Dirichlet L-function

$$L(s,\chi) = \sum_n \chi(n)\, n^{-s};$$

We always assume χ primitive to modul q. Functional equation: $a = \frac{1-\chi(-1)}{2}$, $|\varepsilon|=1$, write:

$$\phi(s,\chi) = \varepsilon \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi),$$

then

$$\phi(s,\chi) = \overline{\phi(1-\bar{s},\chi)}.$$

Implies $\phi(s,\chi)$ real for $s=\frac{1}{2}+it$, t real.

For simplicity I only consider case $\chi(-1)=1$. Case of odd χ can be handled similarly. If we have n distinct even χ_j and form

$$F(s) = \sum_{j=1}^{n} c_j \varepsilon_j q_j^{\frac{s}{2}} L(s,\chi_j); \quad c_j \text{ real } \neq 0,$$

(or alternatively $\left(F(s) = \sum_j^{n} c_j \varepsilon_j (1+q_j^{s-\frac{1}{2}}) L(s,\chi_j)\right)$),

then $\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) F(s)$ is real for $s=\frac{1}{2}+it$.

Apart from trivial zeros implied by the functional equation, the zeros of $F(s)$ are confined to a vertical strip, and if we let $N(T,F)$ denote the number of these zeros with imaginary part in $(0,T)$, then for large T we have

$$N(T,F) = \frac{T}{2\pi}(\log T + B) + O(\log T),$$

where B is a constant dependent on F.

It is conjectured that almost all of these zeros have real part $\frac{1}{2}$. A proof of this can be given if one assumes some other plausible but at present unverifiable conjectures.

For the single L-function it has been proved that a positive proportion of the zeros have real part $\frac{1}{2}$. More precisely, if we denote the number of zeros with imaginary part in $(0,T)$ and real part $\frac{1}{2}$ by $N_0(T,F)$ for a general linear combination $F(s)$, then it has been shown that

$$N_0(T,L) > cT\log T \quad \text{for } T > Aq^2,$$

where c and A are positive constants independent of χ.

For a general linear combination, there are some results in the literature. $N_0(T,F) > cT$ is implied in the work of Hardy-Littlewood, recently

A. A. Karatsuba has looked at linear combinations of the form

$$F(s) = \varepsilon L(s,\chi) + \bar{\varepsilon} L(s,\bar{\chi}),$$

where χ is a complex character, and has shown

$$N_0(T,F) > T(\log T)^{\frac{1}{2}} e^{-c\sqrt{\log\log T}}, \quad T > T_0$$

where c is a positive constant (1993 or 1994). For a more general linear combination he claims a much weaker and much more complicated result.

I shall sketch a proof that for the general combination $F(s)$ we have

(1) <u>$N_0(T,F) > c(n)\, T \log T$ for $T > T_0(F)$, where $c(n)$ is positive and depends on n only.</u>

(2) <u>If $\omega(t) \to \infty$ with t, then $F(\frac{1}{2}+it)$ has a zero in the interval $\left(t, t+\frac{\omega(t)}{\log t}\right)$ for almost all t.</u>

We go back to the method that was used to show these results for a single L-function.

We write for $s = \frac{1}{2} + it$,

$$\vartheta(t) = \arg \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right),$$

and

$$X(t,\chi) = \varepsilon_\chi\, q^{\frac{it}{2}} e^{i\vartheta(t)} L(s,\chi).$$

We need the "approximate functional equation" for $L(s,\chi)$: Assume $t>0$, then

$$L(s,\chi) = \sum_{n<\sqrt{\frac{tq}{2\pi}}} \chi(n)\, n^{-s} +$$

$$+ \bar{\varepsilon}_\chi^{\,2}\, q^{-it} e^{-2i\vartheta(t)} \sum_{n<\sqrt{\frac{tq}{2\pi}}} \bar{\chi}(n)\, n^{s-1} + O\left(\left(\frac{q}{t}\right)^{\frac{1}{4}}\right).$$

We also write

$$(\zeta(s))^{-\frac{1}{2}} = \sum_n \frac{\alpha_n}{n^s}\ ;\ \alpha_1 = 1;\ (L(s,\chi))^{\frac{1}{2}} = \sum_n \frac{\chi(n)\,\alpha_n}{n^s},$$

and for $T \le t \le 2T$; $\xi = T^{\frac{1}{10}}$, $T > q^2$;

we write

$$\eta(t,\chi) = \sum_{n<\xi} \frac{\alpha_n}{n^s}\left(1 - \frac{\log n}{\log \xi}\right).$$

We now consider for

$$\frac{1}{\log T} \le H \le \frac{\log\log T}{\log T},$$

the three expressions:

$$I_\chi(t,H) = \int_t^{t+H} X(u,\chi)\,|\eta(u,\chi)|^2\,du,$$

$$M_\chi(t,H) = \int_t^{t+H} L(\tfrac{1}{2}+iu,\chi)\,\eta^2(u,\chi)\,du - H,$$

and

$$J_\chi(t,H) = \int_t^{t+H} |X(u,\chi)\,\eta^2(u,\chi)|\,du.$$

It is clear that if

$$J_\chi(t,H) > |I_\chi(t,H)|,$$

then $X(u,\chi)$ changes sign in $(t,t+H)$, and so has at least one zero there.

