case $\lambda = \frac{1}{2}$.

Dirichlet's L-functions, χ primitive character mod q (incl. case $q=1$ and $\chi(n) \equiv 1$).

$$L(s,\chi) = \sum_n \frac{\chi(n)}{n^s},$$

integral function for $q \neq 1$, $(s-1)L(s)$ integral for $q=1$.

Write $a = \dfrac{1-\chi(-1)}{2}$, and

$$\phi(s,\chi) = \varepsilon_\chi q^{\frac{s}{2}} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi),$$

then $\phi(s,\chi) = \overline{\phi(1-\bar{s},\chi)}$.

For simplicity we look at the case χ even $(a=0)$, the odd case can be handled in the same way.

Let χ_j, $j=1,2,\cdots,m$. be m distinct primitive even characters, the c_j real and $\neq 0$, and form

$$F(s) = \sum_{j=1}^{m} c_j \varepsilon_j q_j^{\frac{s}{2}} L(s,\chi_j)$$

(alternatively we might consider
$$F^*(s) = \sum_{j=1}^{m} c_j \varepsilon_j \left(1+q_j^{s-\frac{1}{2}}\right) L(s,\chi_j)\ ).$$

Then $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)F(s)$ or $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)F^*(s)$

are real for $s = \frac{1}{2} + it$, t real.
We have in this case

$$N(T, F) = \frac{T}{2\pi}(\log T + B) + O(\log T).$$

For the single $L(s, \chi)$ it has been proved that a positive proportion of the zeros have real part $\frac{1}{2}$, more precisely:

$$N_0(T, L) > c\, T \log T \text{ for } T > Aq^2,$$

where c and A are absolute constants.
For the general linear combination some results are implied in the literature.
From Hardy-Littlewoods work follows

$$N_0(T, F) > cT \text{ for } T > T_0(F).$$

Recently A.A. Karatsuba has considered combinations

$$F(s) = \varepsilon_\chi L(s, \chi) + \bar{\varepsilon}_\chi L(s, \bar{\chi}),$$

where χ is a complex character and obtained the result (1994)

$$N_0(T, F) > T(\log T)^{\varepsilon} e^{-c\sqrt{\log\log T}}, \quad T > T_0;$$

with some positive constant c (for more general combinations he has a much weaker and more complicated result.

I shall sketch a proof that for the general combination $F(s)$ we have

(I) $N_0(T, F) > c(m)\, T \log T$ for $T > T_0(F)$, where $c(m)$ depends on m only. Also:

(II) If $\omega(t) \to \infty$ as $t \to \infty$ then $F(\frac{1}{2}+it)$ has a zero in the interval $(t, t+\frac{\omega(t)}{\log t})$ for almost all t.

First let us see how these results are proved for the single L-function.

Let $s = \frac{1}{2}+it$, $t > 0$, $\vartheta(t) = \arg \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ and write

$$X(t, \chi) = \varepsilon_\chi q^{\frac{it}{2}} e^{i\vartheta(t)} L(s, \chi).$$

"Approximate functional equation"

$$X(t,\chi) = \varepsilon_\chi q^{\frac{it}{2}} e^{i\vartheta(t)} \sum_{m < \sqrt{\frac{tq}{2\pi}}} \chi(m) m^{-s} +$$

$$+ \bar{\varepsilon}_\chi q^{-\frac{it}{2}} e^{-i\vartheta(t)} \sum_{n < \sqrt{\frac{tq}{2\pi}}} \overline{\chi(n)}\, n^{s-1} + O\left(\left(\frac{q}{t}\right)^{\frac{1}{4}}\right).$$

