Dirichlet L-function

$$L(s,\chi) = \sum_n \chi(n) n^{-s};$$

we always assume χ primitive to modulus q. Functional equation: $a = \dfrac{1-\chi(-1)}{2}$, $|\varepsilon| = 1$, write:

$$\phi(s,\chi) = \varepsilon \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi),$$

then

$$\phi(s,\chi) = \overline{\phi(1-\bar{s},\chi)}.$$

Implies $\phi(s,\chi)$ real for $s = \dfrac{1}{2} + it$, t real.

For simplicity I only consider case $\chi(-1) = 1$. Case of odd χ can be handled similarly.

If we have n distinct even χ; and form

$$F(s) = \sum_{j=1}^{n} c_j \varepsilon_j q_j^{\frac{s}{2}} L(s,\chi_j); \quad c_j \ \text{real} \ \neq 0,$$

$$\left(\text{or alternatively} \quad F(s) = \sum_{j=1}^{n} c_j \varepsilon_j \left(1 + q_j^{s-\frac{1}{2}}\right) L(s,\chi_j)\right),$$

then $\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) F(s)$ is real for $s = \dfrac{1}{2} + it$.

Apart from trivial zeros implied by the functional equation, the zeros of $F(s)$ are confined to a vertical strip, and if we let $N(T,F)$ denote the number of these zeros with imaginary part in $(0,T)$, then for large T we have

$$N(T,F) = \frac{T}{2\pi}(\log T + B) + O(\log T),$$

where B is a constant dependent on F.

It is conjectured that almost all of these zeros have real part $\dfrac{1}{2}$. A proof of this can be given if one assumes some other plausible but at present unverifiable conjectures.

For the single L-function it has been proved that a positive proportion of the zeros have real part $\frac{1}{2}$. More precisely, if we denote the number of zeros with imaginary part in $(0,T)$ and real part $\frac{1}{2}$ by $N_0(T,F)$ for a general linear combination $F(s)$, then it has been shown that

$$N_0(T,L) > CT\log T \quad \text{for } T > Aq^2,$$

where c and A are positive constants independent of χ.

For a general linear combination, there are some results in the literature. $N_0(T,F) > cT$ is implied in the work of Hardy-Littlewood. Recently A. A. Karatsuba has looked at linear combinations of the form

$$F(s) = \varepsilon L(s,\chi) + \bar{\varepsilon} L(s,\overline{\chi}),$$

where χ is a complex character, and has shown

$$N_0(T,F) > T(\log T)^{\frac{1}{2}} e^{-c\sqrt{\log\log T}}, \quad T > T_0,$$

where c is a positive constant (1993 or 1994). For a more general linear combination he claims a much weaker and much more complicated result.

I shall sketch a proof that for the general combination $F(s)$ we have

(1) $N_0(T,F) > c(n)T\log T$ for $T > T_0(F)$, where $c(n)$ is positive and depends on n only.

(2) If $\omega(t) \to \infty$ with t, then $F(\frac{1}{2}+it)$ has a zero in the interval $(t, t+\frac{\omega(t)}{\log t})$ for almost all t.

We go back to the method that was used to show these results for a single L-function. We write for $s = \frac{1}{2} + it$,

$$\vartheta(t) = \arg \pi^{-\frac{t}{2}} \Gamma\left(\frac{s}{2}\right),$$

and

$$X(t,\chi) = \varepsilon_\chi q^{\frac{it}{2}} e^{i\vartheta(t)} L(s,\chi).$$

We need the "approximate functional equation" for $L(s,\chi)$: assume $t > 0$, then

$$L(s,\chi) = \sum_{n<\sqrt{\frac{tq}{2\pi}}} \chi(n)n^{-s} + \overline{\varepsilon_\chi}^2 q^{-it} e^{-2i\vartheta(t)} \sum_{n<\sqrt{\frac{tq}{2\pi}}} \overline{\chi}(n) n^{s-1} + O\left(\left(\frac{q}{t}\right)^{\frac{1}{4}}\right).$$

