write $\tilde{E}(z, s, \chi)$ for the column vector with the $\varkappa_1$ components $\tilde{E}_i(z, s, \chi)$, $1 \le i \le \varkappa_1$; ~~and~~ similarly we write $E(z,s,\chi)$ for the column vector with components $(E_i(z,s,\chi))$, and $\Phi(s; \chi)$ for the matrix

$$\Phi(s, \chi) = (\varphi_{ij}(s, \chi)) = \sqrt{\pi}\, \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, L_0(s, \chi).$$

And take further for simplicity $A_1 = A_2 = \ldots = A_{\varkappa_1} = A$; (actually one can show that they may all be chosen equal to 1, but at any rate if $A$ is sufficiently large, we can clearly make the $A_i = A$ for $1 \le i \le \varkappa_1$). Then (7.43) may be written as a matrix equation

$$(7.43') \quad \iint_{\mathcal{D}} \tilde{E}(z, \sigma+ir, \chi)\, \overline{\tilde{E}(z, \sigma+ir, \chi)}'\, \frac{dx\,dy}{y^2}$$

$$= \frac{1}{2\sigma-1}\left\{ A^{2\sigma-1} E - \Phi(\sigma+ir, \chi)\, \overline{\Phi(\sigma+ir,\chi)}'\, A^{1-2\sigma} \right\}$$

$$+ \frac{\overline{\Phi(\sigma+ir, \chi)}'\, A^{2ir} - \Phi(\sigma+ir, \chi)\, A^{-2ir}}{2ir}$$

where $E$ is the $\varkappa_1 \times \varkappa_1$ identity matrix and the dash $'$ means transposition. From (7.43') we can now proceed in a similar way as from (7.42) before.

Denoting by $\varphi(s,x)$ the value of the determinant $|\Phi(s,x)|$, we now get the following results:

$E(z,s,\chi)$ is regular ~~in~~ the region $\sigma \geq \frac{1}{2}$, except possibly for a finite number of simple poles on ~~the~~ interval of the real axis $\frac{1}{2} < s \leq 1$.

If $\frac{1}{2} < \sigma_0 \leq 1$ is a pole of $E(z,s,\chi)$, then the components of the residue of $E(z,s,\chi)$ at $s = \sigma_0$, gives rise to $\nu_{\sigma_0}$ linearly independent ~~square integrable~~ eigenfunctions of our problem, where $\nu_{\sigma_0} \leq \chi_1$ is the order of the pole of $\varphi(s,\chi)$ at $s = \sigma_0$. These eigenfunctions $u(z)$ satisfy $y^2 \Delta u + \sigma_0(1-\sigma_0) u = 0$, and $u(Mz) = \chi(M) u(z)$. and are square integrable over the fundamental domain $D$. The poles at the points $\sigma_0$ are the only poles of $\varphi(s,\chi)$ in the region $\sigma \geq \frac{1}{2}$.

## § 8.

Preliminaries to the proof of the Trace formula in the case of a singular $\chi$.

~~We shall use~~ For use in the next paragraph we shall need to develop certain estimations for $E(z,s,x)$ in the region $\frac{1}{2} \le \sigma \le \frac{3}{2}$, and also certain properties of the function $\varphi(s,x)$. ~~F~~

We first develop some inequalities for the function

$$\int_0^\infty t^{s-\frac{1}{2}} e^{-y(t+\frac{1}{t})} \frac{dt}{t}.$$

Let $\frac{1}{2} \le \sigma \le \frac{3}{2}$, and $s = \sigma + ir$, and suppose at first that $r \geq 0$; turning the line of integration an angle $\frac{\pi}{2} - \delta$, $0 \leq \delta \leq \frac{\pi}{2}$ in the positive direction, writing $t e^{(\frac{\pi}{2}-\delta)i}$ instead of $t$, we get

$$\int_0^\infty t^{s-\frac{1}{2}} e^{-y(t+\frac{1}{t})} \frac{dt}{t} = \cancel{e^{(\frac{\pi}{2}-\delta)i(s-\frac{1}{2})}}$$

$$= e^{-(\frac{\pi}{2}-\delta)r + (\frac{\pi}{2}-\delta)(\sigma-\frac{1}{2})i} \int_0^\infty t^{s-\frac{1}{2}} e^{-y(t+\frac{1}{t})\sin\delta - iy(t-\frac{1}{t})\cos\delta} \frac{dt}{t},$$

or taking absolute values

$$(8.1) \quad \left| \int_0^\infty t^{s-\frac{1}{2}} e^{-y(t+\frac{1}{t})} \frac{dt}{t} \right| \le e^{-(\frac{\pi}{2}-\delta)r} \int_0^\infty t^{\sigma-\frac{1}{2}} e^{-y\sin\delta(t+\frac{1}{t})} \frac{dt}{t}$$

$$\le 2 e^{-(\frac{\pi}{2}-\delta)r} \int_1^\infty t^{\sigma-\frac{1}{2}} e^{-y\sin\delta(t+\frac{1}{t})} \frac{dt}{t}$$

$$\le 2 e^{-(\frac{\pi}{2}-\delta)r} \int_1^\infty e^{-y\sin\delta\, t}\, dt = \frac{2 e^{-(\frac{\pi}{2}-\delta)r - y\sin\delta}}{y\sin\delta}$$

NB!)

or since $\sin\delta \geqslant \frac{2}{\pi}\delta > \frac{\delta}{2}$, $\frac{2e^{-y}}{y}$

$$(8.1') \quad \left| \int_0^\infty t^{s-\frac{1}{2}} e^{-y(t+\frac{1}{t})} \frac{dt}{t} \right| < 4\, \frac{e^{-(\frac{\pi}{2}-\delta)|r| - \frac{\delta}{2}y}}{\delta y}$$

In the form (8.1') the inequality is obviously true also for $r < 0$.

Now consider for $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$, and $\eta > 0$, the expression

$$\int_\eta^\infty (y^2-\eta^2)^{ir} y^{\frac{3}{2}-\sigma-ir}\, dy \int_0^\infty t^{\sigma-\frac{1}{2}+ir} e^{-y(t+\frac{1}{t})} \frac{dt}{t}$$

$$= \frac{1}{2}\int_\eta^\infty (y^2-\eta^2)^{ir}\, d(y^2-\eta^2) \int_0^\infty t^{\sigma-\frac{1}{2}+ir} e^{-y^2 t - \frac{1}{t}} \frac{dt}{t}$$

$$= \frac{1}{2}\int_0^\infty t^{\sigma-\frac{1}{2}+ir} e^{-\eta^2 t - \frac{1}{t}} \frac{dt}{t} \int_\eta (y^2-\eta^2)^{ir} e^{-(y^2-\eta^2)t}\, d(y^2-\eta^2)$$

$$= \frac{1}{2}\,\Gamma(1+ir) \int_0^\infty t^{\sigma-\frac{3}{2}} e^{-\eta^2 t - \frac{1}{t}} \frac{dt}{t}.$$

$$= \frac{1}{2}\,\frac{\Gamma(1+ir)}{\eta^{\sigma-\frac{3}{2}}} \int_0^\infty t^{\sigma-\frac{3}{2}} e^{-\eta(t+\frac{1}{t})} \frac{dt}{t}.$$

From this we get

$$\int_\eta^\infty y^{\frac{3}{2}-\sigma}\, dy \left| \int_0^\infty t^{\sigma-\frac{1}{2}+ir} e^{-y(t+\frac{1}{t})} \frac{dt}{t} \right|$$

$$\geq \frac{1}{2} \frac{|\Gamma(1+ir)|}{\eta^{\sigma-\frac{3}{2}}} \int_0^\infty t^{\sigma-\frac{3}{2}} e^{-\eta(t+\frac{1}{t})} \frac{dt}{t}$$

$$\geq \frac{1}{2} e^{-\frac{\pi}{2}|r|} \eta^{\frac{3}{2}-\sigma} \int_0^\infty e^{-\eta(t+\frac{1}{t})} \frac{dt}{t}$$

$$= e^{-\frac{\pi}{2}|r|} \eta^{\frac{3}{2}-\sigma} \int_1^\infty e^{-\eta(t+\frac{1}{t})} \frac{dt}{t}$$

$$> e^{-\frac{\pi}{2}|r|} \eta^{\frac{3}{2}-\sigma} \int_1^\infty e^{-(2\eta+1)t}\, dt$$

$$> e^{-\frac{\pi}{2}|r|-2\eta-1} \cdot \frac{\eta^{\frac{3}{2}-\sigma}}{2\eta+1},$$

or since the left-hand side obviously ~~decreases~~ increases with decreasing $\eta$, we get easily

$$\int_\eta^\infty y^{\frac{3}{2}-\sigma} \left| \int_0^\infty t^{\sigma-\frac{1}{2}+ir} e^{-y(t+\frac{1}{t})} \frac{dt}{t} \right| dy >$$

$$> \cancel{6}\, e^{-\frac{\pi}{2}|r|-3\eta-c},$$

where $c$ is a sufficiently large positive absolute constant.

