**Selberg Paper.** Norsk. Mathematatisk Tidsskrift, **26**, 71-78 (1944)

Write

$$\Delta(u) = \Delta(u_1, \dots, u_p) = \prod_{i<j}^{p} (u_j - u_i).$$

**Theorem:** *For integer p and complex x, y, z with*

$$\mathrm{Re}\,(x) > 0,\ \mathrm{Re}\,(y) > 0,\ \mathrm{Re}\,(z) > -\min\left[\frac{1}{p}, \frac{\mathrm{Re}\,(x)}{p-1}, \frac{\mathrm{Re}\,(y)}{p-1}\right],$$

*we have*

$$\begin{aligned} I &= \int_0^1 \cdots \int_0^1 \left(\prod_1^p u_j\right)^{x-1} \left(\prod_1^p (1-u_j)\right)^{y-1} |\Delta(u)|^{2z}\, du_1 \cdots du_p \\ &= \prod_{\nu=1}^{p} \left[\frac{\Gamma(1+\nu z)\Gamma(x+(\nu-1)z)\Gamma(y+(\nu-1)z)}{\Gamma(1+z)\Gamma(x+y+(p+\nu-2)z)}\right]. \end{aligned}$$

**Proof:** For $p = 1$ this reduces to the well-known Euler integral

$$\int_0^1 u^{x-1}(1-u)^{y-1} du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

so we assume $p > 1$.

Consider first the case when $z$ is a positive integer. Then

$$|\Delta(u)|^{2z} = \sum C_{\alpha_1 \alpha_2 \cdots \alpha_p} u_1^{\alpha_1} \cdots u_p^{\alpha_p}$$

with integer coefficients $c$. Therefore the integral $I$ is a linear combination of terms

$$\prod_{\nu=1}^{p} \left[\frac{\Gamma(x+\alpha_\nu)\Gamma(y)}{\Gamma(x+y+\alpha_\nu)}\right],$$

where, without loss of generality, we may suppose that $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_p$. Since $\Delta(u)$ is homogeneous of degree $\frac{1}{2}p(p-1)$, we have

$$\sum_1^p \alpha_\nu = p(p-1)z,$$

$$\alpha_p \ge (p-1)z.$$

In the same way, since $\Delta(u_1, \dots, u_p)$ is divisible by $\Delta(u_1, \dots, u_\nu)$ for each $\nu$, we have

$$\alpha_\nu \ge (\nu-1)z.$$

Now

$$|\Delta(u)|^{2z} = \left(\prod_1^p u_j\right)^{2(p-1)z} |\Delta\,(1/u)|^{2z}\ ,$$

and therefore the exponents $\alpha'_\nu = 2(p-1)z - \alpha_{p+1-\nu}$ satisfy the same inequalities

$$\alpha'_\nu \geq (\nu - 1)z\,.$$

Therefore

$$\alpha_\nu \leq 2(p-1)z - (p-\nu)z = (p+\nu-2)z\,.$$

This means that

$$\frac{\Gamma(x+\alpha_\nu)}{\Gamma(x+y+\alpha_\nu)} = \frac{\Gamma(x+(\nu-1)z)}{\Gamma(x+y+(p+\nu-2)z)} q_{\alpha_\nu}(x,y)$$

where $q_{\alpha_\nu}(x,y)$ is a polynomial in $x$ and $y$ with degree $[(p+\nu-2)z - \alpha_\nu]$ in $y$. Thus

$$\prod_{\nu=1}^{p} \frac{\Gamma(x+\alpha_\nu)\Gamma(y)}{\Gamma(x+y+\alpha_\nu)} = Q_\alpha(x,y) \prod_{\nu=1}^{p} \left[\frac{\Gamma(x+(\nu-1)z)\Gamma(y)}{\Gamma(x+y+(p+\nu-2)z)}\right]$$

where $Q_\alpha(x,y)$ is a polynomial in $x$ and $y$ with degree in $y$

$$\sum_{\nu=1}^{p}[(p+\nu-2)z - \alpha_\nu] = \frac{1}{2}p(p-1)z\,.$$

Since $I$ is a linear combination of such terms,

$$\begin{aligned} I &= Q(x,y) \prod_{\nu=1}^{p} \left[\frac{\Gamma(x+(\nu-1)z)\Gamma(y)}{\Gamma(x+y+(p+\nu-2)z)}\right] \\ &= \frac{Q(x,y)}{R(y)} \prod_{\nu=1}^{p} \left[\frac{\Gamma(x+(\nu-1)z)\Gamma(y+(\nu-1)z)}{\Gamma(x+y+(p+\nu-2)z)}\right] \end{aligned}$$