Write

$$\{\zeta(s)\}^{-\frac{1}{2}} = \sum_m \frac{\alpha_m}{m^s}\ ; \ \alpha_1 = 1;\ (L(s,\chi))^{-\frac{1}{2}} = \sum_m \chi(m) \frac{\alpha_m}{m^s}.$$

$$\beta_m = \alpha_m \left(1 - \frac{\log m}{\log \xi}\right)$$

and for $T \leq t \leq 2T$; $\xi = T^{\frac{1}{10}}$; $T > 16q^3$ write

$$\eta(s,\chi) = \sum_{n<\xi} \chi(n) \frac{\alpha_n}{n^s}\left(1 - \frac{\log n}{\log \xi}\right).$$

(we shall often write $\eta(t,\chi)$ for $\eta(\frac{1}{2}+it,\chi)$)

For $\frac{1}{\log T} \leq H \leq \frac{\log\log T}{\log T}$, consider the three expressions:

$$I_\chi(t,H) = \int_t^{t+H} X(u,\chi)\,|\eta(u,\chi)|^2\,du,$$

$$M_\chi(t,H) = \int_t^{t+H} L(\tfrac{1}{2}+iu,\chi)\,\eta^2(u,\chi)\,du - H,$$

and

$$J_\chi(t,H) = \int_t^{t+H} |X(u,\chi)\,\eta^2(u,\chi)|\,du.$$

If $J_\chi(t,H) > |I_\chi(t,H)|$,

$X(t,\chi)$ changes sign in $(t, t+H)$ and so has at least one zero there.

Also $J_\chi(t,H) \geq H - |M_\chi(t,H)|$,

so if

$$|M_\chi(t,H)| + |I_\chi(t,H)| < H,$$

there is a zero in $(t, t+H)$.

Using the approximate functional equation for $L(s,\chi)$ or $X(t,\chi)$, we can show

(1) $$\int_T^{2T} |I_\chi(t,H)|^2 dt = O\left(T \frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right),$$

(2) $$\int_T^{2T} |M_\chi(t,H)|^2 dt = O\left(T \frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right),$$

and (as we shall use much later)

(3) $$\int_T^{2T} |X(t,\chi)\eta^2(t,\chi)|^2 dt = O(T).$$

The constants implied by the O are absolute.

We see now that $|I_\chi(t,H)| \leq \frac{H}{3}$ and $|M_\chi(t,H)| \leq \frac{H}{3}$ holds except in a subset of $(T, 2T)$ of measure $O\left(\frac{T}{\sqrt{H}\log T}\right)$; Choosing now $H = \frac{\lambda}{\log T}$ with λ a large enough constant, we get statement (I) for $N_0(T, L_\chi)$, and by choosing $\lambda = (\omega(T))^{\frac{1}{2}}$ we get statement (II).

7*.

Fourier analyse using

$$\theta_\chi(x) = \sum_{n=-\infty}^{\infty} \chi(n) e^{-\pi \frac{n^2}{q} x}$$

$$\varepsilon_\chi x^{\frac{1}{4}} \theta_\chi(x) = \bar{\varepsilon}_\chi x^{-\frac{1}{4}} \theta_{\bar\chi}\left(\frac{1}{x}\right).$$

$$x = i \exp(i\delta - 2y) \qquad ; \delta = \frac{1}{T}$$

$$f_\chi(y) = -\varepsilon_\chi x^{\frac{1}{4}} \sum_{n=1}^{\infty} \sum_{\mu,\nu} \frac{\beta_\mu \beta_\nu}{\nu} \exp\left(-\frac{\pi x n^2 \mu^2}{q \nu^2}\right) \chi(\mu\nu) \bar\chi(\mu)$$

$$\int_T^{2T} | I_\chi(t,H)|^2 dt \le c \int_0^\infty |f_\chi(y)|^2 \left(\frac{\sin \frac{yH}{2}}{y}\right)^2 dy$$

$$\le cH^2 \int_0^{\frac{1}{H}} |f_\chi(y)|^2 dy + c \int_{\frac{1}{H}}^\infty |f_\chi(y)|^2 \frac{dy}{y^2}$$

$$= O\left(T \frac{H}{\log \xi}\right) = O\left(T \frac{H}{\log T}\right).$$

To adapt this idea to the linear combination $F(s)$, we need some results about the value distribution of $\log|L(\frac{1}{2}+it,\chi)|$ or $\log|X(t,\chi)|$.