Clearly

$$J_\chi(t,H) \geqslant H - |M_\chi(t,H)|.$$

Thus if

$$|M_\chi(t,H)| + |I_\chi(t,H)| < H,$$

there is a zero in $(t,t+H)$.

Using the approximate functional-equation for $L(s,\chi)$ it is possible to show

$$\int_T^{2T} |I_\chi(t,H)|^2 dt = O\left(T \frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right),$$

and

$$\int_T^{2T} |M_\chi(t,H)|^2 dt = O\left(T \frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right),$$

where the constants implied by the O-symbols are independent of χ.

From this it follows that we have $|M_\chi(t,H)| < \frac{H}{3}$ and $|I_\chi(t,H)| < \frac{H}{3}$ in $(T,2T)$ except in a subset of measure $O\left(\frac{T}{\sqrt{H \log T}}\right)$.

Thus, for all t in $(T,2T)$ except for a subset of measure $O\left(\frac{T}{\sqrt{H \log T}}\right)$, the interval $(t,t+H)$ contains a zero of $L(\frac{1}{2}+it,\chi)$, choosing $H = \frac{\lambda}{\log T}$ with λ a large enough constant, both statements made earlier follow easily for the single L-function.

In order to adapt this idea to the general linear combination, we need some general results that can be proved about the value distribution of

$\log |L(s,\chi)|$ for $s = \frac{1}{2} + it$.

We can show that for $T > q^2$, k a positive integer, and

$$T^{\frac{1}{2k}} \leq x \leq T^{\frac{1}{k}}$$

then

$$\int_T^{2T} \left| \log|L(s,\chi)| - \mathcal{R} \sum_{p<x} \frac{\chi(p)}{p^s} \right|^{2k} dt =$$

$$= O\left(k^{4k} e^{Ak} T \right).$$

Again the constant implied by the O is independent of χ.

From this formula, it is possible to prove that

$$\frac{\log|L(s,\chi)|}{\sqrt{\pi \log\log t}},$$

has a normal Gaussian distribution.

More precisely, if $\chi_{a,b}(u)$ denotes the characteristic function of the interval (a,b), we have

$$\int_T^{2T} \chi_{a,b}\left(\frac{\log|L(s,\chi)|}{\sqrt{\pi \log\log T}}\right)dt =$$

$$= T\int_a^b e^{-\pi u^2}du + O\left(T\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right).$$

If we have two L-functions $L(s,\chi)$ and $L(s,\chi')$, similar results hold for the difference

$$\log|L(s,\chi)| - \log|L(s,\chi')|,$$

only here we have to divide by

$$\sqrt{2\pi \log\log t}$$

to get the normal Gaussian distribution.

From this we see that the subset of $(T, 2T)$ where

$$\big|\log|L(s,\chi)| - \log|L(s,\chi')|\big| \le$$
$$\le (\log\log T)^{\frac{1}{4}},$$

has measure which is $O(T(\log\log T)^{-\frac{1}{4}})$.
Thus in $(T, 2T)$ for our n L-functions we see that when $j \neq k$,
$$\big|\log|L(s,\chi_j)| - \log|L(s,\chi_k)|\big| > (\log\log T)^{\frac{1}{4}}$$
except for a set of t of measure
$$O(T(\log\log T)^{-\frac{1}{4}}).$$

Outside of this exceptional set, which we will call $\mathcal{E}_1$, we see that one single term is decisively dominant in the linear combination that forms $F(s)$.

We shall see that this dominance is in general fairly stable over longer (compared with $\frac{1}{\log T}$) stretches.

For $|h| < \frac{1}{\sqrt{\log T}}$, we can show that
$$\int_T^{2T} \big|\log|L(s+h,\chi)| - \log|L(s,\chi)|\big|^{2k} =$$
$$= O\left(T k^k \left(\log(e+|h|\log T)\right)^k + T k^{4k} e^{Ak}\right),$$
which shows that the amplitude of the oscillations of $X(t,\chi)$ is fairly

stable over longer stretches.

Put $\Delta = \frac{1}{\log T} e^{(\log\log T)^{\frac{1}{5}}}$,

and let us compare $\log|L(\frac{1}{2}+it, \chi)|$,

with the average

$$\Delta(t,\chi) = \frac{1}{2\Delta}\int_{t-\Delta}^{t+\Delta} \log|L(\tfrac{1}{2}+iu, \chi)|\,du.$$

Using the previous estimate with k a sufficiently large positive integer, we find that

$$\left|\log|L(\tfrac{1}{2}+it,\chi)| - \Delta(t,\chi)\right| < (\log\log T)^{\frac{1}{5}},$$

except in a subset E_2 of measure that is $O\left(\frac{T}{(\log\log T)^N}\right)$ for any fixed positive N.

