Gram's law and the Riemann zeta-function

I will first briefly give an account of the definition and properties of the zeta-function

Let $s = \sigma + it$; $\sigma > 1$

(1) $\zeta(s) = \sum_{1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1}$

by analytic continuation, $\zeta(s)$ defined as a regular function in whole plane apart from simple pole at $s = 1$. Further

(2) $\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \pi^{\frac{s-1}{2}} \Gamma(\frac{1-s}{2}) \zeta(1-s)$.

From (1) and (2) we find that $\zeta(s)$ has no zeros in $\sigma > 1$ and in $\sigma < 0$ only "trivial" zeros $s = -2, -4, -6, \dots$ while all other zeros must lie in critical strip $0 \leq \sigma \leq 1$. symmetrically to real axis and to the critical line $\sigma = \frac{1}{2}$. zeros $\beta + i\gamma$

$N(T)$ number of γ's with $0 < \gamma \leq T$

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right)$$

where $S(T) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + iT)$

or $N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$.

Riemann conjectured that all the zeros of the zeta-function actually lie on the critical line itself. Later numerical investigations have given considerable evidence in this direction, so it would seem that

Write for $\rho = \frac{1}{2} + it$; $t > 0$

$$\vartheta = \vartheta(t) = \arg \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right).$$

Then it follows from (2) that

$$\mathcal{X}(t) = e^{i\vartheta} \xi\left(\tfrac{1}{2} + it\right)$$

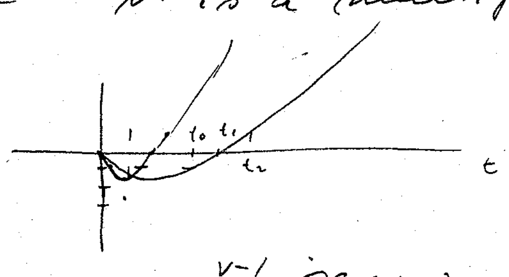
is real for real t. For large t we have

$$\vartheta = \frac{t}{2} \log \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O\left(\frac{1}{t}\right).$$

Gram used the points where $\xi(\frac{1}{2}+it)$ is real that is where ϑ is a multiple of π.

$\vartheta(t_\nu) = (\nu - 1)\pi$

$\vartheta'(t_\nu) > 0$.



$$\xi\left(\tfrac{1}{2} + it_\nu\right) = (-1)^{\nu-1} \mathcal{X}(t_\nu) \ldots$$

1). $t_{\nu-1} < \gamma_\nu < t_\nu$. 3). ; $\Delta_\nu = 0$

2) $\xi(\frac{1}{2}+it_\nu)$ is positive

3) $\xi(\frac{1}{2}+it_{\nu-1})$ and $\xi(\frac{1}{2}+it_\nu)$ has the same sign

$t_{\mu-1} < \gamma_\nu < t_\mu$; $\Delta_\nu = \nu - \mu$

Gram, Backlund, Hutchinson, Titchmarsh.

Riemann - Siegel $t > 0$

$$\mathcal{Z}(t) = \sum_{n \le \sqrt{\frac{t}{2\pi}}} \frac{\cos(\vartheta - t\log n)}{\sqrt{n}} + O(t^{-\frac{1}{4}})$$

$t > 2\pi$; t_ν

$$\xi(\tfrac{1}{2}+it_\nu) = 1 + \sum_{1 < n \le \sqrt{\frac{t_\nu}{2\pi}}} \frac{\cos(t_\nu \log n)}{\sqrt{n}} + O(t_\nu^{-\frac{1}{4}})$$

Titchmarsh: $\xi(\frac{1}{2}+it_\nu)$ negative for an infinity of ν

and $\overline{\lim}\ \Delta_\nu = \infty$; $\underline{\lim}\ \Delta_\nu = -\infty$

$N(T) = \frac{1}{\pi}\vartheta(T) + 1 + S(T)$

$\nu = \frac{\pi}{\pi}(\mu - \theta) + 1 + S(\gamma_\nu)$ $\quad 0 < \theta < 1$

$\nu - \mu = S(\gamma_\nu) + 1 - \theta$; $\Delta_\nu = S(\gamma_\nu) + \theta$

$\xi(\frac{1}{2}+it_{\nu-1})$ and $\xi(\frac{1}{2}+it_\nu)$ has same sign.