For $T > 16q^3$, k a positive integer and $T^{\frac{1}{2k}} \le x \le T^{\frac{1}{k}}$, we can show

$$\int_T^{2T} \left| \log|X(t,\chi)| - R\sum_{p<x} \chi(p)\, p^{-\frac{1}{2}-it} \right|^{2k} dt =$$

$$= O\left(T\, k^{4k} e^{Ak}\right),$$

The constants implied by the O are again absolute.

From this we can prove that

$$\frac{\log|X(t,\chi)|}{\sqrt{\pi \log\log t}}$$

has a normal gaussian distribution. More precisely: let $\chi_{a,b}$ denote the characteristic function of the interval (a,b), then

$$\int_T^{2T} \chi_{a,b}\left(\frac{\log|X(t,\chi)|}{\sqrt{\pi\log\log T}}\right) dt = T\int_a^b e^{-\pi u^2}\,du + O\left(T\,\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right).$$

Also: for two distinct characters χ and χ' similar results hold for the difference

$$\log|X(t,\chi)| - \log|X(t,\chi')|,$$

only here we must divide by $\sqrt{2\pi \log\log t}$ to get the normal gaussian distribution. Thus if $0 < \delta < \frac{1}{2}$, the set in $(T, 2T)$ where

$$\big|\log|X(t,\chi)| - \log|X(t,\chi')|\big| \leq (\log\log T)^{\delta},$$

has measure

$$O\left(T\,(\log\log T)^{-\frac{1}{2}+\delta}\right).$$

Thus most of the time one $X(t,\chi_j)$ dominates all the other decisively. This dominance is somewhat persistant over stretches long compared to $\frac{1}{\log T}$. Define

$$\Delta_\chi(t,H) = \frac{1}{H}\int_t^{t+H} \log|X(u,\chi)|\,du,$$

For $0 \leq h \leq H$, we can show for any positive integer k that:

$$\int_T^{2T} \left(\Delta_\chi(t,H) - \log|X(t+h,\chi)| \right)^{2k} dt =$$

omit

$$= O\left(T e^{Ak} \left(k^k \log^k (H \log T) + k^{4k} \right) \right).$$

Integrating over h we get

$$\int_T^{2T} \int_0^H \left(\Delta_\chi(t,H) - \log|X(t+h,\chi)| \right)^{2k} dh\, dt =$$

$$= O\left(T H e^{Ak} \left(k^k (\log\log\log T)^k + k^{4k} \right) \right).$$

If we denote by $W(t,\chi)$ the measure of the subset of h for which

$$\left| \Delta_\chi(t,H) - \log|X(t+h,\chi)| \right| > (\log\log T)^{\frac{\delta}{2}},$$

we find choosing k so large that $k\delta > 2N+1$, that

$$W(t,\chi) \leq \frac{H}{(\log\log T)^N},$$

except for a subset of t in $(T, 2T)$ of measure $O\left(\frac{T}{(\log\log T)^N} \right)$.

We also can show that for $\chi \neq \chi'$

$$\left| \Delta_\chi(t,H) - \Delta_{\chi'}(t,H) \right| > (\log\log T)^\delta$$

11

except for a subset of $(T, 2T)$ of measure $O(T(\log\log T)^{-\frac{1}{2}+\delta})$.