From (8.1') we get, taking $\delta = \frac{\pi}{2}$, and $T = \pi|r| + 6\eta + 2c + 6$, that

$$\int_T^\infty y^{\frac{3}{2}-\sigma} \left| \int_0^\infty t^{\sigma-\frac{1}{2}+ir} e^{-y(t+\frac{1}{t})} \frac{dt}{t} \right| dy$$

$$< e^{-\frac{\pi}{2}|r| - 3\eta - c - 1},$$

Thus we get

$$\int_\eta^T y^{\frac{3}{2}-\sigma} \left| \int_0^\infty t^{\sigma-\frac{1}{2}+ir} e^{-y(t+\frac{1}{t})} \frac{dt}{t} \right| dy$$

$$> e^{-\frac{\pi}{2}|r| - 3\eta - c - 1},$$

using Schwartz inequality we get from this:

$$(8.2) \quad \int_\eta^\infty \left| \int_0^\infty t^{\sigma-\frac{1}{2}+ir} e^{-y(t+\frac{1}{t})} \frac{dt}{t} \right|^2 \frac{dy}{y} >$$

$$> \frac{\left( \int_\eta^T y^{\frac{3}{2}-\sigma} \left| \int_0^\infty t^{\sigma-\frac{1}{2}+ir} e^{-y(t+\frac{1}{t})} \frac{dt}{t} \right| dy \right)^2}{\int_\eta^T y^{4-2\sigma}\, dy} >$$

$$> \frac{5-2\sigma}{T^{5-2\sigma}} e^{-\pi|r| - 6\eta - 2c - 2}$$

$$\geq \frac{1}{T^{4}} e^{-\pi|r| - 6\eta - 2c - 2}$$

$$> \frac{1}{(1+|r|+\eta)^{4}} e^{-\pi|r| - 6\eta - c'},$$

where $c'$ is an absolute positive constant.

We now turn to the function $\varphi(s, \chi)$, from the previous ~~section~~ paragraph we know that $\varphi(s,\chi)$ is meromorphic, and can be written as a quotient between two integral functions of $s$ each of order at most 4, further that

(8.3) $\varphi(s, \chi)\, \varphi(1-s, \chi) = 1$

and that $\varphi(s,\chi)$ is regular for $\sigma \geq \frac{1}{2}$ except possibly for a finite number of poles in the interval $\frac{1}{2} < s \leq 1$ of the real axis. Finally

(8.4) $\varphi(s, \chi) = \left\{ \sqrt{\pi}\, \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \right\}^{\varkappa_1} l_0(s, \chi),$

where $l_0(s, \chi)$ is a Dirichlet series with real coefficients (actually the determinant of the matrix $L_0(s, \chi)$) which is absolutely convergent for $\sigma > 1$.

Now looking at (7.43) with $i=j$, and all $A_i = A$ $1 \le i \le \varkappa_1$, and observing that the left hand side is obviously positive, we get

$$\frac{1}{2\sigma-1}\left(A^{2\sigma-1} - A^{1-2\sigma}\sum_{k=1}^{\varkappa_1} |\varphi_{ik}(\sigma+ir, x)|^2\right)$$
$$+ \frac{\overline{\varphi_{ii}(\sigma+ir,x)}\,A^{2ir} - \varphi_{ii}(\sigma+ir,x)A^{-2ir}}{2ir} > 0.$$

For $|r| \geq 1$, this gives

$$\sum_{k=1}^{\varkappa_1} |\varphi_{ik}(\sigma+ir,x)|^2 \le A^{4\sigma-2} + \frac{2\sigma-1}{|r|} A^{2\sigma-1} |\varphi_{ii}(\sigma+ir,x)|$$

for $1 \le i \le \varkappa_1$, assuming also $\frac{1}{2} \le \sigma \le \frac{3}{2}$, we obtain that all $\varphi_{ik}(\sigma+ir, x)$ are uniformly bounded for $|r| \geq 1$, $\frac{1}{2} \le \sigma \le \frac{3}{2}$, thus also $\varphi(s, x)$ is uniformly bounded for $s = \sigma + ir$, $|r| \geq 1$, $\frac{1}{2} \le \sigma \le \frac{3}{2}$. Now ~~denote by~~ write

$$(8.5) \quad l_0(s) = \sum_{n=0}^{\infty} \frac{a_n}{C_n^{2s}}; \; a_0 \neq 0; \; C_0 < C_1 < C_2, \ldots$$

and denote by $\sigma_i$ $1 \le i \le N$ the poles of $\varphi(s, x)$ in the interval $\frac{1}{2} < s \le 1$, each counted with its proper multiplicity. Consider now the function

$$(8.6) \quad \varphi^*(s,x) = C_0^{2s-1}\left(\prod_{i=1}^{N} \frac{s-\sigma_i}{s-1+\sigma_i}\right) \varphi(s,x)$$

This function is ~~now~~ regular for $\sigma \geq \frac{1}{2}$, and we get from (8.4) and our result about the boundedness of $\varphi(s, x)$ for $|r| \geq 1$, $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$, that $\varphi^*(s, x)$ is uniformly bounded in the halfplane $\sigma \geq \frac{1}{2}$, and since

(8.7) $|\varphi^*(\tfrac{1}{2} + ir, x)| = |\varphi(\tfrac{1}{2} + ir, x)| = 1$, by (8.6) and (8.3), we get

$$(8.8) \qquad |\varphi^*(s, x)| \leq 1, \text{ for } \sigma \geq \tfrac{1}{2}.$$

Denoting by $\rho = \beta + i\gamma$ ; $\beta < \frac{1}{2}$ the poles of $\varphi(s, x)$ in the halfplane $\sigma < \frac{1}{2}$, which will also be the poles of $\varphi^*(s, x)$ in this region, ~~we~~ can prove from (8.4) and (8.7), using well known methods for estimating ~~number~~ number of zeros in a rectangle; that

$$(8.9) \qquad \sum_{|\gamma| < T} \left(\tfrac{1}{2} - \beta\right) = O(T \log T),$$

( Actually one can even prove

$$\sum_{|\gamma| \leq T} \left(\tfrac{1}{2} - \beta\right) = \overset{\frac{\alpha_1}{4\pi} T \log \frac{T}{2\pi e}}{\alpha_1 T \log T} + \alpha_2 T + O(\log T)$$

where $\alpha_1 > 0$, and $\alpha_2$ are constants, but this will not be needed.)

From this one shows easily that the product *)

$$\prod_{\rho} \frac{s-1+\bar{\rho}}{s-\rho}$$

converges absolutely ~~absolutely~~ if we combine the term containing $\rho$ with the one containing $\bar{\rho}$ for the complex $\rho$'s. Therefore, since $\varphi^*$ is at most of order 4, ~~and~~

$$\varphi^*(s, \chi) = \pm e^{\alpha_3 (s-\frac{1}{2}) + \alpha_4 (s-\frac{1}{2})^3} \prod_{\rho} \frac{s-1+\bar{\rho}}{s-\rho},$$

since $\varphi^*(s,\chi)\,\varphi^*(1-s,\chi) = 1$. An investigation of the behaviour of $|\varphi^*(s,\chi)|$ as $s \to \infty$, from (8.6) and (8.4), shows that $\alpha_3 = \alpha_4 = 0$, so that

$$(8.10) \qquad \varphi^*(s,\chi) = \pm \prod_{\rho} \frac{s-1+\bar{\rho}}{s-\rho},$$

and ~~[illegible]~~

$$(8.10') \qquad \varphi(s,\chi) = \pm\, c_0^{1-2s} \prod_{i=1}^{N} \frac{s-1+\sigma_i}{s-\sigma_i} \prod_{\rho} \frac{s-1+\bar{\rho}}{s-\rho}.$$

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*) We could here also avoid any ~~[illegible]~~ combination of terms by putting in the factor $e^{i\frac{2\beta-1}{\gamma}}$, or by taking the factor $\left(1+\frac{s-\frac{1}{2}}{\bar{\rho}-\frac{1}{2}}\right) \Big/ \left(1-\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}\right)$ instead of $\frac{s-1+\bar{\rho}}{s-\rho}$.

Forming from (8.10)

$$(8.11)\qquad -\frac{\varphi^{*\prime}}{\varphi^{*}}\left(\tfrac{1}{2}+ir,\chi\right) = \sum_{\rho} \frac{\overset{1-2\beta}{\cancel{2\beta-1}}}{(\beta-\frac{1}{2})^2+(r-\gamma)^2},$$

we observe that this expression is positive and defining

$$(8.12)\qquad \omega(r) = 1-\frac{\varphi^{*\prime}}{\varphi^{*}}\left(\tfrac{1}{2}+ir,\chi\right),$$

we ~~get~~ have that $\omega(r)\geq 1$ ~~is positive~~ and further that

$$\int_{-R}^{R}\omega(r)\,dr = 2R+\sum_{\rho}\int_{-R}^{R}\frac{\overset{1-2\beta}{\cancel{2\beta-1}}}{(\beta-\frac{1}{2})^2+(r-\gamma)^2}\,dr$$

$$\leq 2R+\sum_{|\gamma|\leq 2R}\int_{-\infty}^{\infty}\frac{\overset{1-2\beta}{\cancel{2\beta-1}}}{(\beta-\frac{1}{2})^2+(r-\gamma)^2}\,dr$$

$$+\,8R\sum_{|\gamma|>2R}\cancel{2\int_{\frac{|\gamma|}{2}}^{\infty}\frac{2\beta-1}{(\beta-\frac{1}{2})^2+r^2}\,dr}\quad\frac{1-2\beta}{|\gamma|^2}$$

$$\leq 2R+2\pi\sum_{|\gamma|\leq 2R}1+8R\sum_{|\gamma|>2R}\frac{\overset{1-2\beta}{\cancel{2\beta-1}}}{|\gamma|^2},$$

here the last term on the right-hand side is $O(R)$ by (8.9) and the second term is $O(R^5)$ since $\varphi(s,\chi)$ and so $\varphi^{*}(s,\chi)$ is a ratio of integral functions each of order at most 4. Thus we have

$$(8\cdot 13)\qquad \int_{-R}^{R} \omega(r)\,dr = O(R^5).$$

Furthermore since

$$\sum_{1\le i,j\le \varkappa_1} |\varphi_{ij}(s,x)|^2$$

is the trace of the positive Hermitian matrix $\Phi(s,x)\,\overline{\Phi(s,x)}'$ and $|\varphi(s,x)|^2$ its determinant, we have

$$\sum_{i,j} |\varphi_{ij}(s,x)|^2 \ge \varkappa_1 |\varphi(s,x)|^{\frac{2}{\varkappa_1}},$$

and so

$$\varkappa_1 - \sum_{i,j} |\varphi_{i,j}(s,x)|^2 \le \varkappa_1 \left(1 - |\varphi(s,x)|^{\frac{2}{\varkappa_1}}\right).$$

