case $\Lambda = \frac{1}{2}$.

Dirichlet $L$-functions, $X$ primitive character mod $q$ (incl. case $q=1$ and $X(m) \equiv 1$).

$$L(s, X) = \sum_{m} \frac{X(m)}{m^s},$$

integral function for $q \neq 1$, $(s-1) L(s)$ integral for $q=1$.

Write $a = \frac{1-X(-1)}{2}$, and

$$\phi(s, X) = \sum_{m} \frac{a}{m} \Gamma\left(\frac{s+\frac{a}{2}}{2}\right) L\left(s, \frac{a}{2}\right),$$

then $\phi(s, X) = \phi(1-a, X)$.

For simplicity we look at the case $X$ even ($a = 0$), the odd case can be handled in the same way.

Let $X_j$, $j = 1, 2, \ldots, m$, be $m$ distinct primitive even characters, the $c_j$ real and $\neq 0$, and form

$$f(s) = \sum_{j=1}^{m} c_j X_j q_j^{s} L(s, X_j).$$

Alternatively, we might consider

$$f^*(s) = \sum_{j=1}^{m} c_j X_j \left(1 + q_j^{-\frac{a}{2}}\right) L\left(s, \frac{a}{2}\right).$$

Then $\pi^{-\frac{a}{2}} \Gamma\left(\frac{a}{2}\right) f(s) \sim \pi^{-\frac{a}{2}} \Gamma\left(\frac{a}{2}\right) f^*(s)$.
are real for \( s = \frac{1}{2} + it \), \( t \) real.

We have in this case

\[
\mathcal{N}(T, F) = \frac{1}{2\pi} \left( \log T + B \right) + O(\log T).
\]

For the single \( L(0, \chi) \) it has been proved that a positive proportion of the zeros have real part \( \frac{1}{2} \); more precisely:

\[
\mathcal{N}_0(T, \chi) > c T \log T \text{ for } T > Aq^2,
\]

where \( c \) and \( A \) are absolute constants.

For the general linear combination some results are implied in the literature.

From Hardy-Littlewood we have follows

\[
\mathcal{N}_0(T, F) > c T \text{ for } T > T_0(F).
\]

Recently A.A. Karatsuba has considered combinations

\[
F(s) = \sum_{\chi} L(0, \chi) + \sum_{\chi} L(s, \chi^*),
\]

where \( \chi \) is a complex character and obtained the result (1994)

\[
\mathcal{N}_0(T, F) > T (\log T)^{1/2} e^{-c \sqrt{\log T}}, \quad T > T_0;
\]

with some positive constant \( c \) (for more general combinations he has a much weaker and more complicated result).
I shall sketch a proof that for the general combination \( F(s) \) we have

\((1)\) \( N_0(T, F) > c(M) T \log T \) for \( T > T_0(F) \),
where \( c(M) \) depends on \( M \) only. Also:

\((2)\) \( f(t) \to \infty \) as \( t \to \infty \) then
\( F(\frac{1}{2} + it) \) has a zero in the interval
\( (t, t + \frac{\omega(t)}{2 \log t}) \) for almost all \( t \).

First let us recall how these results are proved for the single \( L \)-function.

Let \( s = \frac{1}{2} + it, \ t > 0, \ \omega(t) = \frac{1}{2} \pi \frac{\log \pi - \frac{2}{3} \log a}{\log t}, \ \) and write

\[
X(t, \chi) = \sum_{n < \sqrt{t}} \frac{\chi(n) \log n}{n^{1/2}} + O\left( \frac{\log t}{t} \right)
\]

**Approximate functional equation**

\[
X(t, \chi) = \sum_{n < \sqrt{t}} \frac{\chi(n) \log n}{n^{1/2}} + \sum_{n < \sqrt{t}} \frac{\chi(n) \log n}{n^{1/2}} + O\left( \frac{\log t}{t} \right)
\]

