3. We may ask what is the situation with the large sieve, if we instead of an interval use the setting of the weighted sets W.

Around 1951, while discussing Renyi's papers on the arge sieve, S. Gál and I developed a simpler proof of his results via the Bessel inequality route.

Essentially our argument gave:

Suppose that for $q \leqq Q^2$

$$N_{l,q}(W) = \frac{N(W)}{q} + R_{l,q}, \tag{3.1}$$

with

$$|R_{l,q}| \leqq 1. \tag{3.1'}$$

Write

$$A_{l,q} = \sum_{n \equiv l(q)} a_n w_n,$$

then

$$\sum_{q \leqq Q} q \sum_{e=0}^{q-1} \left| \sum_{d/q} \frac{\mu(d)}{d} A_{l,\frac{q}{d}} \right|^2 \leqq (N(W) + Q^3) \sum_n |a_n|^2 w_n. \tag{3.2}$$

It is doubtful whether Q^3 can be replaced by something of lower order in (3.2) without restricting W further than the requirements (3.1) and (3.1′) do.*

The weighted set W satisfying (2.13) and (2.13′) with $z = Q^2$, leads to very simple proofs of the various versions of the large sieve. The system of exponentials $e^{2\pi i \frac{l}{q} n}$ with $q \leqq Q$ form an orthogonal system on W; similarly do the primitive characters $\chi(u)$ belonging to moduls $q \leqq Q$. Thus the character version can be proved directly without first treating the case of the exponentials. We get

$$\sum_{\substack{q \leqq Q \\ (l,q)=1}} \left| \sum_n a_n e^{2\pi i \frac{l}{q} n} \right|^2 \leqq \hat{F}(o) \sum_n \frac{|a_n|^2}{w_n} \tag{3.3}$$

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* At the time we were interested in the case when a_n takes only the values 0 and 1. This did not enter in the argument however. Unfortunately, though we, besides using the characteristic functions of residue classes (which led to the form (3.2)), also experimented with the functions $e^{2\pi i \frac{l}{q} n}$ and $\chi(n)$, we missed the sharper form for the interval by a slip up in our estimations when using the exponentials. (The χ led naturally to Q^3 again.)

Only a year or so after the appearance of Bombieri's paper on the large sieve (Mathematika, 12 (1965), pp. 201–225) did I return to this idea and get first $c\,Q^2 \log Q$, then $c\,Q^2$, for the interval, using essentially the original Bessel inequality approach.

provided

$$\sum_n \frac{|a_n|^2}{w_n} < \infty .$$

The inequality (3.3) is sharp. That is, equality can occur without the a_n being all zero. These a_n however do not vanish outside an interval I_x. The usual form for I_x is obtained by letting the $a_n = 0$ for n not in I_x, observing that $\hat{F}(o)$ can be replaced by $x + c\,Q^2$, and that $w_n = F(n) \geqq 1$ for $n \in I_x$.

The system of the primitive characters χ_q with $q \leqq Q$ can be supplemented by certain pseudocharacters (multiplicative functions depending on the residue class of n with respect to some modul). Define for q square free:

$$\psi_q(n) = \mu(d)\,\varphi(d) \tag{3.4}$$

where $d = (q, n)$ and φ is Euler's function. The set of pseudocharacters

$$\chi_{q_1}(n)\,\psi_{q_2}(n) \tag{3.5}$$

where $(q_1, q_2) = 1$, $q_1 q_2 \leqq Q$, q_2 square free and χ_{q_1} a primitive character mod q_1, still forms an orthogonal set. For the interval I_x we get

$$\sum_{q_1 q_2 \leqq Q} \frac{q_1}{\varphi(q_1 q_2)} \left| \sum_{n \in I_x} a_n\,\chi_{q_1}(n)\,\psi_{q_2}(n) \right|^2 \leqq (x + c\,Q^2) \sum_{n \in I_x} |a_n|^2 . \tag{3.6}$$

The system (3.5) can still be enlarged keeping the orthogonality but they are then no longer all pseudocharacters.

The introduction of the $\psi_q(n)$ in (3.6) is useful for some applications. Thus it is possible to show that for $T \geqq 1$, $Q \geqq 1$, $\sigma > \frac{1}{2}$

$$\sum_{q \leqq Q}' N_\chi(\sigma, T) \leqq c(\varepsilon)\,(T^{3+\varepsilon} Q^{5+\varepsilon})^{1-\sigma} . \tag{3.7}$$

Here the expression on the left hand side counts the number of zeros $\beta + i\gamma$ of all $L(s, \chi)$ with primitive characters belonging to $q \leqq Q$, in the region $\beta \geqq \sigma$, $|\gamma| \leqq T$. ε is an arbitrary fixed positive number and $c(\varepsilon)$ depends on ε only.

A theorem of this type, but with an unspecified large constant instead of 3 and 5 in (3.7) has earlier been obtained by P. X. Gallagher.*

* P. X. Gallagher: A large sieve density estimate near $\sigma = 1$, Inventiones Math., 11 (1970), pp. 329–339.