For $|r|\ge 1$, $\frac{1}{2}\le \sigma \le \frac{3}{2}$, $s=\sigma+ir$, we get from this

$$\varkappa_1 - \sum_{i,j} |\varphi_{i,j}(s,x)|^2 \le \varkappa_1 \left(1-|\varphi^*(s,x)|^{\frac{2}{\varkappa_1}}\right)$$

$$+\ \varkappa_1\left(|\varphi^*(s,x)|^{\frac{2}{\varkappa_1}} - |\varphi(s,x)|^{\frac{2}{\varkappa_1}}\right)$$

$$\le 1 - |\varphi^*(s,x)|^2 + \varkappa_1\left(1 - c_0^{\frac{2-4\sigma}{\varkappa_1}}\right),$$

since $|\varphi^*(s,x)| \le 1$ by (8.8) and $|\varphi(s,x)| \geq c_0^{1-2\sigma} |\varphi^*(s,x)|$ by (8.6). Thus

$$(8.14) \qquad \varkappa_1 - \sum_{i,j} |\varphi_{ij}(s,x)|^2 \le O(\sigma - \tfrac{1}{2}) + 1 - |\varphi^*(s,x)|^2,$$

Now, by (8.10)

$$1 - |\varphi^*(s,x)|^2 = 1 - \prod_\rho \left(1 - (2\sigma - 1)\frac{1-2\beta}{(\sigma-\beta)^2 + (r-\gamma)^2}\right)$$

$$\le (2\sigma-1)\sum_\rho \frac{1-2\beta}{(\sigma-\beta)^2+(r-\gamma)^2} \le (2\sigma-1)\sum_\rho \frac{1-2\beta}{(\beta-\frac{1}{2})^2+(r-\gamma)^2}$$

$$= -(2\sigma-1)\frac{\varphi^{*\prime}}{\varphi^*}(\tfrac{1}{2}+ir, x).$$

We now get from (8.14) and (8.12)

$$(8.15) \quad \varkappa_1 - \sum_{i,j} |\varphi_{ij}(s,x)|^2 \le O\left((\sigma - \tfrac{1}{2})\left(1 - \frac{\varphi^{*\prime}}{\varphi^*}(\tfrac{1}{2}+ir, x)\right)\right)$$

$$= O\left((\sigma - \tfrac{1}{2})\,\omega(r)\right).$$

In order to estimate the components of $E(z,s,x)$, we first obtain ~~estimates~~ estimates for the fouriercoefficients $\alpha_m^{(i,j)}(y_j, s, x)$ in the expansions (7.5), (7.11), (7.14) for $1 \le i \le \varkappa_1$, $1 \le j \le \varkappa$. For this purpose we shall go back to formula (7.43'). First consider the regions $y_j \geq \sqrt[10]{\eta}$; $|x_j| \le \frac{1}{2}$,

for $1 \le j \le \varkappa$,
in the notations of the previous paragraph, and take $\eta > 0$, so small that the regions together cover $\mathfrak{D}$ completely. Next consider the region $S_j^{(\eta)}$ defined by $y_j \geq \eta$, $|x_j| \le \frac{1}{2}$ for a fixed $j$, $1 \le j \le \varkappa$, ~~and~~ this region in general will not be contained in $\mathfrak{D}$, but by adding to $\mathfrak{D}$ a finite number of images of $\mathfrak{D}$ by suitable transformations of $\Gamma$, we can obtain a domain $D^{(\eta)}$ which contains it. Therefore, extending the notation $\tilde{E}(z, s, \chi)$, to the images of $\mathfrak{D}$ by the relation $\tilde{E}(Mz, s, \chi) = \chi(M)\, \tilde{E}(z, s, \chi)$, we get from (7.43′), taking the trace,

$$\iint_{D^{(\eta)}} \overline{\tilde{E}(z, \sigma + ir, \chi)}' \, \tilde{E}(z, \sigma + ir, \chi) \frac{dx\,dy}{y^2}$$

$$= \mathcal{O}\Big( \frac{1}{2\sigma - 1} \Big\{ A^{2\sigma-1} \varkappa_1 - A^{1-2\sigma} \sum_{1 \le i, k \le \varkappa_1} |\varphi_{ik}(\sigma+ir, \chi)|^2 \Big\}$$

$$+ \sum_{1 \le i \le \varkappa_1} \frac{\overline{\varphi_{ii}(\sigma+ir, \chi)}\, A^{2ir} - \varphi_{ii}(\sigma+ir, \chi)\, A^{-2ir}}{2ir} \Big),$$

or assuming $|r| \geq 1$; $\frac{1}{2} < \sigma \le \frac{3}{2}$,

$$(8.16) \quad \iint_{D^{(\eta)}} \overline{\tilde{E}(z, \sigma + ir, \chi)}' \, \tilde{E}(z, \sigma + ir, \chi) \frac{dx\,dy}{y^2} =$$

$$= \mathcal{O}\left( \frac{1}{2\sigma-1} \, \kappa_1 \left( A^{2\sigma-1} - A^{1-2\sigma} \right) \right) +$$

$$+ \mathcal{O}\left( \frac{1}{2\sigma-1} \, A^{1-2\sigma} \left( \kappa_1 - \sum_{1 \le i,k \le \kappa_1} |\varphi_{ik}(\sigma+ir, \chi)|^2 \right) \right)$$

$$+ \mathcal{O}(1)$$

$$= \mathcal{O}(1) + \mathcal{O}(\omega(r)) + \mathcal{O}(1) = \mathcal{O}(\omega(r)),$$

by (8.15), and since $\varphi_{ii}$ is uniformly bounded in the region for $s$ being considered. We now take our constant $A$ so large that in $S_j^{(\eta)}$ $\tilde{E}(z,s,\chi)$ differs from $E(z,s,\chi)$ only for $y_j \ge A$, (8.16) ~~then~~ gives since the integrand is non-negative

$$(8.17) \quad \iint_{S_j^{(\eta)}} \overline{\tilde{E}(z,\sigma+ir,\chi)}' \, \tilde{E}(z,\sigma+ir,\chi) \frac{dx\,dy}{y^2} = \mathcal{O}(\omega(r))$$

or for $1 \le i \le \kappa_1$,

$$(8.17') \quad \iint_{S_j^{(\eta)}} |\tilde{E}_i(z,\sigma+ir,\chi)|^2 \frac{dx\,dy}{y^2} = \mathcal{O}(\omega(r)).$$

Suppose first that $1 \le j \le \kappa_1$, then from (7.5) and (7.11) we get for $n \ne 0$,

$$(8.18) \qquad \int_{\eta}^{\infty} |\alpha_{ij}(y_j, \sigma + ir, x)|^2 \frac{dy}{y^2} = O(\omega(r))$$

Writing now

$$(N.B). \qquad \alpha_{ij}^{(ij)}(y, s, x) = \beta_{ij}^{(i,j)}(s, x)\sqrt{y} \int_0^{\infty} t^{s-\frac{1}{2}} e^{-\pi|m|y(t+\frac{1}{t})} \frac{dt}{t},$$

We get from (8,18)

$$|\beta_{ij}^{(ij)}(\sigma + ir, x)|^2 \int_{\eta}^{\infty} \left| \int_0^{\infty} t^{\sigma + ir - \frac{1}{2}} e^{-\pi|m|y(t+\frac{1}{t})} \frac{dt}{t} \right|^2 \frac{dy}{y}$$

$$= O(\omega(r)).$$

From (8.2), we now obtain

$$\int_{\eta}^{\infty} \left| \int_0^{\infty} t^{\sigma - \frac{1}{2} + ir} e^{-\pi|m|y(t+\frac{1}{t})} \frac{dt}{t} \right|^2 \frac{dy}{y}$$

$$= \int_{\pi|m|\eta}^{\infty} \left| \int_0^{\infty} t^{\sigma - \frac{1}{2} + ir} e^{-y(t+\frac{1}{t})} \frac{dt}{t} \right|^2 \frac{dy}{y}$$

$$> \frac{1}{(1 + |r| + \pi|m|\eta)^4} e^{-\pi|r| - 6\pi|m|\eta - c'},$$

inserting this in the previous inequality we get

$$|\beta^{(i,j)}_{m\eta}(\sigma+ir,x)|^2 = O\left(\omega(r)\, e^{\pi|r|+6\pi|m|\eta}\,(1+|r|+|m|\eta)^4\right)$$

$$= O\left(\omega(r)\,(1+|r|)^4\, e^{\pi|r|+20|m|\eta}\right).$$

Thus for $1 \le i, j \le \varkappa_1$

$$(8.19)\quad \beta^{(i,j)}_{m\eta}(\sigma+ir,x) = O\left(\sqrt{\omega(r)}\,(1+|r|)^2\, e^{\frac{\pi}{2}|r|+10|m|\eta}\right)$$