For $x_1 \dots x_n$ we now define, $j \neq k$, $S_{j,k}$ as the subset of $(T, 2T)$ where

$$|\Delta_{x_j}(t, H) - \Delta_{x_k}(t, H)| \leq (\log\log T)^{\delta},$$

we have $m(S_{j,k}) = O(T(\log\log T)^{-\frac{1}{2}+\delta})$

If we exclude all of these subsets from $(T, 2T)$, the rest consists of n sets S_j such that in S_j for $k \neq j$

$$\Delta_{x_j}(t, H) > \Delta_{x_k}(t, H) + (\log\log T)^{\delta}.$$

If we also from each S_j exclude all t for which for any k

$$w(t, x_k) > \frac{H}{(\log\log T)^{N}}, \text{ we get}$$

that $(T, 2T)$ except for a subset of measure $O(T(\log\log T)^{-\frac{1}{2}+\delta})$ is divided into n subsets S_j^* such that $\sum_m m(S_j^*) = T - O(T(\log T)^{-\frac{1}{2}+\delta})$

and for each t in S_j^* we have for $k \neq j$ that

$$\log|X(t+h, \chi_j)| - \log|X(t+h, \chi_k)| > (\log\log T)^{\delta} - 2(\log\log T)^{\frac{\delta}{2}} > \frac{1}{2}(\log\log T)^{\delta},$$

except for a set ~~of~~ of h in $(0, H)$ of measure $O\left(\frac{H}{(\log\log T)^N}\right)$.

From

$$\int_T^{2T} |X(t,\chi)\,\eta^2(t,\chi)|^2\,dt = O(T)$$

we see that

$$\int_t^{t+H} |X(u,\chi_j)\,\eta^2(u,\chi_j)|^2\,du < H\log\log T$$

except for a subset of t of measure $O\left(\frac{T}{\log\log T}\right)$, we exclude also these t from the S_j^* (without renaming them).

Now look at

$$I^*_{\chi_j}(t,H),\ M^*_{\chi_j}(t,H) \text{ and } J^*_{\chi_j}(t,H)$$

which are for t in S_j^* the integrals I_{χ_j}, M_{χ_j} and J_{χ_j} but with the bad subset removed. They differ from these at most by

$$O\left(\sqrt{\frac{H}{(\log\log T)^N}}\cdot\sqrt{H\log\log T}\right)$$

$$= O\left(\frac{H}{\log\log T}\right),$$

taking $N=3$. We see now that we get a signchange of $X(u,\chi_j)$ in $(t,t+H)$ for t in S_j^* and

$J^*_{\chi_j}(t,H) > |I^*_{\chi_j}(t,H)|$ which is equivalent to

$$H > |I_{\chi_j}(t,H)| + |M_{\chi_j}(t,H)| + O\left(\frac{H}{\log\log T}\right)$$

For T large enough this holds outside a ~~subset~~ measure $O\left(\frac{T}{\sqrt{H\log T}}\right)$ and so in most of S_j^* if $H = \frac{\lambda m^2}{\log T}$ with λ a large enough constant. Produces more than $\frac{c}{m^2}T\log T$ sign changes of

$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) F(s)$; $s = \frac{1}{2} + it$ in S_j^*.
adding up over j we get more than $\frac{c}{m^{*}} T \log T$ or

$$N_0(T, F) > \frac{c}{m^{*}} T \log T \quad \text{for } T > T_0(F).$$

can be improved to

$$N_0(T, F) > \frac{c(\alpha)}{m^{\alpha}} T \log T \quad \text{for any } \alpha > 1.$$

list (Hafner)

Case $n = 1$. Some cases of such $\mathcal{L}(s)$ have been handled and

$$N_0(T, \mathcal{L}) > c T \log T \quad \text{for } T > T_0(\mathcal{L})$$

proved. Essentially what is required is that one can estimate expressions like

$$\int_T^{2T} |\mathcal{L}(\tfrac{1}{2}+it) \, P(\tfrac{1}{2}+it)|^2 \, dt$$

where P is a dirichlet polynomial $P(s) = \sum_{n \leq \xi} \frac{\lambda_n}{n^s}$, where ξ is like some small power of T. In these cases the $I_\chi(t, H)$ have been handled but not the $M_\chi(t, H)$ which were avoided using a simpler device which does not work for the linear combination. By slightly modifying the way one defines the analog of $\eta(s)$ for $\mathcal{L}(s)$, we put

$$\left(\mathcal{L}(s)\right)^{-\frac{1}{2}} = \sum_n \frac{\alpha_n}{n^s} \; ; \alpha_1 = 1$$

and

$$\eta(s) = \sum_{n < \xi^{\frac{1}{2}}} \frac{\alpha_n}{n^s} + \sum_{\xi^{\frac{1}{2}} \le n < \xi} \frac{\alpha_n}{n^s}\left(2 - 2\frac{\log n}{\log \xi}\right).$$

should work.