We also write

$$(\zeta(s))^{-\frac{1}{2}} = \sum_n \frac{\alpha_n}{n^s}; \quad \alpha_1 = 1; \quad (L(s,\chi))^{-\frac{1}{2}} = \sum_n \frac{\chi(n)\alpha_n}{n^s},$$

and for $T \le t \le 2T$; $\xi = T^{\frac{1}{10}}$, $T > q^2$; we write

$$\eta(t,\chi) = \sum_{n<\xi} \frac{\chi(n)\alpha_n}{n^s}\left(1 - \frac{\log n}{\log \xi}\right).$$

We now consider for

$$\frac{1}{\log T} \le H \le \frac{\log\log T}{\log T},$$

the three expressions:

$$I_\chi(t,H) = \int_t^{t+H} X(u,\chi)|\eta(u,\chi)|^2\, du,$$

$$M_\chi(t,H) = \int_t^{t+H} L(\frac{1}{2} + iu,\chi)\eta^2(u,\chi)\, du - H,$$

and

$$J_\chi(t,H) = \int_t^{t+H} |X(u,\chi)\eta^2(u,\chi)|\, du.$$

It is clear that if

$$J_\chi(t,H) > |I_\chi(t,H)|,$$

then $X(u,\chi)$ changes sign in $(t, t+H)$, and so has at least one zero there. Clearly

$$J_\chi(t,H) \ge H - |M_\chi(t,H)|.$$

Thus if

$$|M_\chi(t,H)| + |I_\chi(t,H)| < H,$$

there is a zero in $(t, t+H)$.

Using the approximate functional equation for $L(s,\chi)$ it is possible to show

$$\int_T^{2T} |I_\chi(t,H)|^2\, dt = \mathcal{O}\left(T\frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right),$$

and

$$\int_T^{2T} |M_\chi(t,H)|^2\, dt = \mathcal{O}\left(T\frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right),$$

where the constants implied by the $\mathcal{O}$-symbols are independent of χ. From this it follows that we have $|M_\chi(t,H)| < \frac{H}{3}$ and $|I_\chi(t,H)| < \frac{H}{3}$ in $(T,2T)$ except in a subset of measure $\mathcal{O}\left(\frac{T}{\sqrt{H\log T}}\right)$. Thus, for all t in $(T,2T)$ except for a subset of measure $\mathcal{O}\left(\frac{T}{\sqrt{H\log T}}\right)$, the interval $(t,t+H)$ contains a zero of $L(\frac{1}{2}+it,\chi)$. Choosing $H = \frac{\lambda}{\log T}$ with λ a large enough constant, both statements made earlier follow easily for the single L-function.

In order to adapt this idea to the general linear combination, we need some general results that can be proved about the value distribution of

$$\log|L(s,\chi)| \quad \text{for } s = \frac{1}{2}+it.$$

We can show that for $T > q^2$, k a positive integer, and

$$T^{\frac{1}{2k}} \le x \le T^{\frac{1}{k}},$$

then

$$\int_T^{2T} \left|\log|L(s,\chi)| - \Re\sum_{p<x}\frac{\chi(p)}{p^s}\right|^{2k} dt = \mathcal{O}\left(k^{4k}e^{Ak}T\right).$$

Again the constant implied by the $\mathcal{O}$ is independent of χ.

From this formula, it is possible to prove that

$$\frac{\log|L(s,\chi)|}{\sqrt{\pi\log\log t}}$$

has a normal Gaussian distribution. More precisely, if $\mathcal{U}_{a,b}(u)$ denotes the characteristic function of the interval (a,b), we have

$$\int_T^{2T} \mathcal{U}_{a,b}\left(\frac{\log|L(s,\chi)|}{\sqrt{\pi\log\log T}}\right) dt = T\int_a^b e^{-\pi u^2}\, du + \mathcal{O}\left(T\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right).$$

If we have two L-functions $L(s,\chi)$ and $L(s,\chi')$, similar results hold for the difference

$$\log|L(s,\chi)| - \log|L(s,\chi')|.$$

Only here we have to divide by $\sqrt{2\pi\log\log t}$ to get the normal Gaussian distribution.

