General Problem

Class (Amalfi lecture 89)

$\mathcal{L}(s) = \sum_n \frac{a_n}{n^s}$; $a_1 = 1$, $a_n = O(n^\delta)$ for any $\delta > 0$.

"Eulerproduct"

$\log \mathcal{L}(s) = \sum_n \frac{b_n}{n^s}$, where $b_n = 0$ unless $n = p^\alpha$; $\alpha > 0$. $(s-1)^m \mathcal{L}(s)$ integral function of finite order for some integer $m \geq 0$.

$b_n = O(n^\theta)$ with $\theta < \frac{1}{2}$.

Functional equation: Let

$\phi(s) = \varepsilon Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j), \mathcal{L}(s)$, with constants $|\varepsilon| = 1$, $Q > 0$, $\lambda_j > 0$, $R\,\mu_j \geq 0$.

and

$$\phi(s) = \overline{\phi(1-\bar{s})} .$$

$\phi(\frac{1}{2}+it)$ real for real t.

Suppose we have n distinct $\mathcal{L}_j$ with the same Γ-factors and form with real constants $c_j \neq 0$

$$F(s) = \sum_{j=1}^{r} c_j \varepsilon_j Q_j^s \mathcal{L}_j(s),$$

then $\prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)\ F(s)$ is real for $s = \frac{1}{2} + it$; t real.

Conjecture: Almost all "non-trivial" zeros of $F(s)$ are on the line $\sigma = \frac{1}{2}$, $s = \sigma + it$. The non-trivial zeros lie in a strip $-A < \sigma < A$ and the number with imaginary part in $(0, T)$ is

$$N(T, F) = \frac{\Lambda}{\pi} T(\log T + B) + O(\log T),$$

where $\Lambda = \sum_j \lambda_j$. If we denote the number of zeros of $F(s)$ with real part $\frac{1}{2}$ in $0 < t < T$ by $N_0(T, F)$, the conjecture

$$N_0(T, F) \sim N(T, F)$$ can be proved if certain plausible conjectures are assumed (none of which can today be proved for even a single function in our class).

What can be proved without using any hypothesis?

It is clear that we could only hope to prove anything of significance for the linear combination $F(s)$, where we can prove something significant for the single $\mathcal{L}(s)$. This has been done only in the case $\Lambda = \frac{1}{2}$, and for some cases with $\Lambda = 1$.

case $\Lambda = \frac{1}{2}$.

Dirichlets L-functions, χ primitive character mod q (incl. case $q=1$ and $\chi(n) \equiv 1$).

$$L(s,\chi) = \sum_n \frac{\chi(n)}{n^s},$$

integral function for $q \neq 1$, $(s-1)L(s)$ integral for $q=1$.

Write $a = \dfrac{1-\chi(-1)}{2}$, and

$$\phi(s,\chi) = \varepsilon_\chi q^{\frac{s}{2}} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi),$$

then $\phi(s,\chi) = \overline{\phi(1-\bar{s},\chi)}$.

For simplicity we look at the case χ even ($a=0$), the odd case can be handled in the same way.

Let χ_j, $j=1,2,\cdots,m$. be m distinct primitive even characters, the c_j real and $\neq 0$, and form

$$F(s) = \sum_{j=1}^{m} c_j \varepsilon_j q_j^{\frac{s}{2}} L(s,\chi_j)$$

(alternatively we might consider
$$F^*(s) = \sum_{j=1}^{m} c_j \varepsilon_j (1+q_j^{s-\frac{1}{2}}) L(s,\chi_j).)$$