where

$$R(y) = \prod_{\nu=1}^{p} [y(y+1)\cdots(y+(\nu-1)z-1)]\,,$$

and $Q(x,y)$ is a polynomial in $x, y$ of degree at most $\frac{1}{2}p(p-1)z$ in $y$. It follows from $\Delta(u) = \pm\,\Delta(1-u)$ that $I$ is symmetric in $x$ and $y$. Therefore

$$\frac{Q(x,y)}{R(y)} = \frac{Q(y,x)}{R(x)}\,.$$

But the right side of this identity is a polynomial in $y$, and therefore $Q(x,y)$ must be divisible by $R(y)$. Since the degree of $Q(x,y)$ in $y$ is equal to the degree of $R(y)$, the quotient must be independent of $y$. By symmetry, the quotient is also independent of $x$. That is to say

$$I = c_p(z) \prod_{\nu=1}^{p} \left[ \frac{\Gamma(x+(\nu-1)z)\Gamma(y+(\nu-1)z)}{\Gamma(x+y+(p+\nu-2)z)} \right]$$

To determine $c_p(z)$, we take $x = y = 1$. Then

$$J_p = \int_0^1 \cdots \int_0^1 |\Delta(u)|^{2z} \, du_1 \cdots du_p = c_p(z) \prod_{\nu=1}^{p} \left[ \frac{(\Gamma(1+(\nu-1)z))^2}{\Gamma(2+(p+\nu-2)z)} \right] .$$

Now we let $w$ be the largest of the $u_j$ and take for the other $u_j$

$$u_j = w\, v_j \, , \quad 0 \leq v_j \leq 1 \, .$$

Then

$$\begin{aligned} J_p &= p \int_0^1 w^{p-1} \int_0^1 \cdots \int_0^1 |\Delta(u)|^{2z} \, dv_1 \cdots dv_{p-1} \, dw \\ &= p \int_0^1 w^{p-1+zp(p-1)} \int_0^1 \cdots \int_0^1 \left| \prod_{\nu=1}^{p-1} (1-v_\nu)\Delta(v) \right|^{2z} dv_1 \cdots dv_{p-1} \, dw \\ &= \frac{1}{(p-1)z+1} \, I' \end{aligned}$$

where $I'$ is the integral $I$ with $x = 1, y = 2z+1$ and $p-1$ for $p$. That is to say,

$$c_p(z) \prod_{\nu=1}^{p} \left[ \frac{(\Gamma(1+(\nu-1)z))^2}{\Gamma(2+(p+\nu-2)z)} \right] = \frac{c_{p-1}(z)}{(p-1)z+1} \prod_{\nu=1}^{p-1} \left[ \frac{\Gamma(1+(\nu-1)z)\Gamma(1+(\nu+1)z)}{\Gamma(2+(p+\nu-2)z)} \right]$$

This reduces to

$$\frac{c_p(z)}{c_{p-1}(z)} = \frac{\Gamma(2+(p-1)z)\Gamma(1+pz)}{((p-1)z+1)\Gamma(1+(p-1)z)\Gamma(1+z)} = \frac{\Gamma(1+pz)}{\Gamma(1+z)} .$$

Since $c_1(z) = 1$, we have

$$c_p(z) = \prod_{\nu=1}^{p} \left[ \frac{\Gamma(1+\nu z)}{\Gamma(1+z)} \right] ,$$

which completes the proof for integer $z$.

The proof extends to complex $z$ with $\mathrm{Re}\, z > 0$ by a standard argument using Carlson's theorem. Finally, by analytic continuation, it extends to all complex $x, y, z$ for which the integral $I$ is well-defined.