Write

\[
\frac{1}{\zeta(s)} = \sum_m \frac{x_m}{m^s}; \ \chi_i = 1; (L(s, \chi_i))^{-\frac{s}{2}} \sum_{m < \sqrt{t}} \frac{\chi(m) \log m}{m^{1/2}}
\]

\[
\beta_m = x_m \left( 1 - \frac{\log m}{\log \frac{1}{\sqrt{t}}} \right)
\]
and for \( T \leq t \leq 2T; \quad \xi = T^{\frac{1}{10}}; \quad T > 16q^{3} \)
write
\[
\gamma(t, x) = \sum_{m \leq q} \mu(m) \frac{x}{m^{2}} (1 - \frac{\log m}{\log \xi}).
\]
(we shall often write \( \gamma(t, x) \) for \( \gamma(t + it, x) \))
for \( \frac{1}{\log T} \leq h \leq \frac{\log \log T}{\log 3} \), consider
the three expressions:
\[
\begin{align*}
I_{x}(t, h) &= \int_{t}^{t+\frac{1}{\log T}} x(u, x) |\gamma(u, x)|^{2} du, \\
M_{x}(t, h) &= \int_{t}^{t+\frac{1}{\log T}} L\left(\frac{1}{2} + it, x\right) \gamma^{2}(u, x) du - h, \\
J_{x}(t, h) &= \int_{t}^{t+\frac{1}{\log T}} x(u, x) \gamma^{2}(u, x) du.
\end{align*}
\]
Since \( J_{x}(t, h) > 1 I_{x}(t, h) \),
\( x(t, x) \) changes sign in \((t, t + h)\) and
so has at least one zero here.
Also \( J_{x}(t, h) \geq h - 1 M_{x}(t, h) \),
so if
\[ |M_{x}(t, h)| + |I_{x}(t, h)| < h, \]
then is a zero in \((t, t + h)\).
Using the approximate functional equation for \( L(s, x) \) or \( X(t, x) \), we can show

\[
\begin{align*}
(1) \quad & \int_{T}^{2T} \left| I_X(t, H) \right|^2 dt = O(T \frac{H^{\frac{3}{2}}}{\sqrt{\log T}}), \\
(2) \quad & \int_{T}^{2T} \left| M_X(t, H) \right|^2 dt = O(T \frac{H^{\frac{3}{2}}}{\sqrt{\log T}}), \\
\end{align*}
\]

and (as we shall use much later)

\[
\begin{align*}
(3) \quad & \int_{T}^{2T} \left| X(t, x) \right|^2 dt = O(T).
\end{align*}
\]

The constants implied by the \( O \) are absolute.

We see now that \( \left| I_X(t, H) \right| \leq \frac{H}{3} \) and \( \left| M_X(t, H) \right| \leq \frac{H}{3} \) holds except in a subset of \((T, 2T)\) of measure \( O\left( \frac{T}{\sqrt{H \log T}} \right) \); Choosing now \( H = \frac{T}{\log T} \) with \( \lambda \) a large enough constant we get statement (I) for \( N_0(T, L_X) \), and by choosing \( \lambda = (\omega(T)) \) we get statement (II).
Formal analysis using

\[ \Phi(x) = \sum_{n=1}^{\infty} \varphi(n) \frac{\exp\left(-\frac{n^2}{4}\right)}{\sqrt{x}} \]

\[ \sum_{k=1}^{\infty} \frac{1}{k^x} \Phi(x) = \sum_{k=1}^{\infty} \frac{1}{k^x} \Phi\left(\frac{1}{x}\right) \]

\[ x = \exp\left(18 - 20y\right) \quad ; \quad \delta = \frac{7}{4} \]

\[ f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \beta^2 / \nu^2 \right) \exp\left(-\frac{x m^2 / \nu^2}{2} \right) \chi(m) \chi(n) \]

\[ \int_{t}^{T} \left| I_{\chi}(t,H) \right|^2 \, dt \leq C \int_{0}^{\infty} \left( f(x,y) \right)^2 \left( \frac{\ln x \, y}{y} \right) \, dy \]

\[ \leq C \int_{0}^{\infty} \left( f(x,y) \right)^2 \, dy + C \int_{0}^{H} \frac{1}{y} \left( f(x,y) \right)^2 \, dy \]

\[ = O \left( T \frac{H}{\ln x} \right) = O \left( T \frac{H}{\log T} \right) \]
To adapt this idea to the linear combination $F(x)$, we need some results about the value distribution of 
$\log |L(\frac{1}{2}+it, \chi)|$ or $\log |X(t, \chi)|$.