Then $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})F(s)$ or $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})F^*(s)$

are real for $s = \frac{1}{2} + it$, t real.
We have in this case

$$N(T, F) = \frac{T}{2\pi}(\log T + B) + O(\log T).$$

For the single $L(s, \chi)$ it has been proved that a positive proportion of the zeros have real part $\frac{1}{2}$, more precisely:

$$N_0(T, L) > c\, T \log T \text{ for } T > Aq^2,$$

where c and A are absolute constants.
For the general linear combination some results are implied in the literature.
From Hardy-Littlewoods work follows

$$N_0(T, F) > cT \text{ for } T > T_0(F).$$

Recently A.A. Karatsuba has considered combinations

$$F(s) = \varepsilon_\chi L(s, \chi) + \bar{\varepsilon}_\chi L(s, \bar{\chi}),$$

where χ is a complex character and obtained the result (1994)

$$N_0(T, F) > T (\log T)^{\frac{1}{2}} e^{-c\sqrt{\log\log T}}, \quad T > T_0;$$

with some positive constant c (for more general combinations he has a much weaker and more complicated result.

I shall sketch a proof that for the general combination $F(s)$ we have

(I) $N_0(T,F) > c(n)\, T \log T$ for $T > T_0(F)$, where $c(n)$ depends on n only. Also:

(II) If $\omega(t) \to \infty$ as $t \to \infty$ then $F(\frac{1}{2}+it)$ has a zero in the interval $\left(t, t+\frac{\omega(t)}{\log t}\right)$ for almost all t.

First let us see how these results are proved for the single L-function.

Let $s = \frac{1}{2}+it$, $t>0$, $\vartheta(t) = \arg \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})$ and write

$$X(t,\chi) = \varepsilon_\chi q^{\frac{it}{2}} e^{i\vartheta(t)} L(s,\chi).$$

"Approximate functional equation"

$$X(t,\chi) = \varepsilon_\chi q^{\frac{it}{2}} e^{i\vartheta(t)} \sum_{n<\sqrt{\frac{tq}{2\pi}}} \chi(n) n^{-s} +$$

$$+ \bar{\varepsilon}_\chi q^{-\frac{it}{2}} e^{-i\vartheta(t)} \sum_{n<\sqrt{\frac{tq}{2\pi}}} \overline{\chi(n)}\, n^{s-1} + O\left(\left(\frac{q}{t}\right)^{\frac{1}{4}}\right).$$

Write

$$\{\zeta(s)\}^{-\frac{1}{2}} = \sum_n \frac{\alpha_n}{n^s}\ ;\ \alpha_1 = 1;\ (L(s,\chi))^{-\frac{1}{2}} = \sum_n \chi(n) \frac{\alpha_n}{n^s},$$

and for $T \leq t \leq 2T$; $\xi = T^{\frac{1}{10}}$; $T > 16q^3$

write –

$$\eta(s,\chi) = \sum_{n<\xi} \chi(n) \frac{\alpha_n}{n^s}\left(1 - \frac{\log n}{\log \xi}\right).$$

(we shall often write $\eta(t,\chi)$ for $\eta(\frac{1}{2}+it,\chi)$)

For $\frac{1}{\log T} \leq H \leq \frac{\log\log T}{\log T}$, consider the three expressions:

$$I_\chi(t,H) = \int_t^{t+H} X(u,\chi)\,|\eta(u,\chi)|^2\,du,$$

$$M_\chi(t,H) = \int_t^{t+H} L(\tfrac{1}{2}+iu,\chi)\,\eta^2(u,\chi)\,du - H,$$

and

$$J_\chi(t,H) = \int_t^{t+H} |X(u,\chi)\,\eta^2(u,\chi)|\,du.$$

If $J_\chi(t,H) > |I_\chi(t,H)|$,

$X(t,\chi)$ changes sign in $(t,t+H)$ and so has at least one zero there.

Also $J_\chi(t,H) \geq H - |M_\chi(t,H)|$,

so if

$$|M_\chi(t,H)| + |I_\chi(t,H)| < H,$$

there is a zero in $(t,t+H)$.