We can now conclude that $(T, 2T)$ apart from a subset of measure

$$O(T(\log\log T)^{-\frac{1}{4}}),$$

which we call E, can be divided into n sets S_j such that in S_j

$\log|L(\frac{1}{2}+it, \chi_j)|$ exceeds all the other $\log|L(\frac{1}{2}+it,\chi_k)|$ with $k\neq j$ by at least $(\log\log T)^{\frac{1}{4}}$ and in addition such that in the interval $(t, t+H)$ where $\frac{1}{\log T} < H < \frac{\log\log T}{\log T}$ and t in S_j we have

$$\log|L(\tfrac{1}{2}+it',\chi_j)| > \log|L(\tfrac{1}{2}+it',\chi_k| + \tfrac{1}{2}(\log\log T)^{\frac{1}{4}},$$

for $t \leq t' \leq t+H$, except in a subset of $(t, t+H)$ of measure

$$O\left(\frac{H}{(\log\log T)^{\frac{1}{5}}}\right).$$

The measure of S_j can be shown to be

$$\geq \frac{T}{m} - O(T(\log\log T)^{-\frac{1}{4}})$$

(The $\log|L(s,\chi_j)|$ are statistically independent so each dominates about equally often).

We also have for all j that

$$\int_T^{2T} |L(\tfrac{1}{2}+it, \chi_j)\, \eta^2(t,\chi_j)|^2\, dt = O(T).$$

This shows that we have

$$\int_t^{t+H} |L(\tfrac{1}{2}+it, \chi_j)\, \eta^2(t,\chi_j)|^2\, dt < H(\log\log T)^{\frac{1}{6}},$$

except for a subset E_j' of measure

$$O(T(\log\log T)^{-\frac{1}{6}}).$$

If we now exclude from S_j the t that may be in E_j' we get a set S_j' of measure $> \frac{T}{m} - O(T(\log\log T)^{-\frac{1}{6}})$,

and such that for t in S_j' we have

$$\log|L(\tfrac{1}{2}+iu, \chi_j)| > \log|L(\tfrac{1}{2}+iu, \chi_k)| + \tfrac{1}{2}(\log\log T)^{\frac{1}{4}},$$

for $t \le u \le t+H$, except for an exceptional set of measure at most

$$O(H(\log\log T)^{-\frac{1}{5}}).$$

This exceptional set in $(t, t+H)$ can contribute at most

$$O\left(H^{\frac{1}{2}}(\log\log T)^{-\frac{1}{10}}\right) \cdot O\left(H^{\frac{1}{2}}(\log\log)^{\frac{1}{12}}\right)$$

$$= O\left(H(\log\log T)^{-\frac{1}{60}}\right),$$

to the integrals

$$I_{\chi_j}(t,H), \quad M_{\chi_j}(t,H) \text{ and } J_{\chi_j}(t,H).$$

Calling these integrals with the exceptional set excluded

$$I^*_{\chi_j}(t,H), \; M^*_{\chi_j}(t,H) \text{ and } J^*_{\chi_j}(t,H),$$

we see that we have a sign change of $X_{\chi_j}(t)$ in $(t, t+H)$ outside of the excluded subset if

$$J^*_{\chi_j}(t,H) > |I^*_{\chi_j}(t,H)|,$$

which is equivalent to

$$H > |I_{\chi_j}(t,H)| + |M_{\chi_j}(t,H)| + O\left(H(\log\log T)^{-\frac{1}{60}}\right).$$

For T large enough this holds in $(T, 2T)$ outside a set of measure

$$O\left(\frac{T}{\sqrt{H \log T}}\right),$$

and therefore in most of S_j^i if $H = \frac{\lambda n^2}{\log T}$ with λ a large enough constant.

This produces more than

$$\frac{c}{n^3} T \log T$$

signchanges in $\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) F(s)$ for $s = \frac{1}{2} + it$ and t in $S_j^{\prime}$, adding up over the j, we get in all more than

$$\frac{c}{n^2} T \log T$$

signchanges for $T > T_0(F)$. This proves our statement (1); (2) also follows simply by taking $\lambda = (\omega(2T))^{\frac{1}{2}}$.

The dependence on n can be sharpened by improving the estimations for $\int_T^{2T} |I_\chi(t, H)|^2 dt$ and $\int_T^{2T} |M_\chi(t, H)|^2 dt$, If one replaces the estimation

$$O\left(T\,\frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right)$$

by

$$O\left(T\,\frac{H^{2}}{(H\log T)^{\alpha}}\right)$$

one gets

$$> c\,n^{-\frac{1}{\alpha}}\,T\log T$$

signchanges. This can be proved for any $\alpha < 1$, I cannot yet show that it holds for $\alpha = 1$ (which I think is the best one can hope for). So far my best result is $(n > 1)$

$$N_0(T, F) > \frac{c}{n\log n}\,T\log T \text{ for } T > T_0(F).$$

Problem of generalizations to L-functions with higher Γ-factors.