Similarly if we for $\varkappa_1 \le j \le \varkappa$, write

$$(N.B)\quad \alpha^{(ij)}_{m\eta}(y,s,x) = \beta^{(ij)}_{m\eta}(s,x)\sqrt{y}\int_0^\infty t^{s-\frac{1}{2}} e^{-\pi|m+\alpha_j|y(t+\frac{1}{t})}\frac{dt}{t},$$

we get for all $m$.

$$(8.19')\quad \beta^{(i,j)}_{m}(s,x) = O\left(\sqrt{\omega(r)}\,(1+|r|)^2\, e^{\frac{\pi}{2}|r|+10|m+\alpha_j|\eta}\right).$$

From (8.1) and (8.19) we now get for $y_j \ge 10\eta$, assuming first $1 \le j \le \varkappa_1$,

$$(8.20)\quad \alpha^{(i,j)}_{m}(y_j,\sigma+ir,x) = O\left(\sqrt{\omega(r)}\, e^{2|r|-2m y_j}\right),$$

for $m \ne 0$,

and similarly for $\varkappa_1 < j \le \varkappa$, and all $m$

$$(8.20')\quad \alpha^{(i,j)}_{m}(y_j,\sigma+ir,x) = O\left(\sqrt{\omega(r)}\, e^{2|r|-2|m+\alpha_j|y_j}\right).$$

Inserting this in the fourier expansion for $E_i(z, s, \chi)$, we obtain first for $1 \leq j \leq \varkappa_1$, and $y_j > 10\eta$,

$$(8.21) \quad E_i(z, s, \chi) = \delta_{ij}\, y_j^{s} + \varphi_{ij}(s, \chi)\, y_j^{1-s} + O\left(\sqrt{\omega(r)}\, e^{2|r| - 2y_j}\right),$$

where $\delta_{ij} = 1$ if $i = j$ and $0$ otherwise. Similarly for $\varkappa_1 < j \leq \varkappa$, $y_j > 10\eta$, we get

$$(8.21') \quad E_i(z, s, \chi) = O\left(\sqrt{\omega(r)}\, e^{2|r| - 2|\alpha_j| y_j}\right)$$

if $\alpha_j \neq 0$, in $e^{2\pi i \alpha_j} = \chi(S_j)$ is chosen so that $|\alpha_j| \leq \frac{1}{2}$. Since the regions $y_j \geq 10\eta$; $|x_j| \leq \frac{1}{2}$ together cover $D$ completely (8.21) and (8.21′) give us estimations for the $E_i$ and so for $E(z, s, \chi)$ everywhere in $D$. It is clear that the estimations (8.21) and (8.21′) still hold if we instead of the region $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$, $|r| \geq 1$, consider any region obtained from $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$, by removing $N$ small circular regions with the poles $s = \sigma_i$ ; $1 \leq i \leq N$ as centers. In particular this is true for the region $\frac{1}{2} \le \sigma \le \theta$ if $\theta > \frac{1}{2}$ is small enough (that is $\theta < \sigma_i$, for $1 \le i \le N$.).

§ 9.

The trace formula in the case of a singular $\chi$.

We shall have to consider in addition to the kernels $K(z, z'; \chi) = \sum_{M \subset \Gamma} \chi(M) k(z, Mz')$, an other set

$$(9.1) \quad H(z, z', \chi) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \overline{E(z', \tfrac{1}{2}+ir, \chi)}\, E(z, \tfrac{1}{2}+ir, \chi) h(r)\, dr$$

where $h(r)$ as before is an even analytic function, regular in some strip $|\mathcal{I}(r)| \le \frac{1}{2} + \varepsilon$, $\varepsilon > 0$ and for the present we will assume that

$$(9.2) \qquad h(r) = O(e^{-5|r|})$$

in this strip. ~~as $|r| \to \infty$~~

First, the existence of the integral on the right-hand side of (9.1), follows at once from the estimations (8.21) and (8.21') combined with (8.13). We shall next look at the behaviour of $H(z,z',\varkappa)$ as $z$ or $z'$ or both tend towards the cusps of the fundamental domain $\mathfrak{D}$. Let us first consider the case that both $z$ and $z'$ lie in the region $y_j \geq 10\eta$, $|x_j| \leq \frac{1}{2}$ for some $j$ $1 \leq j \leq \varkappa_1$, ~~Writing~~

$$(9.3) \quad E_i(z,s,\varkappa) = \delta_{ij}\, y_j^s + \varphi_{ij}(s,\varkappa)\, y_j^{1-s} + E_i^*(z,s,\varkappa),$$

we have according to (8.21)

$$(9.4) \quad E_i^*(z,s,\varkappa) = O(\sqrt{\omega(r)}\, e^{+2|r|-2y_j}),$$

uniformly for $\frac{1}{2} \leq \sigma \leq \theta$, $\theta > \frac{1}{2}$. Inserting the expressions (9.3) in

$$H(z,z',\varkappa) = \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r) \sum_{1\leq i\leq \varkappa_1} E_i(z,\tfrac{1}{2}+ir,\varkappa)\, \overline{E_i(z',\tfrac{1}{2}+ir,\varkappa)}\, dr,$$

~~We obt~~ the left-hand side breaks up into 9 terms, which we shall consider separately. First there is the term ~~that contains~~

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r) \sum_{1\leq i\leq \varkappa_1} E_i^*(z,\tfrac{1}{2}+ir,\varkappa)\, \overline{E_i^*(z',\tfrac{1}{2}+ir,\varkappa)}\, dr,$$

From (9.4) and (8.13), we get immediately that this term is

$$O\left(e^{-2(y_j + y_j')}\right) = O(1).$$

Then there are 4 terms that contain only one $E^*$ factor, namely

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\, y_j^{\frac{1}{2}+ir}\, \overline{E_j^*(z', \tfrac{1}{2}+ir, \chi)}\, dr,$$

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\, y_j^{\frac{1}{2}-ir} \sum_{1\le i\le \varkappa_1} \varphi_{ij}(\tfrac{1}{2}+ir, \chi)\, \overline{E_i^*(z', \tfrac{1}{2}+ir, \chi)}\, dr$$

and the remaining two are obtained from these by interchanging $z$ and $z'$ and ~~taking~~ putting $\bar{\chi}$ instead of $\chi$. Now since

$$E_j(z', \tfrac{1}{2}+ir, \bar{\chi}) = \sum_{1\le i\le \varkappa_1} \varphi_{ji}(\tfrac{1}{2}+ir, \bar{\chi})\, E_i(z', \tfrac{1}{2}-ir, \bar{\chi})$$

and $\varphi_{ji}(\tfrac{1}{2}+ir, \bar{\chi}) = \varphi_{ij}(\tfrac{1}{2}+ir, \chi)$,

we have

$$\sum_{1\le i\le \varkappa_1} \varphi_{ij}(\tfrac{1}{2}+ir, \chi)\, \overline{E_i^*(z', \tfrac{1}{2}+ir, \chi)} = \overline{E_j^*(z', \tfrac{1}{2}-ir, \chi)},$$

so that the second of the above two ~~terms~~ integrals goes over into the first by writing $-r$ instead of $r$. We therefore look only at the first integral, and moving the line of integration

slightly
upwards by writing $i(\theta-\frac{1}{2})+r$ instead of $r$, we obtain from (9.4) and (8.13), by taking absolute values that the integral is

$$O\left(y_j^{1-\theta} e^{-2y_j'}\right) = O\left(\left(y_j^{\cancel{+2\theta}} y_j'\right)^{1-\theta}\right)$$

and so are the three other terms.

There remain the terms that do not contain any $E^*$, of these there are 4. We first consider the two terms

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\, \cancel{y_j^{\frac{1}{2}+ir}}\, \overline{\varphi_{jj}(\tfrac{1}{2}+ir, \chi)}\, (y_j\, y_j')^{\frac{1}{2}+ir}\, dr$$

and

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\, \varphi_{jj}(\tfrac{1}{2}+ir, \chi)\, (y_j\, y_j')^{\frac{1}{2}-ir}\, dr,$$

since $\overline{\varphi_{jj}(\frac{1}{2}+ir, \chi)} = \varphi_{jj}(\frac{1}{2}-ir, \bar{\chi}) = \varphi_{jj}(\frac{1}{2}-ir, \chi)$,

the second integral goes over into the first if we ~~replace~~ write $-r$ instead of $r$. Moving the line of integration slightly upwards by writing $i(\theta-\frac{1}{2})+r$ instead of $r$, and observing that $\varphi_{jj}(s, \bar{\chi})$ is uniformly bounded for $\frac{1}{2} \leq \sigma \leq \theta$, we obtain again that these terms are $O((y_j y_j')^{1-\theta})$.

