Introduction 1.

1. Form of large sieve with pseudocharacter

χ prim. character belong to q_1, $\psi_{q_2}(n) = \mu((q_2,n))\,\varphi((q_2,n))$

q_2 squarefree. q_1, q_2 rel prime.

$$\sum_{q_1 q_2 < Q} \frac{q_1}{\varphi(q_1 q_2)} \Big| \sum_{N \le n \le N+x} a_n \chi_{q_1}(n) \psi_{q_2}(n) \Big|^2$$

$$\le (x + Q^2) \sum_{N \le n \le N+x} |a_n|^2 .$$

combine with integral inequality (Gallagher essentially) for use on Dirichlet series. to estimate

for $S_{\chi\psi}(s) = \sum \frac{a_n}{n^s} \chi_{q_1}(n) \psi_{q_2}(n)$

$T \geq 2;\ Q \geq 1$

$$\sum_{q_1 q_2 < Q} \frac{q_1}{\varphi(q_1 q_2)} \int_{-T}^{T} |S_{\chi\psi}(\sigma+it)|^2\, dt$$

$$\le 13 \Big(\sum \frac{|a_n|^2}{n^{2\sigma-1}} + TQ^2 \sum \frac{|a_n|^2}{n^{2\sigma}} \Big)$$

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proof:

$$S_{\chi\psi}(s) = \int x^{-s}\, dA_{\chi\psi}(x)\, dx$$

$$= s \int \frac{A_{\chi\psi}(x)}{x^{s}} \frac{dx}{x}$$

or

$$\frac{e^{\delta s}-1}{s} S_{\chi\psi}(s) = \int \frac{A_{\chi\psi}(e^{\delta}x) - A_{\chi\psi}(x)}{x^s} \frac{dx}{x}$$

By parsevals formula $s = \sigma + it$

$$\int_{-\infty}^{\infty} \left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2 |S_{\chi_4}(\sigma+it)|^2\, dt$$

$$= 2\pi \int_0 \frac{|A_{\chi_4}(e^{\delta}x) - A_{\chi_4}(x)|^2}{x^{2\sigma}} \frac{dx}{x}$$

take $e^{\delta} = 1+\frac{1}{T}$; $|t| \le T$

then

$$\left|\frac{e^{\delta(\sigma+it)}-1}{\sigma+it}\right|^2 > \frac{3}{4}\frac{1}{T}$$

$$\left|e^{\delta}\right|^2 > \frac{1}{2}\frac{1}{T^2}$$

$$\int_{-T}^{T} |S_{\chi_4}(\sigma+it)|^2\, dt < 4\pi T^2 \int \frac{|A_{\chi_4}(x+\frac{x}{T}) - A(x)|^2}{x^{2\sigma}} \frac{dx}{x}$$

thus

$$\sum_{q_1 q_2 < Q} \frac{q_1}{\varphi(q_1 q_2)} \int_{-T}^{T} |S_{\chi_4}(\sigma+it)|^2\, dt$$

$$< 4\pi T^2 \int \frac{\left(\frac{x}{T}+Q^2\right)\left(A_2(x+\frac{x}{T}) - A_2(x)\right)}{x^{2\sigma}} \frac{dx}{x}$$

$$= 4\pi T \int \frac{A_2(x+\frac{x}{T}) - A_2(x)}{x^{2\sigma-1}} \frac{dx}{x}$$

$$+ 4\pi T^2 Q^2 \int \frac{A_2(x+\frac{x}{T}) - A_2(x)}{x^{2\sigma}} \frac{dx}{x}$$

let $A_2(x) = \sum_{n \le x} |a_n|^2$

$$= 4\pi T \frac{(1+\frac{1}{T})^{2\sigma-1}-1}{2\sigma-1} \sum \frac{|a_n|^2}{n^{2\sigma-1}}$$

$$+ 4\pi T^2 Q^2 \frac{(1+\frac{1}{T})^{2\sigma}-1}{2\sigma} \sum \frac{|a_n|^2}{n^{2\sigma}}$$

$$< 13\left(\sum \frac{|a_n|^2}{n^{2\sigma-1}} + TQ^2 \sum \frac{|a_n|^2}{n^{2\sigma}} \right)$$

Auxilliary function.

Look at

$$\sum \frac{\chi(n)\psi_{q_1}(n)}{n^s} \overbrace{\sum_{d|n} \lambda_d}^{a_n}.$$

$$= \sum \frac{\lambda_d\,\chi(d)\psi_{q_1}(d)}{d^s} \sum \frac{\chi(\nu)\frac{\psi_q(d\nu)}{\psi_q(d)}}{\nu^s}.$$

$$= \sum \frac{\lambda_d\,\chi(d)\psi_{q_1}(d)}{d^s} \prod_{p \nmid \frac{q}{(d,q)}} \frac{1}{1-\frac{\chi(p)}{p^s}} \cdot \prod_{p / \frac{q}{(d,q)}} \frac{1-\chi(p)p^{1-s}}{1-\chi(p)p^{-s}}$$

$$\text{or} = L(s,\chi) \cdot \underbrace{\sum_d \frac{\lambda_d\,\chi(d)\psi_{q_1}(d)}{d^s} \prod_{p / \frac{q_2}{(d,q)}} \left(1-\chi(p)p^{1-s}\right)}_{M(s,\,\chi\psi)}$$

assume $|\lambda_d| \le 1$ for all d and $\lambda_d = 0$ for $d > zy$

and $\lambda_d = \mu(d)$ for $d \le z$.