Using the approximate functional equation for $L(s, \chi)$ or $X(t, \chi)$, we can show

$$(1) \quad \int_T^{2T} |I_\chi(t,H)|^2 dt = O\left(T \frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right),$$

$$(2) \quad \int_T^{2T} |M_\chi(t,H)|^2 dt = O\left(T \frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right),$$

and (as we shall use much later)

$$(3) \quad \int_T^{2T} |X(t,\chi)\eta^2(t,\chi)|^2 dt = O(T).$$

The constants implied by the O are absolute.

We see now that $|I_\chi(t,H)| \leq \frac{H}{3}$ and $|M_\chi(t,H)| \leq \frac{H}{3}$ holds except in a subset of $(T, 2T)$ of measure $O\left(\frac{T}{\sqrt{H}\log T}\right)$; Choosing now $H = \frac{\lambda}{\log T}$ with λ a large enough constant we get statement (I) for $N_0(T, L_\chi)$, and by choosing $\lambda = (\omega(T))^{\frac{1}{2}}$ we get statement (II).

To adapt this idea to the linear combination $F(s)$, we need some results about the value distribution of $\log|L(\frac{1}{2}+it,\chi)|$ or $\log|X(t,\chi)|$.

For $T > 16q^3$, k a positive integer and $T^{\frac{1}{2k}} \le x \le T^{k}$, we can show

$$\int_T^{2T} \left|\log|X(t,\chi)| - R\sum_{p<x} \chi(p)p^{-\frac{1}{2}-it}\right|^{2k} dt =$$

$$= O(T\,k^{4k}e^{Ak}),$$

The constants implied by the O are again absolute.

From this we can prove that

$$\frac{\log|X(t,\chi)|}{\sqrt{\pi\log\log t}}$$

has a normal gaussian distribution. More precisely: let $\chi_{a,b}$ denote the characteristic function of the interval (a,b), then

$$\int_T^{2T} \chi_{a,b}\left(\frac{\log|X(t,\chi)|}{\sqrt{\pi\log\log T}}\right)dt = T\int_a^b e^{-\pi u^2}du + O\left(T\,\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right).$$

Also: for two distinct characters χ and χ' similar results hold for the difference

$$\log|X(t,\chi)| - \log|X(t,\chi')|,$$

only here we must divide by $\sqrt{2\pi \log\log t}$ to get the normal gaussian distribution. Thus if $0<\delta<\frac{1}{2}$, the set in $(T,2T)$ where

$$\big|\log|X(t,\chi)| - \log|X(t,\chi')|\big| \le (\log\log T)^{\delta},$$

has measure

$$O\left(T(\log\log T)^{-\frac{1}{2}+\delta}\right).$$

Thus most of the time one $X(t,\chi_j)$ dominates all the others decisively. This dominance is somewhat persistent over stretches long compared to $\frac{1}{\log T}$. Define

$$\Delta_{\chi}(t,H) = \frac{1}{H}\int_{t}^{t+H} \log|X(u,\chi)|\,du.$$

For $0\le h\le H$, we can show for any positive integer k that:

$$\int_T^{2T} \left(\Delta_\chi(t,H) - \log|X(t+h,\chi)| \right)^{2k} dt =$$

$$= O\left(T e^{Ak} \left(k^k \log^k(H \log T) + k^{4k} \right) \right).$$

Integrating over h we get

$$\int_T^{2T} \int_0^H \left(\Delta_\chi(t,H) - \log|X(t+h,\chi)| \right)^{2k} dh\, dt =$$

$$= O\left(T H e^{Ak} \left(k^k (\log\log\log T)^k + k^{4k} \right) \right).$$

If we denote by $W(t,\chi)$ the measure of the subset of h for which

$$|\Delta_\chi(t,H) - \log|X(t+h,\chi)|| > (\log\log T)^{\frac{\delta}{2}},$$

we find choosing k so large that $k\delta > 2N+1$, that

$$W(t,\chi) \leq \frac{H}{(\log\log T)^N},$$

except for a subset of t in $(T, 2T)$ of measure $O\left(\frac{T}{(\log\log T)^N} \right)$.