From this we see that the subset of $(T, 2T)$ where

$$\left|\log|L(s,\chi)| - \log|L(s,\chi')|\right| \leq (\log\log T)^{\frac{1}{4}},$$

has measure which is $\mathcal{O}\left(T(\log\log T)^{-\frac{1}{4}}\right)$. Thus in $(T, 2T)$ for our n L-functions we see that when $j \neq k$,

$$\left|\log|L(s,\chi_j)| - \log|L(s,\chi_k)|\right| > (\log\log T)^{\frac{1}{4}}$$

except for a set of t of measure

$$\mathcal{O}(T(\log\log T)^{-\frac{1}{4}}).$$

Outside of this exceptional set, which we will call E_1, we see that one single term is decisively dominant in the linear combination that forms $F(s)$.

We shall see that this dominance is in general fairly stable over longer (compared with $\frac{1}{\log T}$) stretches.

For $|h| < \frac{1}{\sqrt{\log T}}$, we can show that

$$\int_T^{2T} \left|\log|L(s+h,\chi)| - \log|L(s,\chi)|\right|^{2k} ds$$
$$= \mathcal{O}\left(Tk^k(\log(e+|h|\log T))^k + Tk^{4k}e^{Ak}\right),$$

which shows that the amplitude of the oscillations of $X(t,\chi)$ is fairly stable over longer stretches.

Put

$$\Delta = \frac{1}{\log T} e^{(\log\log T)^{\frac{1}{5}}},$$

and let us compare $\log\left|L(\frac{1}{2}+it,\chi)\right|$ with the average

$$\Delta(t,\chi) = \frac{1}{2\Delta}\int_{t-\Delta}^{t+\Delta} \log\left|L(\frac{1}{2}+iu,\chi)\right| du.$$

Using the previous estimate with k a sufficiently large positive integer, we find that

$$\left|\log\left|L(\frac{1}{2}+it,\chi)\right| - \Delta(t,\chi)\right| < (\log\log T)^{\frac{1}{5}},$$

except in a subset E_2 of measure that is $\mathcal{O}\left(\frac{T}{(\log\log T)^N}\right)$ for any fixed positive N.

We can now conclude that $(T, 2T)$ apart from a subset of measure

$$\mathcal{O}\left(T(\log\log T)^{-\frac{1}{4}}\right),$$

which we call E, can be divided into n sets S_j such that in S_j $\log\left|L(\frac{1}{2}+it,\chi_j)\right|$ exceeds all the other $\log\left|L(\frac{1}{2}+it,\chi_k)\right|$ with $k \neq j$ by at least $(\log\log T)^{\frac{1}{4}}$ and in addition such that in the interval $(t, t+H)$ where $\frac{1}{\log T} < H < \frac{\log\log T}{\log T}$ and t in S_j we have

$$\log\left|L(\frac{1}{2}+it',\chi_j)\right| > \log\left|L(\frac{1}{2}+it',\chi_k)\right| + \frac{1}{2}(\log\log T)^{\frac{1}{4}},$$

for $t \leq t' \leq t+H$, except in a subset of $(t, t+H)$ of measure

$$\mathcal{O}\left(\frac{H}{(\log\log T)^{\frac{1}{5}}}\right).$$

The measure of S_j can be shown to be

$$\geq \frac{T}{n} - \mathcal{O}\left(T(\log\log T)^{-\frac{1}{4}}\right).$$

(The $\log|L(s,\chi_j)|$ are statistically independent, so each dominates about equally often.)