1) prove that $\nu \le N$ opposite sign in $> KN$.

enough to prove that a positive proportion of intervals $(t_{\nu-1}, t_\nu)$ are without any γ. This is the case if $S(t_\nu) < S(t_{\nu-1})$; since

$$N(t) = \frac{1}{\pi}\vartheta(t) + 1 + S(t)$$

$$\underline{N(t_\nu) = \nu + S(t_\nu)}$$

Enough to prove that for some fixed r we have

$$S(t_{\nu+r}) < S(t_\nu) \quad \text{for more than } KN, \nu\text{'s}.$$

Now we have

$$\frac{1}{N}\sum_{\nu\le N}\left|S(t_\nu) + \frac{1}{\pi}\sum_{p\le X}\frac{\sin t_\nu \log p}{\sqrt{p}}\right|^{2k} = O(1)$$

for $N^a \le X \le N^{\frac{1}{k}}$; $a>0$.

From this we get in the first place

$$\frac{1}{N}\sum_{\nu=1}^{N}\left(S(t_{\nu+r}) - S(t_\nu)\right)^2 = c\log(1+r) + O(1) + O(\sqrt{\log(1+r)}).$$

on the other hand we have

$$\frac{1}{N}\sum_{\nu=1}^{N}\left(S(t_{\nu+r}) - S(t_\nu)\right) = o(1)$$

and

$$\frac{1}{N} \sum_{\nu=1}^{N} (S(t_{\nu+r}) - S(t_\nu))^3 = O(\log(1+r)).$$

$$\underline{\underline{r}} + S(t_{\nu+r}) - S(t_\nu) \geq 0.$$

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2) prove that $\xi(\frac{1}{2}+it_{\nu-1})$ and $\xi(\frac{1}{2}+it_\nu)$ has same sign or that $\mathcal{X}(t_{\nu-1})$ and $\mathcal{X}(t_\nu)$ opposite sign.

$$\cancel{\sum_{\nu=1}^{N}} \left\{ \mathcal{X}(t_\nu) + \mathcal{X}(t_{\nu+1}) \cdots + \mathcal{X}(t_{\nu+r}) \right\}^2$$

$$\cancel{\sum_{\nu=1}^{N}} \left\{ \mathcal{X}(t_\nu) - \mathcal{X}(t_{\nu+1}) + \cdots + (-1)^r \mathcal{X}(t_{\nu+r}) \right\}^2$$

$$I(\nu, r) = \mathcal{X}(t_\nu) + \mathcal{X}(t_{\nu+1}) + \cdots + \mathcal{X}(t_{\nu+r})$$

$$(-1)^{\nu-1} J(\nu, r) = \mathcal{X}(t_\nu) - \mathcal{X}(t_{\nu+1}) + \cdots + (-1)^r \mathcal{X}(t_{\nu+r})$$

if $|I(\nu,r)| < |J(\nu,r)|$ then not all $\mathcal{X}(t_{\nu+i})$ $i=0,\ldots r$ of same sign.

Investigate

$$\sum_{\nu=1}^{N} (I(\nu,r))^2 \quad ; \quad \sum_{\nu=1}^{N} (J(\nu,r))^2$$

and $\displaystyle\sum_{\nu=1}^{N} J(\nu, r) > K r N.$

$$\sum_{n=1}^{N} \Delta_n^{2k} = \frac{2k!}{k!(2\pi)^{2k}} N(\log\log N)^k + O(N(\log\log N)^{k-\frac{1}{2}})$$

$$\sum_{n \le N} \Delta_n^{k} = \int_1^{t_N} S^k(t)\, dN(t) =$$

$$= \frac{1}{2\pi}\int_1^{t_N} S^k(t) \log\frac{t}{2\pi}\, dt + \int_1^{t_N} S^k(t)\, dS(t)$$

$$\sum_{\nu=1}^{N} \left(S(t_{\nu+r}) - S(t_\nu)\right)^k$$

$$\frac{1}{N}\sum_{\nu=1}^{N} \left(S(t_{\nu+r}) - S(t_\nu)\right)^{2} = c\log(1+r) + O(1)$$

$$r + S(t_{\nu+r}) - S(t_\nu) \ge 0$$

$$\mathcal{X}_1(t) = \mathcal{X}(t)\,|\eta(t)|^2$$

$$\sum_{\nu=1}^{N} \{\mathcal{X}_1(t_\nu) + \mathcal{X}_1(t_{\nu+1}) \cdots + \mathcal{X}_1(t_{\nu+r})\}^2$$

$$\sum_{\nu=1}^{N} \{\mathcal{X}_1(t_\nu) - \mathcal{X}_1(t_{\nu+1}) \cdots + (-1)^r \mathcal{X}_1(t_{\nu+r})\}^2$$

$$\sum (-1)^{\nu} \mathcal{X}_1(t_\nu) - \mathcal{X}_1(t_{\nu+1}) \cdots + (-1)^{\nu+r} \mathcal{X}_1(t_{\nu+r})$$