For $T > \log \log T$, $k$ a positive integer and $T^{\frac{1}{2k}} \leq x \leq T$, we can show

$$\int_0^T \left| \log |X(t, \chi)| - R \sum_{p \leq x} \chi(p) p^{-\frac{1}{2}} - it \right|^2 dt =$$

$$= O \left( T^{1+\frac{1}{4k}} e^{A_1 T} \right),$$

The constants implied by the $O$ are again absolute.

From this we can prove that

$$\frac{\log |X(t, \chi)|}{\sqrt{\frac{\log \log t}{T}}}$$

has a normal Gaussian distribution.

More precisely: let $\mu(a, b)$ denote the characteristic function of the interval $(a, b)$, then

$$\int_a^b \frac{\log |X(t, \chi)|}{\sqrt{\frac{\log \log t}{T}}} dt = T \mu \left( -\frac{\log \log t}{T} \right) +$$

$$\int_a^b \left( \frac{\log |X(t, \chi)|}{\sqrt{\frac{\log \log t}{T}}} \right) dt = T \mu \left( -\frac{\log \log t}{T} \right) + O \left( \frac{T}{\sqrt{\log \log T}} \right),$$

$$\int_a^b \left( \frac{\log |X(t, \chi)|}{\sqrt{\frac{\log \log t}{T}}} \right) dt = T \mu \left( -\frac{\log \log t}{T} \right) + O \left( \frac{T}{\sqrt{\log \log T}} \right).$$
Also, for two distinct characters X and X', similar results hold for the difference
\[ \log |X(t, X)| - \log |X(t, X')|, \]
only here we cannot divide by \( \sqrt{2\pi \log \log t} \) to get the normal gaussian distribution. Thus if
\[ 0 < \delta < \frac{1}{2}, \]
the set \( \text{im}(T, 2T) \) where
\[ \log |X(t, X)| - \log |X(t, X')| \leq (\log \log T), \]
has measure
\[ O\left( T \left( \log \log T \right)^{-\frac{1}{2} + \delta} \right). \]

Thus most of the time one \( X(t, X) \) dominates all the other decisively. This dominance is somewhat persistent even though long compared to \( \frac{1}{\log T} \). Define
\[ \Delta X(t, X) = \frac{1}{H} \int_{t}^{t+H} \log |X(m, X)| dm, \]
for \( 0 \leq m \leq H \), we can show for any positive integer \( K \) that:
\[
\frac{2T}{t} \left( \Delta \chi(t, H) - \log \left| X(t+\theta, X) \right| \right)^{2k} \, dt = \mathcal{O} \left( \Theta e^{Ak} \left( \frac{t}{k} \log \left( \Theta \log T \right) + k^{4k} \right) \right).
\]

Integrating over \( t \) we get

\[
\frac{2T}{t} \int \int \left( \Delta \chi(t, H) - \log \left| X(t+\theta, X) \right| \right) \, d\theta \, dt = \mathcal{O} \left( \Theta e^{Ak} \left( \frac{t}{k} \log \left( \Theta \log T \right) + k^{4k} \right) \right).
\]

If we denote by \( W(t, X) \) the subset of \( t \) for which

\[
\left| \Delta \chi(t, H) - \log \left| X(t+\theta, X) \right| \right| > \left( \log \log T \right)^{\varepsilon}
\]

we find choosing \( k \) so large that

\[
k^{\varepsilon} > 2N+1,
\]

that

\[
W(t, X) \leq \frac{H}{\left( \log \log T \right)^{\varepsilon}},
\]

except for a subset of \( t \) in \((\frac{1}{2}, 2T)\) of measure \( \mathcal{O} \left( \frac{1}{\left( \log \log T \right)^{\varepsilon}} \right) \).

We also can show that for \( \chi \neq \chi' \)

\[
\left| \Delta \chi(t, H) - \Delta \chi'(t, H) \right| > \left( \log \log T \right)^{\varepsilon}
\]

\]

\]
except for a subset of $(T, 2T)$ of measure $O(T (\log \log T)^{-\frac{1}{2}+\delta})$.

For $x_j, \ldots, x_m$ we now define $j \neq k$, $S_j, k$ as the subset of $(T, 2T)$ where
\[ |\Delta x_j(t, H) - \Delta x_k(t, H)| \leq (\log \log T)^{\delta}, \]
we have $m(S_j, k) = O(T (\log \log T)^{-\frac{1}{2}+\delta})$.