We also can show that for $\chi \neq \chi'$

$$|\Delta_\chi(t,H) - \Delta_{\chi'}(t,H)| > (\log\log T)^\delta$$

except for a subset of $(T, 2T)$ of measure $O(T(\log\log T)^{-\frac{1}{2}+\delta})$.

For $\chi_1, \ldots \chi_m$ we now define for $j \neq k$, $S_{j,k}$ as the subset of $(T, 2T)$ where

$$|\Delta_{\chi_j}(t, H) - \Delta_{\chi_k}(t, H)| \leq (\log\log T)^{\delta},$$

we have $m(S_{j,k}) = O(T(\log\log T)^{-\frac{1}{2}+\delta})$

If we exclude all of these subsets from $(T, 2T)$, the rest consists of m sets S_j such that in S_j for $k \neq j$

$$\Delta_{\chi_j}(t, H) > \Delta_{\chi_k}(t, H) + (\log\log T)^{\delta}.$$

If we also from each S_j exclude all t for which for any k

$$W(t, \chi_k) > \frac{H}{(\log\log T)^N}$$

, we get that $(T, 2T)$ except for a subset of measure $O(T(\log\log T)^{-\frac{1}{2}+\delta})$ is divided into m subsets S_j^* such that $\sum_n m(S_j^*) = T - O(T(\log T)^{-\frac{1}{2}+\delta})$

and for each t in S_j^* we have for $k \neq j$ that

$\log|X(t+h, x_j)| - \log|X(t+h, x_k)| >$
$(\log\log T)^{\delta} - 2(\log\log T)^{\frac{\delta}{2}} > \frac{1}{2}(\log\log T)^{\delta}$,

except for a set ~~of~~ h in $(0, H)$ of measure $O\left(\frac{H}{(\log\log T)^N}\right)$.

From
$$\int_T^{2T} |X(t,x)\,\eta^2(t,x)|^2 dt = O(T)$$
we see that
$$\int_t^{t+H} |X(u, x_j)\,\eta^2(u, x_j)|^2 du < H \log\log T$$
except for a subset of t of measure $O\left(\frac{T}{\log\log T}\right)$, we exclude also these t from the S_j^* (without renaming them).

Now look at

$I_{x_j}^*(t,H)$, $M_{x_j}^*(t,H)$ and $J_{x_j}^*(t,H)$

which are for t in S_j^* the integrals I_{χ_j}, M_{χ_j} and J_{χ_j} but with the bad subset removed. They differ from these at most by

$$O\left(\sqrt{\frac{H}{(\log\log T)^N}}\cdot\sqrt{H\log\log T}\right)$$

$$= O\left(\frac{H}{\log\log T}\right),$$

taking $N=3$. We see now that we get a signchange of $X(u,\chi_j)$ in $(t, t+H)$ for t in S_j^* and

$J^*_{\chi_j}(t,H) > |I^*_{\chi_j}(t,H)|$ which is equivalent to

$$H > |I_{\chi_j}(t,H)| + |M_{\chi_j}(t,H)| + O\left(\frac{H}{\log\log T}\right)$$

For T large enough this holds outside a set of measure $O\left(\frac{1}{\sqrt{H\log T}}\right)$ and so in most of S_j^* if $H = \frac{\lambda m^2}{\log T}$ with λ a large enough constant. Produces more than $\frac{c}{m^3} T\log T$ sign changes of