Finally we consider the terms

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\, y_j^{\frac{1}{2}+ir}\, y_j'^{\,\frac{1}{2}-ir}\, dr$$

and

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\, y_j^{\frac{1}{2}-ir}\, y_j'^{\,\frac{1}{2}+ir} \sum_{1\le i\le \varkappa_1} |\varphi_{ij}(\tfrac{1}{2}+ir, \chi)|^2\, dr.$$

Observing that

$$\sum_{1\le i\le \varkappa_1} |\varphi_{ij}(\tfrac{1}{2}+ir, \chi)|^2 = 1,$$

since $\Phi(\frac{1}{2}+ir, \chi)$ is unitary, we get by writing $-r$ instead of $r$ in the last integral, that the two terms together give

$$\frac{\sqrt{y_j y_j'}}{2\pi}\int_{-\infty}^{\infty} h(r)\, e^{ir\log\frac{y_j}{y_j'}}\, dr$$

$$= \sqrt{y_j y_j'}\; g(\log \frac{y_j}{y_j'}).$$

Combining these results we obtain

(9.5) $\quad H(z, z', \chi) = \sqrt{y_j y_j'}\; g(\log \frac{y_j}{y_j'}) + O((y_j y_j')^{1-\theta}),$

with $1 > \theta > \frac{1}{2}$, for $z_j$ and $z_j'$ in the region $y_j \geq 10\eta$ ; $|x_j| \leq \frac{1}{2}$, for some $j$, $1 \leq j \leq \varkappa_1$.

In the same way one shows that for $\varkappa_1 < j \leq \varkappa$, we have if $z$ and $z'$ both lie in the region $y_j \geq 10\eta$ ; $|x_j| \leq \frac{1}{2}$, that

$$(9.5') \quad H(z, z', \chi) = O(1) = O((y_j \, y_j')^{1-\theta}).$$

Finally if $i \neq j$ $1 \leq i, j \leq \varkappa$, and $z$ lies in the region $y_i \geq 10\eta$ ; $|x_i| \leq \frac{1}{2}$, and $z'$ in the region $y_j' \geq 10\eta$, $|x_j'| \leq \frac{1}{2}$, we obtain in a similar way

$$(9.5'') \quad H(z, z', \chi) = O((y_i \, y_j')^{1-\theta}).$$

We now consider the integral operator

$$(9.6) \quad \iint_{\mathcal{D}} \left(K(z, z', \chi) - H(z, z', \chi)\right) f(z') \frac{dx' dy'}{y'^2}$$

where $K$ and $H$ are derived from the same function $h(r)$. Combining our previous results (6.5') about the behaviour of $K(z, z', \chi)$ with the ones just obtained for $H(z, z', \chi)$, we see that for $z$ and $z'$ both in $\mathcal{D}$, ~~we~~

(9.7) $$K(z,z';\chi) - H(z,z',\chi) = O\Big( \sum_{i,j=1}^{\kappa} y_i^{1-\theta} y_j'^{1-\theta} \Big).$$

This implies that ~~the integral~~

(9.8) $$\iint_{\mathcal{D}\,\mathcal{D}} |K(z,z';\chi) - H(z,z';\chi)|^2 \frac{dx\,dy}{y^2}\frac{dx'\,dy'}{y'^2} < \infty.$$

~~exists.~~

Now consider ~~let~~ $u(z)$ ~~be~~ a square-integrable solution of

(9.9) $$y^2\Delta u + (\tfrac{1}{4}+r^2)u = 0$$

and such that $u(Mz) = \chi(M)u(z)$, then one easily sees (from the fourier-expansions of $u$) that

$$u(z) = O(y_i^{\alpha}),$$

with $\alpha < \frac{1}{2}$ as $y_i \to \infty$.

Therefore the integral

$$\iint_{\mathcal{D}} u(z)\,\overline{E(z,\tfrac{1}{2}+ir',\chi)}\,\frac{dx\,dy}{y^2}$$

exists, and the value is easily seen to be zero, by first assuming $r' \neq \pm r$, and considering the expression

$$\iint_{\mathcal{D}} \left( u(z)\, \Delta \overline{E(z, \tfrac{1}{2}+ir, \chi)} - \overline{E(z, \tfrac{1}{2}+ir, \chi)}\, \Delta u(z) \right) dx\,dy$$

which is zero by Greens theorem, making $r' \to r$ we obtain the result

$$\iint_{\mathcal{D}} u(z)\, \overline{E(z, \tfrac{1}{2}+ir', \chi)}\, \frac{dx\,dy}{y^2} = 0$$

for all real $r'$. Therefore we also get

$$\iint_{\mathcal{D}} H(z, z', \chi)\, u(z')\, \frac{dx'dy'}{y'^2} = 0,$$

furthermore since*)

$$\iint_{\mathcal{D}} K(z, z', \chi)\, u(z')\, \frac{dx'dy'}{y'^2} = h(r)\, u(z),$$

we obtain

$$(9.10) \qquad \iint_{\mathcal{D}} \left( K(z, z', \chi) - H(z, z', \chi) \right) u(z')\, \frac{dx'dy'}{y'^2} = h(r)\, u(z).$$

---

*) Earlier we have actually only shown that if a function $f(z)$ satisfies (9.9) and also is an eigenfunction of $\iint_{\mathcal{D}} k(z,z')\, f(z')\, \frac{dx'dy'}{y'^2}$ then the latter expression has the value $h(r)\, f(z)$. However that $u(z)$ is actually an eigenfunction of $\iint_{\mathcal{D}} K(z,z',\chi)\, u(z')\, \frac{dx'dy'}{y'^2} = \iint_{H} k(z,z')\, u(z')\, \frac{dx'dy'}{y'^2}$ can be seen for instance from one of its fourierexpansion, where each term is an eigenfunction of the last operator with eigenvalue $h(r)$.

On the other hand ~~since~~ the operator $y^2\Delta$ commutes with the operator

$$\iint_{\mathcal{D}} \left(K(z,z',\chi) - H(z,z')\right) f(z')\, \frac{dx'dy'}{y'^2}$$

because (denoting by subscript to $\Delta$ the variable upon which the differential operators acts)

$$y^2 \Delta_z \left(K(z,z';\chi) - H(z,z';\chi)\right)$$
$$= y'^2 \Delta_{z'} \left(K(z,z';\chi) - H(z,z';\chi)\right)$$

and because of Greens theorem. ~~Therefore~~ Furthermore because of (9.8), the ~~set of~~ eigenfunctions of (9.6) which do not belong to the eigenvalue zero are square integrable over $\mathcal{D}$, and the set of eigenvalues has zero as its only point of accumulation and to each eigenvalue $\neq 0$, belongs only a finite number of linearly independent eigenfunctions. Therefore since the operator (9.6) commutes with $y^2\Delta$, it follows easily that ~~the~~ orthonormal system of eigenfunctions of (9.6) can be chosen so that they are all also eigenfunctions of $y^2\Delta$. Thus the square integrable solutions of (9.9) give us all eigenfunctions of (9.6) that do not belong to the eigenvalue zero, (and possibly a finite or infinite number that belong to the eigenvalue zero, if (9.9) has solutions for values of $r$ that make $h(r) = 0$).

From this we also conclude that the whole class of operators (9.6) commute, and that ~~to~~ the product of two such operators, derived respectively from the functions $h_1(r)$ and $h_2(r)$, is the operator derived from $h_1(r)\,h_2(r)$.

We now proceed to compute twice the trace of the operator (9.6),

$$(9.11) \qquad 2\iint_{\mathcal{D}} \left(K(z,z,x) - H(z,z,x)\right) \frac{dx\,dy}{y^2}$$

This expression on one hand is equal to the sum $\sum_i h(r_i)$ extended over the $\cancel{\sum_r h(r)}$ $r_i$'s for which (9.9) has squareintegrable solutions, ~~on the ass assuming for the fi~~ if ~~we assume that~~ $h(r)$ can be written as a product of two functions of $r$ that each satisfy the conditions assumed fulfilled by $h(r)$ at the beginning of this paragraph, or what is the same, if we ~~assume that~~ replace the condition (9.2) by

$$(9.12) \qquad h(r) = O(e^{-10|r|}),$$

and leave the other conditions unchanged.