If we exclude all of these subsets from $(T, 2T)$, the rest consists of sets $S_j$ such that in $S_j$ for $k \neq j$
\[ \Delta x_j(t, H) > \Delta x_k(t, H) + (\log \log T)^{\delta}. \]

If we also from each $S_j$ exclude all $t$ for which for any $k$
\[ w(t, x_k) > \frac{H}{(\log T)^{1+\delta}}, \]
we get that $(T, 2T)$ except for a subset of measure $O(T (\log \log T)^{-\frac{1}{2}+\delta})$ is divided into $m$ subsets $S_j$ such
that $\sum_{m} \gamma_m (S_j^*) = T - O(T (\log T)^{-\frac{1}{2} + \delta})$

and for each $t$ in $S_j^*$ we have for $k \neq j$

that

$$\log |\chi(t + h, x_j)| - \log |\chi(t + h, x_k)| >$$

$$> \frac{5}{(\log \log T)^2} - 2 \frac{5}{(\log \log T)^{\frac{2}{3}}} > \frac{1}{2} \frac{5}{(\log \log T)^2},$$

except for a set of $h$ in $(0, H)$ of measure $O \left( \frac{H}{(\log \log T)^{\nu}} \right)$.

From $\int_{T}^{2T} \sum_{t} |\chi(t, x) y(t, x)|^2 \, dt = O(T)$

we see that

$$\int_{\theta}^{H} \sum_{t} \chi(u, x_j) \gamma^2 (u, x_j) \, du < H \log \log T$$

except for a subset of $t$ of measure $O \left( \frac{H}{(\log \log T)^{\nu}} \right)$, we exclude also these $t$ from the $S_j^*$ (without renaming them).

Now look at

$\mathcal{I}_{x_j}^*(t, H), \mathcal{M}_{x_j}^*(t, H)$ and $\mathcal{I}_{x_j}^*(t, H)$.
which are for $t$ in $S_j^*$ the integrals $I_{x_j}, M_{x_j}$ and $J_{x_j}$ but with the bad subset removed. They differ from these at most by

$$O\left( \sqrt{\frac{H}{(\log \log T)^N}} \cdot \sqrt{H \log T} \right)$$

$$= O\left( \frac{H}{\log \log T} \right)$$

taking $N = 3$. We see now that we get a sign change of $X(u, x_j)$ in $(t, t + H)$ for $t$ in $S_j^*$ and

$$J_{x_j}^* \gg |I_{x_j}^* (t, H)| \text{ which is equivalent to}$$

$$H > |I_{x_j} (t, H)| + |M_{x_j} (t, H)| + O\left( \frac{H}{\log^2 T} \right)$$

for $T$ large enough this holds outside a measure $O\left( \sqrt{H \log T} \right)$ and so in most of $S_j^*$ if $H = \frac{\lambda m^2}{\log T}$ with $\lambda$ a large enough constant. Produce more than $\frac{c}{m^2} T^4 T$ sign changes of
\[ \pi^{-\frac{1}{2}} P \left( \frac{1}{2} \right) \Phi(s) ; \quad s = \frac{1}{2} + it \text{ in } \mathbb{S}^1_n \]

Adding up over \( j \) we get more than \( \frac{6}{\pi^2} T \theta T \) or

\[ N_0(T, F) > \frac{c}{\log^a T} T \theta T \text{ for } T > T_0(F) \]

can be improved to

\[ N_0(T, F) > \frac{c}{\log^a T} T \theta T \text{ for any } a > 1. \]

Case \( N = 2 \). Some cases of such \( \mathcal{L}(s) \) have been handled and

\[ N_0(T, L) > c \theta T \text{ for } T > T_0(L) \]

proved. Essentially what is required is that one can estimate expressions like

\[ \int_T^{2T} \left| L \left( \frac{1}{2} + it \right) \Phi \left( \frac{1}{2} + it \right) \right|^2 dt \]

where \( \Phi \) is a Dirichlet polynomial

\[ \Phi(s) = \sum \frac{\lambda_m}{m^s} \] where \( \lambda_m \) is like

some small power of \( T \). In these cases the \( I_\chi(t, H) \) have been handled but not the \( \Pi_\chi(t, H) \) which were
avoided using a simpler device which does not work for the linear combination. By slightly modifying the way one defines the analog of $\eta(s)$ for $L(s)$, we put

$$
(L(s))^{\frac{1}{2}} = \sum_{n} \frac{\lambda_{n}}{m_{n}} ; \alpha, \beta = 1
$$

and

$$
\eta(s) = \sum_{\alpha \leq n < 1} \frac{\lambda_{n}}{m_{n}^s} + \sum_{1 \leq m < \infty} \frac{\lambda_{m}}{m^s} (2 - 2 \frac{\lambda_{m}}{m^{\frac{1}{2}}})
$$