This new $\eta(s)$ works equally well for $I.(t, H)$, and much better for

$$M(t, H) = \int_t^{t+H} \mathcal{L}(\tfrac{1}{2}+iu)\,\eta^2(\tfrac{1}{2}+iu)\,du - H,$$

We can actually get

$$\int_T^{2T} |M(t,H)|^2 dt = O\left(\frac{T}{\log^2 T}\right)$$

uniformly for $0 < H < T$. Everything else works as before.

Briefly sketched: We write

$$|M(t,H)| = \Big| \int_t^{t+H} (\mathcal{L}(\tfrac12+iu)\eta^2(\tfrac12+iu)-1)\,du \Big| \leq$$

$$\leq \Big| \int_{\frac12+it}^{2+it} (\mathcal{L}(s)\eta^2(s)-1)\,ds \Big| + \big| (t \to t+H) \big| + O(\xi^{-\frac13}).$$

also

$$\Big| \int_{\frac12+it}^{2+it} (\mathcal{L}(s)\eta^2(s)-1)\,ds \Big|^2 \leq$$

$$\int_{\frac12}^{2} \xi^{\frac14(\frac12-\sigma)}\,d\sigma \cdot \int_{\frac12}^{2} \xi^{\frac14(\sigma-\frac12)} |\mathcal{L}(\sigma+it)\eta^2(\sigma+it)-1|^2\,d\sigma.$$

If we write

$$J_\sigma = \int_T^{2T} |\mathcal{L}(\sigma+it)\eta^2(\sigma+it)-1|^2\,dt,$$

we have

$$J_{\frac12} = O(T) \quad \text{and} \quad J_2 = O(T\xi^{-1}).$$

A convexity argument now gives

$$J_\sigma = O\left(T\,\xi^{-\frac12(\sigma-\frac12)}\right),$$

the estimation

$$\int_T^{2T} |M(t,H)|^2\,dt = O\left(\frac{T}{\log^2 \xi}\right) = O\left(\frac{T}{\log^2 T}\right)$$

now follows easily.

The drawback is that with this $\eta(s)$, we have to go through the estimation of

$$\int_T^{2T} |I(t,H)|^2 dt$$

anew, since it does not follow from our earlier result. The same approach can however be adapted to our old $\eta(s)$ defined as

$$\eta(s) = \sum_{n<\xi} \frac{\alpha_n}{n^s}\left(1 - \frac{\log n}{\log \xi}\right),$$

by observing that the dirichlet series for $\mathcal{L}(s)\eta^2(s)$ is identical with that of

$$\left(1 - \frac{1}{2\log\xi}\frac{\mathcal{L}'}{\mathcal{L}}(s)\right)^2 \quad \text{for } n \le \xi.$$

Subtracting from the expansion of $\mathcal{L}(s)\eta^2(s) - 1$ the terms with $1 < n \le \xi$, and handling this part separately on the line $\sigma = \frac{1}{2}$, the remainder with the method just outlined we get easily for $H > \frac{2}{\log T}$

$$\int_T^{2T} |M(t,H)|^2 \, dt = O\left(T \frac{\log(H \log T)}{\log^2 T}\right)$$

a result only slightly worse (and much better than we actually need).

When we can prove

$$\int_T^{2T} [I(t,H)]^2 \, dt \le a \frac{T \cancel{\log} F}{\log T}, \quad T > T_0$$

uniformly for all L_j constant a get again.

$$N_0(F,T) > \frac{c}{m \, a} T \log T \quad \text{for } T > T_0(F)$$

with absolute c.