For simplicity we shall carry out first in detail the computation of (9.11) in the case that ~~we have only~~ D has only one cusp, which is placed at $\infty$, ~~that~~ the primitive parabolic transformation leaving the cusp fixed is $Sz = z+1$, and that $\chi(M) = 1$ for all $M$. We split the expression $K(z,z) - H(z,z)$ up into three parts

$$(9.13) \quad K(z,z) - H(z,z) = \left\{ \sum_{n=-\infty}^{\infty}{}' k(z, z+n) - H(z,z) \right\} +$$

$$+ \sum_N \sum_{n=-\infty}^{\infty}{}' k(z, NS^n N^{-1} z) + \sum_M{}'' k(z, Mz) ,$$

where the dash $\Sigma'$ denotes that $n=0$ is omitted, $\sum_N$ is taken over a complete set of transformations of $\Gamma$ that do not differ by a power of $S$ on the right side, and finally $\sum_M{}''$ is extended over all non-parabolic transformations of $\Gamma$ (including the identity transformation). From our previous results it follows that the integral (9.11) can be split correspondingly, each of the three resulting integrals existing. The last term is treated precisely as in the compact case, and gives the same contribution from the identity, the elliptic and hyperbolic transformations as before, so we shall only consider the ~~terms~~ contribution from the first two, writing $\bar{\mathcal{D}}$ for the part of $\mathcal{D}$ that lies below the line $y = A$ for large positive $A$, we have

$$(9.14)\quad \iint_{\mathcal{D}} \Big\{ \sum_{-\infty}^{\infty}{}' k(z, z+n) - H(z,z) \Big\} \frac{dx\,dy}{y^2} =$$

$$= \lim_{A\to\infty} \iint_{\mathcal{D}_A} \sum_{-\infty}^{\infty}{}' k(z,z+n) \frac{dx\,dy}{y^2} - \iint_{\mathcal{D}_A} H(z,z) \frac{dx\,dy}{y^2}.$$

Furthermore we have denoting by $\mathcal{S}$ the strip $y > 0$ ; $|x| \le \frac{1}{2}$ (which we may assume contains $\mathcal{D}$.) and by $\mathcal{S} - \mathcal{D}$ the part of $\mathcal{S}$ that does not belong to $\mathcal{D}$, that

$$(9.15)\quad \iint_{\mathcal{D}} \sum_{N} \sum_{n=-\infty}^{\infty}{}' k(z, NS^n N^{-1} z) \frac{dx\,dy}{y^2} =$$

$$= \iint_{\mathcal{S}-\mathcal{D}} \sum_{n=-\infty}^{\infty}{}' k(z, z+n) \frac{dx\,dy}{y^2}.$$

Therefore writing $\bar{\mathcal{S}}$ for the part of $\mathcal{S}$ that lies below the line $y = A$, we obtain combining (9.14) and (9.15) that the contribution of the two first terms in (9.13) to the expression (9.11) is

$$(9.16)\quad \lim_{A\to\infty}\left\{2\iint_{\mathcal{F}}\sum_{n=-\infty}^{\infty}{}' k(z,z+n)\frac{dx\,dy}{y^2}-2\iint_{\mathcal{D}}H(z,z)\frac{dx\,dy}{y^2}\right\}$$

Here we ~~do~~ ~~first~~ consider first

$$(9.17)\quad 2\iint_{\mathcal{D}}H(z,z)\frac{dx\,dy}{y^2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\,dr\iint_{\mathcal{D}}\left|E(z,\tfrac12+ir)\right|^2\frac{dx\,dy}{y^2}.$$

Combining (7.42'), with the fact that for $y \geq A$

$\tilde{E}(z,\frac12+ir) = E(z,\frac12+ir) - y^{\frac12+ir} - \varphi(\frac12+ir)\,y^{\frac12-ir}$

$= O(\sqrt{\omega(r)}\,e^{2|r|-2y})$, we obtain

$$(9.18)\quad \iint_{\mathcal{D}}\left|E(z,\tfrac12+ir)\right|^2\frac{dx\,dy}{y^2}=2\log A-\frac{\varphi'}{\varphi}(\tfrac12+ir)$$

$$+\frac{\overline{\varphi(\frac12+ir)}\,A^{2ir}-\varphi(\frac12+ir)\,A^{-2ir}}{2ir}+O(\omega(r)\,e^{4|r|-4A}).$$

Now

$$\int_{-\infty}^{\infty}|h(r)|\,\omega(r)\,e^{4|r|-4A}\,dr = O(e^{-4A})$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\varphi(\frac12-ir)A^{2ir}-\varphi(\frac12+ir)A^{-2ir}}{2ir}h(r)\,dr=$$

$$= \frac{1}{2\pi}\int_{-t}^{t} \frac{\varphi(\frac{1}{2}-ir)A^{2ir} - \varphi(\frac{1}{2}+ir)A^{-2ir}}{2ir}\, h(r)\,dr + o(1)$$

$$= \frac{1}{2\pi}\int_{-t}^{t} h(0)\,\varphi(\tfrac{1}{2})\,\frac{\sin 2r\log A}{r}\,dr$$

$$+ \frac{1}{2\pi}\int_{-t}^{t} \frac{\varphi(\frac{1}{2}-ir)h(r) - \varphi(\frac{1}{2})h(0)}{2ir}\, A^{2ir}\,dr$$

$$- \frac{1}{2\pi}\int_{-t}^{t} \frac{\varphi(\frac{1}{2}+ir)h(r) - \varphi(\frac{1}{2})h(0)}{2ir}\, A^{-2ir}\,dr + o(1)$$

$$= \frac{1}{2\pi}\, h(0)\,\varphi(\tfrac{1}{2}) \int_{-t}^{t} \frac{\sin 2r\log A}{r}\,dr + o(1)$$

$$= \frac{1}{2\pi}\, h(0)\,\varphi(\tfrac{1}{2}) \int_{-2t\log A}^{2t\log A} \frac{\sin x}{x}\,dx + o(1)$$

$$= \frac{1}{2\pi}\, h(0)\,\varphi(\tfrac{1}{2}) \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx + o(1) = \tfrac{1}{2}\, h(0)\,\varphi(\tfrac{1}{2}) + o(1).$$

as $A \to \infty$

Also

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\,dr = g(0).$$

Thus from (9.17) and (9.18) we get

$$(9.19)\quad 2\iint_{D} H(z,z)\,\frac{dx\,dy}{y^2} = 2g(0)\log A - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\,\frac{\varphi'}{\varphi}(\tfrac{1}{2}+ir)\,dr$$

$$+ \tfrac{1}{2}\, h(0)\,\varphi(\tfrac{1}{2}) + o(1),$$

as $A \to \infty$.

Secondly, we have

$$(9.20) \qquad 2 \iint\limits_{\varphi} \sum_{n=-\infty}^{\infty}{}' \, k(z, z+n) \frac{dx\,dy}{y^2} =$$

$$= 4 \int_0^A \sum_{n=1}^{\infty} k\left(\frac{n^2}{y^2}\right) \frac{dy}{y^2} =$$

$$= 4 \sum_{n=1}^{\infty} \frac{1}{n} \int_{\frac{n}{A}}^{\infty} k(u^2)\, du$$

$$= 4 \int_0^{\infty} k(u^2) \left( \sum_{n < Au} \frac{1}{n} \right) du \, ,$$

here uniformly for $u > 0$,

$$\sum_{n < Au} \frac{1}{n} = \log A + \log u + c + O\left(\frac{1}{\sqrt{Au}}\right)$$

where $c$ is Eulers constant, thus

$$(9.21) \qquad 2 \iint\limits_{\varphi} \sum_{-\infty}^{\infty}{}' \, k(z, z+n) \frac{dx\,dy}{y^2} = 4(\log A + c) \int_0^{\infty} k(u^2)\, du$$

$$+ 4 \int_0^{\infty} \log u \; k(u^2)\, du + O\left(\frac{1}{\sqrt{A}}\right).$$

Here

$$\int_0^{\infty} k(u^2)\, du = \frac{1}{2} \int_0^{\infty} \frac{k(t)}{\sqrt{t}}\, dt = \frac{1}{2}\, g(0).$$

~~and~~ Further

$$4\int_0^\infty \log u \; k(u^2)\, du = \int_0^\infty k(t)\,\frac{\log t}{\sqrt{t}}\, dt$$

Now

$$k(t) = -\frac{1}{\pi}\int_t^\infty \frac{dQ(w)}{\sqrt{w-t}},$$

where $Q(e^{u}+e^{-u}-2) = g(u)$, inserting this we get

$$4\int_0^\infty \log u \; k(u^2)\, du = -\frac{1}{\pi}\int_0^\infty dQ(w)\left\{\int_0^w \frac{\log t}{\sqrt{t(w-t)}}\, dt\right\}.$$

~~where~~ $\cancel{\int_0^{w} \log t}$

Now for $s$ real $> -1$

$$\int_0^w \frac{t^s}{\sqrt{w-t}}\, dt = w^{s+\frac{1}{2}}\int_0^1 \frac{t^s}{\sqrt{1-t}}\, dt = w^{s+\frac{1}{2}}\,\frac{\Gamma(s+1)\,\Gamma(\frac{1}{2})}{\Gamma(s+\frac{3}{2})},$$

differentiating with respect to $s$ and putting $s = -\frac{1}{2}$, we get easily

$$\int_0^w \frac{\log t}{\sqrt{t(w-t)}}\, dt = \pi\,(\log w - 2\log 2).$$

Thus

$$4\int_0^\infty \log u\, k(u^2)\,du = -\int_0^\infty (\log w - 2\log 2)\, dQ(w)$$

$$= -\int_0^\infty \log(e^u + e^{-u} - 2)\, dg(u) - 2\log 2\; g(0) =$$

$$= -2\int_{-\infty}^\infty \log\left(e^{\frac{u}{2}} - e^{-\frac{u}{2}}\right)\, dg(u) - 2\log 2\; g(0)$$

$$= -2\int_0^\infty \log(1-e^{-u})\, dg(u) - \cancel{2}\int_0^\infty u\, dg(u) - 2\log 2\; g(0)$$

$$= -2\int_0^\infty \log(1-e^{-u})\, dg(u) + \int_0^\infty g(u)\, du - 2\log 2\; g(0)$$

$$= -2\int_0^\infty \log(1-e^{-u})\, dg(u) + \tfrac{1}{2} h(0) - 2\log 2\; g(0).$$