I

$$S_n(x,y,z) = \int_{0<t_1<\cdots<t_n<1}\cdots\int (t_1\cdots t_n)^{x-1}((1-t_1)\cdots(1-t_n))^{y-1}|\Delta(t)|^{2z}\,dt_1\cdots dt_n =$$

$$= \prod_{\nu=1}^{n} \frac{\Gamma(\nu z)}{\Gamma(z)} \frac{\Gamma(x+(\nu-1)z)\,\Gamma(y+(\nu-1)z)}{\Gamma(x+y+(n+\nu-2)z)}$$

$$S_n(x,y,z) = \int_{(F)} |F(0)|^{x-1}|F(1)|^{y-1}|\Delta_F|^{z-\frac{1}{2}}\,dF_0\cdots dF_{n-1}$$

$$F(t) = (t-\theta_1)\cdots(t-\theta_n) = \sum_{i=0}^{n} F_i t^i \qquad \left|\frac{\partial F_j}{\partial \theta_i}\right| = \sqrt{\Delta_F} = \prod_{i<j}|\theta_i-\theta_j|.$$

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Lemma. La $\tau_0 < \theta_1 < \tau_1 < \theta_1 \cdots < \theta_n < \tau_n$

$$F(t) = \prod_{i=1}^{n}(t-\theta_i) = \sum_{i=0}^{n} F_i t^i \; ; \; T(t) = \prod_{i=0}^{n}(t-\tau_i)$$

Da har vi

$$\int_{\mathcal{D}_F} \prod_{i=0}^{n} |F(\tau_i)|^{s_i-1}\,dF_0\cdots dF_{n-1} =$$

$$= \frac{\Gamma(s_0)\cdots\Gamma(s_n)}{\Gamma(s_0+\cdots+s_n)} \prod_{i=0}^{n} |T'(\tau_i)|^{s_i-\frac{1}{2}}.$$

Sett $\frac{F(t)}{T(t)} = \sum_{i=0}^{n} \frac{\rho_i}{t-\tau_i}$ ; da er $\rho_i = \frac{F(\tau_i)}{T'(\tau_i)}$

samt $\rho_i > 0$, $\sum_{i=0}^{n} \rho_i = 1$, og ~~til~~ enhvert slikt

set $\rho_i$ svarer et polynom $F$ i $\mathcal{D}_F$. Skriv

integralet med nye variable $\rho_1 \cdots \rho_n$:

~~m~~ Vi har $\left\|\frac{\partial \rho_i}{\partial F_j}\right\| = \prod_{i=0}^{n}|T'(\tau_i)|^{-\frac{1}{2}} = \left|\frac{1}{\sqrt{\Delta_T}}\right|$

~~M~~ $i\neq 0$ Integralet blir

$$\prod_{i=0}^{n} |T'(\tau_i)|^{s_i - \frac{1}{2}} \int \cdots \int_{\substack{\rho_i > 0 \\ \sum_1^n \rho_i < 1}} \rho_1^{s_1-1} \cdots \rho_n^{s_n-1} (1-\rho_1-\cdots-\rho_n)^{s_0-1} d\rho_1 \cdots d\rho_n$$

hvilket gir vårt lemma.

Nu la: $0 < \vartheta_1 < \theta_1 < \vartheta_2 < \cdots < \theta_{n-1} < \vartheta_n < 1$

Sett

$$F(t) = \prod_{i=1}^{n-1} (t-\theta_i) = \sum_{i=0}^{n-1} F_i t^i, \quad G(t) = \prod_{i=1}^{n} (t-\vartheta_i) = \sum_{i=0}^{n} G_i t^i$$

definerer $\mathcal{D}_{F,G}$, betrakter integralet

$$J = \int_{\mathcal{D}_{F,G}} |G(0)|^{x-1} |G(1)|^{y-1} |R(F,G)|^{z-1} dF_0 \cdots dF_{n-2}\, dG_0 \cdots dG_{n-1}$$

Resultanten $|R| = \prod_{i=1}^{n} |F(\vartheta_i)| = \prod_{i=1}^{n-1} |G(\theta_i)|$

får

$$J = S_n(x,y,z) \frac{(\Gamma(z))^n}{\Gamma(nz)}$$ (integrerer ~~først~~ over $F$; $G$ spiller rollen av $T$ i lemmaet)

og

$$J = S_{n-1}(x+z, y+z, z) \frac{(\Gamma(z))^{n-1} \Gamma(x) \Gamma(y)}{\Gamma(x+y+(n-1)z)}$$ (integrerer over $G$, $t(t-1)F$ spiller rollen av $T$ i lemma.)

d.v.s.

$$S_n(x,y,z) = S_{n-1}(x+z, y+z, z) \frac{\Gamma(nz)\, \Gamma(x)\, \Gamma(y)}{\Gamma(z)\, \Gamma(x+y+(n-1)z)}$$