} \). These commute among themselves and are generated by one second order differential operator (dependent on \( k \)).

We may proceed in a similar way as before and prove a more general trace formula. Of particular interest is it that the analytic forms have a rather singular position, in that

Assuming \( k > 2 \), and taking

\[ \kappa \left( \frac{12-21^2}{8y} \right)^{k} \text{ identically, we see that the expression} \]

\[ \frac{g}{(2-21)^{k}} \text{ is analytic in } z. \]

The resulting operator has the property that it reproduces any analytic function \( f(z) \) which in \( \mathcal{H} \) satisfies \( \frac{k}{2} |f| \leq A \) for some \( A > 0 \), apart from a factor depending on \( k \) only.

The trace formula for this specific operator and a group \( \Gamma \) and \( \mathcal{X}(\Gamma) \), will give us the formula for the number of analytic forms of dimension \( k \), that