Here

$$g'(u) = \frac{1}{2\pi}\int_{-\infty}^\infty ir\, h(r)\, e^{iru}\, dr,$$

inserting this we have

$$-2\int_0^\infty \log(1-e^{-u})\, dg(u) = -\frac{i}{\pi}\int_{-\infty}^\infty h(r)\, r\, dr \int_0^\infty e^{iru} \log(1-e^{-u})\, du,$$

and

$$-ir\int_0^\infty e^{iru}\log(1-e^{-u})\, du = \sum_{n=1}^\infty \frac{ir}{n}\int_0^\infty e^{(ir-n)u}\, du$$

$$= \sum_{n=1}^\infty \frac{ir}{n(n-ir)} = \sum_{n=1}^\infty \left(\frac{1}{n-ir} - \frac{1}{n}\right) =$$

$$= -\frac{\Gamma'}{\Gamma}(1-ir) - C,$$

where $C$ is Eulers constant, Thus

$$-2\int_0^\infty \log(1-e^{-u})\,dg(u) = -\frac{1}{\pi}\int_{-\infty}^{\infty} h(r)\left\{\frac{\Gamma'}{\Gamma}(1-ir)+C\right\}dr$$

$$= -2C\,g(0) - \frac{1}{\pi}\int_{-\infty}^{\infty} h(r)\,\frac{\Gamma'}{\Gamma}(1+ir)\,dr.$$

Hence

$$4\int_0^\infty \log u\;k(u^2)\,du = -\frac{1}{\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)\,dr$$

$$-2(C+\log 2)\,g(0) + \frac{1}{2}h(0).$$

Inserting our results in (9.21) we get

$$(9.22)\quad 2\iint_{\mathcal{F}} \sum_{n=-\infty}^{\infty}{}' k(z,z+n)\,\frac{dx\,dy}{y^2} = 2\log A\;g(0)$$

$$-2\log 2\;g(0) + \frac{1}{2}h(0) - \frac{1}{\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)\,dr + O\left(\frac{1}{\sqrt{A}}\right),$$

as $A \to \infty$. Finally inserting (9.19) and (9.22) in (9.16) and letting $A \to \infty$, we find the contribution of the parabolic transformations to be

$$(9.23)\quad \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\,\frac{\varphi'}{\varphi}\left(\tfrac{1}{2}+ir\right)dr - \frac{1}{\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)\,dr$$

$$-2\log 2\, g(0) + \tfrac{1}{2}\left(1-\varphi(\tfrac{1}{2})\right)h(0).$$

In the general case one may proceed in a similar way and obtains then as the contribution of all parabolic transformations (including the ones treated already in §6, where $\chi$ of the primitive transformation $\neq 1$),

$$(9.23')\quad \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\,\frac{\varphi'}{\varphi}\left(\tfrac{1}{2}+ir,\chi\right)dr - \frac{\varkappa_1}{\pi}\int_{-\infty}^{\infty} h(r)\,\frac{\Gamma'}{\Gamma}(1+ir)\,dr$$

$$-2\varkappa_1\log 2\; g(0) + 2\sum_{\varkappa_1<i\le\varkappa}\log\frac{1}{|1-\chi(S_i)|}\; g(0)$$

$$+\tfrac{1}{2}\left(\varkappa_1-\sigma\left(\Phi(\tfrac{1}{2},\chi)\right)\right)h(0).$$

It should be observed that since the eigenvalues of the $\varkappa_1\times\varkappa_1$ matrix $\Phi(\frac{1}{2},\chi)$ are all $\pm 1$, because $\Phi(\frac{1}{2},\chi)$ is unitary and equal to its conjugate transposed, the coefficient of $h(0)$ in (9.23') is always a non-negative integer $\le\varkappa_1$.

We can now write down the trace-formula

 $\left( \mu(\mathcal{D}) = \frac{A(\mathcal{D})}{2\pi} \text{ where } A(\mathcal{D}) \text{ is area} \right)$

$$(9.24) \quad \sum_i h(r_i) = \mu(\mathcal{D}) \int_{-\infty}^{\infty} r \, \frac{e^{\pi r} - e^{-\pi r}}{e^{\pi r} + e^{-\pi r}} \, h(r)\, dr$$

$$+ \sum_{\{R\}_\Gamma} \sum_{k=1}^{m-1} \frac{\chi^k(R)}{m \sin \frac{k\pi}{m}} \int_{-\infty}^{\infty} \frac{e^{-2\pi r \frac{k}{m}}}{1 + e^{-2\pi r}} \, h(r)\, dr$$

$$+ 2 \sum_{\{P\}_\Gamma} \sum_{k=1}^{\infty} \frac{\chi^k(P) \log N\{P\}}{(N\{P\})^{\frac{k}{2}} - (N\{P\})^{-\frac{k}{2}}} \, g(k \log N\{P\})$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\varphi'}{\varphi}\left(\tfrac{1}{2} + ir, \chi\right) dr - \frac{\varkappa_1}{\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'}{\Gamma}(1 + ir)\, dr$$

$$- 2\varkappa_1 \log 2 \cdot g(0) + 2 \sum_{\varkappa_1 < i \le \varkappa} \log \frac{1}{|1 - \chi(S_i)|} \, g(0)$$

$$+ \tfrac{1}{2}\left(\varkappa_1 - \sigma\left(\Phi\left(\tfrac{1}{2}, \chi\right)\right)\right) h(0).$$

This formula has now been proved valid under the condition (9.12) in addition to the earlier conditions imposed upon $h(r)$. We now wish to show that (9.24) is valid under the conditions (i) $h(r) = h(-r)$, (ii) $h(r)$ regular for $|\Im(r)| < \frac{1}{2} + \varepsilon$, ~~and (iii)~~ $\cancel{h(r) = O\left(\frac{1}{|r|^{2+\varepsilon}}\right)}$, for some positive $\varepsilon$, and instead of (9.12) the condition

$$(9.25) \qquad h(r) = O\left(\frac{1}{(1+|r|)^{2+\varepsilon}}\right).$$

For this purpose we move the ~~second~~ fourth term on the righthand side of (9.24) over to the lefthand side, and put in the function

$$h(r) = e^{-\frac{r^2}{R^2}},$$

where $R$ is a large positive number, this function clearly satisfies (9.12) and the other conditions on $h(r)$. Furthermore we have from (8.10')

$$-\frac{\varphi'}{\varphi}\left(\tfrac{1}{2}+ir, x\right) = -\frac{\varphi^{*\prime}}{\varphi^*}\left(\tfrac{1}{2}+ir, x\right) + O(1),$$

where the first term on the righthand side is non negative. Also $\frac{\Gamma'}{\Gamma}(1+ir) = O(\log(2+|r|))$. Using ~~From~~ this we easily get from (9.24)

$$(9.26) \quad \sum_{|r_i|\le R} 1 - \frac{1}{2\pi}\int_{-R}^{R} \frac{\varphi^{*\prime}}{\varphi^*}\left(\tfrac{1}{2}+ir, x\right) dr \le O(R^2),$$

or

$$(9.26') \quad \sum_{|r_i|\le R} 1 + \frac{1}{2\pi}\int_{-R}^{R} \left|\frac{\varphi'}{\varphi}\left(\tfrac{1}{2}+ir, x\right)\right| dr = O(R^2).$$

This implies actually that all series

and infinite series occuring in (9.24) converge absolutely if $h(r)$ satisfies (9.25) and the other standard conditions; also, for a class of functions $h(r)$ that satisfy these conditions uniformly, we ~~have~~ ~~also~~ see that the convergence of the series and integrals is uniform. Taking now for a fixed $h(r)$ the class $h(r)e^{-\varepsilon r^2}$, $1 \geq \varepsilon \geq 0$, these constitute such a class, and for $\varepsilon > 0$, (9.12) is satisfied so that (9.24) is valid for the function $h(r)e^{-\varepsilon r^2}$. Letting $\varepsilon \to 0$, we obtain that the formula (9.24) is valid if $h(r)$ is even, analytic in some ~~region~~ strip $|\mathfrak{I}(r)| < \frac{1}{2} + \varepsilon$, and $h(r) = O\left(\frac{1}{(1+|r|)^{2+\varepsilon}}\right)$, in this strip for some $\varepsilon > 0$.

~~Before we go~~ ~~Consider~~ From (9.26) and (8.11), we get

$$\sum_{\rho} (1-2\beta) \int_{-2R}^{2R} \frac{1}{(\beta-\frac{1}{2})^2+(r-\gamma)^2}\, dr = O(R^2)$$

Here we have, using (8.9)

$$\sum_{|\gamma| \le R} (1-2\beta) \int_{-2R}^{2R} \frac{1}{(\beta-\frac{1}{2})^2+(r-\gamma)^2}\, dr =$$

$$= \sum_{|\gamma| \le R} (1-2\beta) \int_{-\infty}^{\infty} \frac{dr}{(\beta-\frac{1}{2})^2+r^2} + O\Big( \sum_{|\gamma| \le R} \frac{\frac{1}{2}-\beta}{R} \Big)$$

$$= 2\pi \sum_{|\gamma| \le R} 1 + O(\log R).$$

Thus we get

$$(9.27) \quad \sum_{|\gamma| \le R} 1 = O(R^2)$$

which shows that $\varphi(s, \chi)$ actually is a quotient of integral functions at most of order 2, the same result carries over to the $E_i(z, s; \chi)$ as functions of $s$.