We also have for all j that

$$\int_T^{2T}\left|L(\frac{1}{2}+it,\chi_j)\eta^2(t,\chi_j)\right|^2 dt = \mathcal{O}(T).$$

This shows that we have

$$\int_t^{t+H}\left|L(\frac{1}{2}+it,\chi_j)\eta^2(t,\chi_j)\right|^2 dt < H(\log\log T)^{\frac{1}{6}},$$

except for a subset E_j' of measure

$$\mathcal{O}\left(T(\log\log T)^{-\frac{1}{6}}\right).$$

If we now exclude from S_j the t that may be in E_j' we get a set S_j' of measure

$$> \frac{T}{n} - \mathcal{O}\left(T(\log\log T)^{-\frac{1}{6}}\right),$$

and such that for t in S'_j we have

$$\log\left|L(\frac{1}{2}+iu,\chi_j)\right| > \log\left|L(\frac{1}{2}+iu,\chi_k)\right| + \frac{1}{2}(\log\log T)^{\frac{1}{4}},$$

for $t \le u \le t+H$, except for an exceptional set of measure at most

$$\mathcal{O}\left(H(\log\log T)^{-\frac{1}{5}}\right).$$

This exceptional set in $(t,t+H)$ can contribute at most

$$\mathcal{O}\left(H^{\frac{1}{2}}(\log\log T)^{-\frac{1}{10}}\right)\cdot\mathcal{O}\left(H^{\frac{1}{2}}(\log\log)^{\frac{1}{12}}\right)$$
$$=\mathcal{O}\left(H(\log\log T)^{-\frac{1}{60}}\right)$$

to the integrals

$$I_{\chi_j}(t,H),\quad M_{\chi_j}(t,H)\quad\text{and}\quad J_{\chi_j}(t,H).$$

Calling these integrals with the exceptional set excluded

$$I^*_{\chi_j}(t,H),\quad M^*_{\chi_j}(t,H)\quad\text{and}\quad J^*_{\chi_j}(t,H),$$

we see that we have a sign change of $X_{\chi_j}(t)$ in $(t,t+H)$ outside of the excluded subset if

$$J^*_{\chi_j}(t,H) > \left|I^*_{\chi_j}(t,H)\right|,$$

which is equivalent to

$$H > \left|I_{\chi_j}(t,H)\right| + \left|M_{\chi_j}(t,H)\right| + \mathcal{O}\left(H(\log\log T)^{-\frac{1}{60}}\right).$$

For T large enough this holds in $(T,2T)$ outside a set of measure

$$\mathcal{O}\left(\frac{T}{\sqrt{H\log T}}\right),$$

and therefore in most of S'_j if $H=\dfrac{\lambda n^2}{\log T}$ with λ a large enough constant.

This produces more than

$$\frac{c}{n^3}T\log T$$

sign changes in $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)F(s)$ for $s=\dfrac{1}{2}+it$ and t in S'_j. Adding up over the j, we get in all more than

$$\frac{c}{n^2}T\log T$$

sign changes for $T>T_0(F)$. This proves our statement (1); (2) also follows simply by taking $\lambda=(\omega(T))^{\frac{1}{2}}$.

The dependence on n can be sharpened by improving the estimations for

$$\int_T^{2T} |I_\chi(t,H)|^2 \, dt \quad \text{and} \quad \int_T^{2T} |M_\chi(t,H)|^2 \, dt.$$

If one replaces the estimation

$$\mathcal{O}\left(T\frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right)$$

by

$$\mathcal{O}\left(T\frac{H^2}{(H\log T)^\alpha}\right),$$

one gets

$$> cn^{-\frac{1}{\alpha}} T\log T$$

sign changes. This can be proved for any $\alpha < 1$. I cannot yet show that it holds for $\alpha = 1$ (which I think is the best one can hope for). So far my best result is ($n > 1$)

$$N_0(T,F) > \frac{c}{n\log n} T\log T \quad \text{for} \quad T > T_0(F).$$

Problem of generalizations to L-functions with higher Γ-factors.