This new $\eta(s)$ works equally well for $\Gamma(t, H)$, and much better for

$$
M(t, H) = \sum_{\alpha \leq n < 1} \frac{\lambda_{n}}{m_{n}^s} \gamma_{n}^{2} (\frac{t}{n} + iu) \mu_{n} = H,
$$

We can actually get

$$
\int_{T}^{2T} \left| M(t, H) \right|^2 dt = O \left( \frac{1}{\log^{2}T} \right)
$$

uniformly for $0 < H < T$. Everything else works as before.
Briefly sketched: We write
\[ N(t, H) = \left| \int_{2+i\varepsilon}^{\infty} \left( \frac{d(\tau+it)}{d\tau} \right)^2 \left( \frac{\tau + it}{\tau} \right)^2 - 1 \right| \, d\tau \leq \frac{2\pi}{\varepsilon} \left( \frac{d(\tau+it)}{d\tau} \right)^2 \cdot \left( \frac{\tau + it}{\tau} \right)^2 - 1 \right| \, d\tau \]
also
\[ \left| \int_{2+i\varepsilon}^{\infty} \left( \frac{d(\tau+it)}{d\tau} \right)^2 \left( \frac{\tau + it}{\tau} \right)^2 - 1 \right| \, d\tau \right|^2 \leq \frac{2\pi}{\varepsilon} \left( \frac{d(\tau+it)}{d\tau} \right)^2 \cdot \left( \frac{\tau + it}{\tau} \right)^2 - 1 \right| \, d\tau \]
If we write
\[ J_0 = \frac{2\pi}{\varepsilon} \left( \frac{d(\tau+it)}{d\tau} \right)^2 \cdot \left( \frac{\tau + it}{\tau} \right)^2 - 1 \right| \, d\tau \]
we have
\[ J_1 = O(T) \quad \text{and} \quad J_2 = O \left( T \varepsilon^{-\frac{1}{2}} \right) \]
a continuity argument now gives
\[ J_0 = O \left( T \varepsilon^{-\frac{1}{2}} \left( \delta - \frac{1}{2} \right) \right) \]
the estimation
\[ \frac{2\pi}{T} \left( \frac{d(\tau+it)}{d\tau} \right)^2 \cdot \left( \frac{\tau + it}{\tau} \right)^2 - 1 \right| \, d\tau = O \left( \frac{T}{\varepsilon^2} \right) = O \left( \frac{1}{\varepsilon^2 T} \right) \]
now follows easily.
The drawback is that with this $\eta(s)$, we have to go through the estimation of
\[
\frac{2^T}{T} \sum I(t, H)^2 dt
\]
new, since it does not follow from our earlier result. The same approach can however be adapted to our old $\eta(s)$ defined as
\[
\eta(s) = \sum_{m < \delta} \frac{\chi_m}{m^s} \left( 1 - \frac{\log m}{2\pi i} \right),
\]
by observing that the Dirichlet series for $L(s) \eta^2(s)$ is identical with that of
\[
\left( 1 - \frac{1}{2\log \delta} \frac{L'}{L}(s) \right)^2 \text{ for } m < \delta.
\]
Subtracting from the expansion of
\[
L(s) \eta^2(s) - 1
\]
the terms with $1 < m \leq \delta$, and handling
This part separately on the line \( \sigma = \frac{1}{2} \), the remainder with the method just outlined can get easily for \( H > \frac{3}{2} T \)

\[
\frac{2T}{T} \int_0^T |H(t, H)|^2 \, dt = O \left( T \frac{\log(1 + 2e^q T)}{\log^2 T} \right)
\]

A result only slightly worse (and much better than we actually need).

Where we can prove

\[
\int_0^T |I(t, H)|^2 \, dt \leq C \frac{T}{\log T}, \quad T > T_0
\]

with constant a

uniformly for our \( |j| \) get again.

\[
N_0(CF, T) > \frac{c}{\log T} \quad \text{for } T > T_0(F)
\]

with absolute \( c \).