More precisely one can prove from (9.24) that

$$(9.28) \quad \sum_{|r_i| \le R} 1 + \sum_{|\gamma| \le R} 1 = \mu(D) R^2 + O(R \log R).$$

Unfortunately however, we have no means of estimating the two terms on the left-hand side separately, except in some special cases, when the function $\varphi(s, \chi)$ can be expressed in terms of functions that are known from analytic numbertheory. Such a case will be mentioned later in connection with the modular group. It should be mentioned that in all ~~these~~ cases ~~where this can be~~ those have

$$\sum_{|\gamma| \leq R} 1 = O(R \log R),$$

so that

$$\sum_{|r_i| \leq R} 1 = \mu(\mathcal{D}) R^2 + O(R \log R),$$

in these cases. It is an open question whether this is true in the general case.

We may now as in the previous cases study the function

$$Z_p(s, \chi) = \prod_{\{P\}} \prod_{\nu=0}^{\infty} (1 - \chi\{P\} (N\{P\})^{-s-\nu})$$

by inserting in (9.24) the special function

$$h(r) = (s-\tfrac{1}{2}) \left\{ \frac{1}{(s-\frac{1}{2})^2 + r^2} - \frac{1}{(a-\frac{1}{2})^2 + r^2} \right\}$$

where $a$ is a real constant $> 1$, and ~~at first~~ $\mathcal{R}\, s > 1$. The new terms occurring in the traceformula produces some important changes in the ~~function~~ properties of the function. First of all $Z_\rho(s,\chi)$ is no longer an integral function, the ~~third~~ term on the right hand side of (9.24) produces poles at the points $s = n - \frac{1}{2}$, $n = 1, 2, 3 \ldots$ of order $\varkappa_1$, in addition the fifth term produces a pole at $s = \frac{1}{2}$ of order

$$\varkappa_1 - \sigma(\phi(\tfrac{1}{2},\chi)),$$

which however may be superimposed on ~~some~~ a zero if some of the $r_i = 0$. As before the first and sixth term produces zeros at 0 and ~~to~~ the negative integers. The ~~first~~ second term produces zeros of $Z_\rho(s,\chi)$ at the ~~zeros~~ poles $\rho = \beta + i\gamma$ of $\varphi(s,\chi)$ in the halfplane $\Re(s) < \frac{1}{2}$, ~~Finally $Z_\rho(s,\chi)$ has zeros at the points $\frac{1}{2} \pm ir_i$,~~ and poles at the points $1 - \sigma_i$, these poles however are cancelled out by the zeros produced by the term on the lefthand side of (9.24) at the points $\frac{1}{2} + ir_i$, since the ~~points~~ $\pm (\sigma_i - \frac{1}{2}) i$, belong to the set $r_i$, according to our results mentioned at the end of § 7.

One can also from (9.24) find a functional equation for $Z_p(s, \chi)$, which we shall not give here, it differs from the previous case in that it has the form

$$\frac{Z_p(1-s, \chi)}{Z_p(s, \chi)} = \varphi(s, \chi) \text{ times a simpler function}$$

that can be given explicitly as a canonical product.

Let us finally consider as an explicit case the modular group and $\chi(M) = 1$ for all $M$. The formulas in § 8 then easily give that

$$\varphi(s) = \sqrt{\pi} \, \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^{2s}}$$

$$= \sqrt{\pi} \, \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \, \frac{\zeta(2s-1)}{\zeta(2s)},$$

where $\varphi(n)$ is Eulers function $\varphi(n) = n \prod_{p|n} (1-\frac{1}{p})$, and $\zeta$ the Riemann Zetafunction. From this expression we see that the only poles

of $\varphi(s)$ in this case are the points $s = \frac{\rho}{2}$ where $\rho$ runs over the nontrivial zeros of the Riemann zetafunction, and $s = 1$, also ~~in~~ this case $\varphi(\frac{1}{2}) = -1$.
In this case ~~and for certain~~ the formula (9.24) can be generalized considerably ~~See~~ that the operator (9.6) can be combined with one of the socalled "Hecke-operators" $T_n$ (with which it commutes), and the trace of the product computed in a similar way

Also we can in the general case show that if we have a $f(z)$ which is square integrable over $D$ and satisfies $f(Mz) = \chi(M) f(z)$, then $f(z)$ has an expression convergent in the $L_2$ sense,

$$f(z) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \alpha(r)\, E(z, \tfrac{1}{2}+ir, \chi)\, dr + \sum \beta(r_i)\, u_i(z)$$

where the $u_i(z)$ is the normed squareintegrable eigenfunction corresponding to the eigenvalue $r_i$, ~~also a Plancherel formula can be proved~~. Here

$$\alpha(r) = \iint_D f(z)\, \overline{E(z, \tfrac{1}{2}+ir, \chi)}\, \frac{dx\,dy}{y^2}$$

and

$$\beta(r_i) = \iint_D f(z)\, \overline{u_i(z)}\, \frac{dx\,dy}{y^2}$$

also a Plancherel formula

$$\iint_D |f(z)|^2 \frac{dx\,dy}{y^2} = \frac{1}{4\pi}\int_{-\infty}^{\infty} |\alpha(r)|^2 dr + \sum |\beta(r_i)|^2$$

is valid.

~~§~~

~~Concluding remarks.~~

Finally it should be remarked that these results can all be extended to the case that $\chi$ is a representation by $r \times r$ dimensional unitary matrices where $r > 1$, the resulting formulas are of course somewhat more complicated.

§.10

Concluding remarks.

We can generalize the previous theory to cover ~~also the~~ so not only the automorphic functions, but also automorphic forms that is functions transforming in the manner

$$f(Mz) = \chi(M)\,(cz+d)^k f(z)$$

where $k$ is a real number and $Mz = \frac{az+b}{cz+d}$ belongs to $\Gamma$. This is easiest to fit into the general set-up given in the introduction by considering a threedimensional space ~~and~~ with points $(z, \varphi)$, $z = x+iy$, $y>0$ and $\varphi$ real $-\infty < \varphi < \infty$. Our group $G$ consists of

$$(z, \varphi) \to (\mathfrak{M}z,\ \varphi - \arg(cz+d) + \alpha)$$

where $\mathfrak{M}z = \frac{az+b}{cz+d}$, $ad-bc=1$, and $\alpha$ is a real constant, in addition we define the transformation $\mu$ as

$$(z, \varphi) \to (-\bar{z}, -\varphi).$$

The two differential forms

$$\frac{dx^2+dy^2}{y^2} \quad \text{and} \quad d\varphi + \frac{dx}{2y}$$

are invariant under $G$ and the first also under $\mu$, while the second changes sign under $\mu$. Therefore any combination

$$a\,\frac{dx^2+dy^2}{y^2} + b\left(d\varphi + \frac{dx}{2y}\right)^2$$

with positive $a$ and $b$, gives us an invariant metric. ~~The point pair invariants under $G$~~

This space and group $G$ satisfy all the necessary requirements, the point pair invariants are all of the form $k\left(\frac{|z-\bar{z}'|^2}{yy'},\ \varphi-\varphi'-\arg(z-\bar{z}')\right)$

The invariant operators have as a basis for instance the differential operator $\frac{d}{d\varphi}$ and the Laplace operator corresponding to the above metric.

We now from the group $\Gamma$ in the upper half plane construct a discontinuous group in the three dimensional space by associating with the transformation $Mz = \frac{az+b}{cz+d}$ the transformations $M_k$ given by

$$(z, \varphi) \to (Mz,\ \varphi - \arg(cz+d) + k\pi)$$

for $k = 0, \pm 1, \pm 2, \ldots$.

The volume of the fundamental domain of the new group in the three dimensional space is finite if that of $\Gamma$ in $H$ was. We may then ask for functions $f(z, \varphi)$ that satisfy the relations

$$f(Mz,\ \varphi - \arg(cz+d) + k\pi) = \chi(M_k)\, f(z, \varphi),$$

in particular

$$F(z, \varphi + \pi) = e^{\pi\beta i} F(z, \varphi)$$

with some real $\beta$, and ~~in parti~~ also are eigenfunctions of our two fundamental operators, in particular $\frac{d}{d\varphi}$, therefore they must have the form

$$F(z, \varphi) = f(z)\, e^{-ik\varphi}$$

with $k = 2h + \beta$ ; $h$ an integer.

From this we get

$$f(Mz) = \chi(M)\, e^{-ik \arg(cz+d)} f(z)$$

or writing

$$y^{-\frac{k}{2}} f(z) = \phi(z),$$

we have easily

$$\phi(Mz) = \chi(M)(cz+d)^{-k} \phi(z).$$

If we prefer we may translate everything into ~~theta~~ functions, as can be done by for fixed $k$ considering the operators

$$\iint_H k\left(\frac{|z-z'|^2}{4y'}\right)\frac{y'^k}{(z-\bar{z}')^k} f(z')\,\frac{dx'dy'}{y'^2},$$

the integral being extended over the upper halfplane $H$. These commute among themselves and are generated by one second order differential operator (dependent on $k$).

We may proceed in a similar way as before and prove a more general trace-formula. Of particular interest is it that the analytic forms have a rather singular position, ~~in that~~

Assuming $k>2$, and taking

$k\left(\frac{|z-z'|}{4y'}\right)=1$ identically, we see that the expression $\frac{y'^k}{(z-\bar{z}')^k}$ is analytic in $z$.

The resulting operator has the property that it reproduces any analytic function $f(z)$ which in $H$ satisfies $y^{\frac{k}{2}}|f|\leq A$ for some $A>0$, apart from a factor depending on $k$ only.

The trace formula for this specific operator and a group $\Gamma$ and $\chi(M)$, will give us the formula for the number of analytic ~~forsp~~ forms of dimension $k$, that