Estimates: ~~for $\sigma \ge \frac{1}{2}$.~~ We have

on line $\sigma = 0$; $M(s, \chi\psi) \le \sum \cancel{(q_2,d)}\, d\left(\frac{q}{d,q}\right)$

$$< \tau(q_2) \sum_{d < zy} (q_2, d) < \underline{\tau^2(q_2)\; zy}$$

also on $\sigma = 0$

$$L(s,\chi) = O\left((|t|q_1)^{\frac{1}{2}} \log(2+|t|q_1)\right)$$

thus for $\sigma = 0$

$$L(s,\chi)M(s,\chi\psi) < O\left((|t|q_1q_2)^{\frac{1}{2}}\, zy \log(2+|t|q_1)\right)$$

Write for $\sigma > \frac{1}{2}$

$$L(s)\,M(s, \chi\psi) = 1 + S + R$$

where

$$S = \sum_{z}^{x} \frac{a_n \chi(n)\psi(n)}{n^s}\left(1 - \frac{n}{x}\right)$$

We have.

$$1 + S = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{w}}{w(w+1)}\, L(s+w, \chi)\, M(s, \chi\psi)\, dw$$

Shift line of integration to $\Re w = -\sigma$

get residue at $z = 0$ is $L(s, \chi)\, M(s, \chi\psi)$

Thus

$$R = -\frac{1}{2\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty} \frac{x^{w}}{w(w+1)}\, L(s+w, \chi)\, M(s+w, \chi\psi)\, dw$$

or

for $q_1 q_2 < Q$.

$$\underline{|R| \leq c \cdot x^{-\sigma} z y \sqrt{TQ}\, \log TQ}$$

Estimation of

$$\sum N(\alpha, T, \chi) \cdot \sum_{q_2 < \frac{Q}{q}} \frac{1}{\varphi(q_2)}$$

$> \ldots$

Formula:

Littlewoods.

$$\sum_{|\gamma|<T} \beta-\alpha = \int_{-T}^{T} \log|f(\alpha+it)|\,dt + \int \arg f \ldots$$

can avoid argument by using another formula which gives inequality

$$\sum_{\substack{|\gamma|<T \\ \beta>\alpha}} \beta-\alpha \le \int_{-2T}^{2T} \log|f(\alpha+it)|\,dt$$

$$+ \int_{\alpha}^{\infty} \log|f(\sigma \pm 2iT)| \frac{4T}{\pi} \sinh \frac{\sigma-\alpha}{4T}\pi \, d\sigma$$

Shall here look only at first term.

Let α be $\geq \frac{1}{2}$ and chose X so large that

$$R < A X^{-\frac{\sigma}{2Y}} \sqrt{TQ} \log TQ < \frac{1}{4}$$

for $\sigma \geq \alpha$; $|t| < 2T$.

$$\log|1+S+R| + \log\frac{1}{|1+R|} + \log\left|1-\frac{S}{1+R}\right|$$

$$= \log\left|1-\frac{S^2}{(1+R)^2}\right| < \frac{|S|^2}{|1+R|^2} < 2|S|^2$$

choosing $\alpha = \sigma - \frac{1}{\log TQ}$

we have

$$\frac{1}{\log TQ} \sum{}' N(\sigma, T, \chi) \log \frac{Q}{q_1} <$$

$$\sum_{q_1 q_2 < Q}{}' \, 2 \int_{-2T}^{2T} |S_{\chi,\psi}(\alpha + it)|^2 \, dt \quad + \cdots$$

$$< 26 \left(\sum_{z}^{x} \frac{|a_n|^2}{n^{2\alpha - 1}} + TQ^2 \sum_{z}^{x} \frac{|a_n|^2}{n^{2\alpha}} \right)$$

Have $\sum_{z}^{x} \frac{|a_n|^2}{n} = O\left(\frac{\log x}{\log y} \right)$

with our choice of λ_d. $= \underline{O(1)}$

$$26 \left(x^{2-2\alpha} \sum_{z}^{x} \frac{|a_n|^2}{n} + TQ^2 z^{1-2\alpha} \sum_{z}^{x} \frac{|a_n|^2}{n} \right)$$

$$= O\left(x^{2-2\alpha} + TQ^2 z^{1-2\alpha} \right)$$

for $\frac{\log x}{\log y} = O(1)$.

To get result take $Q_1 = Q(TQ)^{\epsilon_1}$

$z = TQ_1^2$, $y = (TQ)^{\epsilon_1}$; $x = (T^3 Q^5)^{\frac{1+\epsilon}{2}}$

for ϵ_1 small enough this gives result for σ near enough to 1.