$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})F(s)$; $s=\frac{1}{2}+it$ in S_j^*.
adding up over j we get more
than $\frac{c}{n^2}T\lg T$ or

$$N_0(T,F) > \frac{c}{n^2}T\lg T \text{ for } T>T_0(F).$$

can be improved to

$$N_0(T,F) > \frac{c(\alpha)}{n^\alpha}T\lg T \text{ for any } \alpha>1.$$

case $n=1$. Some cases of such $\mathcal{L}(s)$
have been handled and

$$N_0(T,\mathcal{L}) > cT\lg T \text{ for } T>T_0(\mathcal{L})$$

proved. Essentially what is required
is that one can estimate expressions
like

$$\int_T^{2T} |\mathcal{L}(\tfrac{1}{2}+it)\,P(\tfrac{1}{2}+it)|^2\,dt$$

where P is a dirichlet polynomial
$P(s) = \sum_{n\leq\xi} \frac{\lambda_n}{n^s}$, where ξ is like
some small power of T. In these
cases the $I_\chi(t,H)$ have been handled
but not the $M_\chi(t,H)$ which were

avoided using a simpler device which does not work for the linear combination. By slightly modifying the way one defines the analog of $\eta(s)$ for $\mathcal{L}(s)$, we put

$$\left(\mathcal{L}(s)\right)^{-\frac{1}{2}} = \sum_{n} \frac{\alpha_n}{n^s} \,;\, \alpha_1 = 1$$

and

$$\eta(s) = \sum_{n < \xi^{\frac{1}{2}}} \frac{\alpha_n}{n^s} + \sum_{\xi^{\frac{1}{2}} \le n < \xi} \frac{\alpha_n}{n^s}\left(2 - 2\frac{\log n}{\log \xi}\right).$$

This new $\eta(s)$ works equally well for $I.(t,H)$, and much better for

$$M(t,H) = \int_t^{t+H} \mathcal{L}(\tfrac{1}{2}+iu)\,\eta^2(\tfrac{1}{2}+iu)\,du - H,$$

We can actually get

$$\int_T^{2T} |M(t,H)|^2\,dt = O\left(\frac{T}{\log^2 T}\right)$$

uniformly for $0 < H < T$. Everything else works as before.

Briefly sketched: We write

$$|M(t,H)| = \left|\int_t^{t+H} \left(\mathcal{L}(\tfrac{1}{2}+iu)\eta^2(\tfrac{1}{2}+iu)-1\right)du\right| \leq$$

$$\leq \left|\int_{\frac{1}{2}+it}^{2+it} \left(\mathcal{L}(s)\eta^2(s)-1\right)ds\right| + \left|\left(t \to t+H\right)\right| + O\left(\xi^{-\frac{1}{3}}\right).$$

also

$$\left|\int_{\frac{1}{2}+it}^{2+it} \left(\mathcal{L}(s)\eta^2(s)-1\right)ds\right|^2 \leq$$

$$\int_{\frac{1}{2}}^{2} \xi^{\frac{1}{4}(\frac{1}{2}-\sigma)}\,d\sigma \cdot \int_{\frac{1}{2}}^{2} \xi^{\frac{1}{4}(\sigma-\frac{1}{2})}\left|\mathcal{L}(\sigma+it)\eta^2(\sigma+it)-1\right|^2 d\sigma.$$

If we write

$$J_\sigma = \int_T^{2T} \left|\mathcal{L}(\sigma+it)\eta^2(\sigma+it)-1\right|^2 dt,$$

we have

$$J_{\frac{1}{2}} = O(T) \quad \text{and} \quad J_2 = O(T\xi^{-1}).$$

a convexity argument now gives

$$J_\sigma = O\left(T\,\xi^{-\frac{1}{2}(\sigma-\frac{1}{2})}\right),$$

the estimation

$$\int_T^{2T} |M(t,H)|^2 dt = O\left(\frac{T}{\log^2\xi}\right) = O\left(\frac{T}{\log^2 T}\right)$$

now follows easily.

The drawback is that with this $\eta(s)$, we have to go through the estimation of

$$\int_T^{2T} |I(t,H)|^2 dt$$

anew, since it does not follow from our earlier result. The same approach can however be adapted to our old $\eta(s)$ defined as

$$\eta(s) = \sum_{n<\xi} \frac{\alpha_n}{n^s}\left(1-\frac{\log n}{\log \xi}\right),$$

by observing that the dirichlet series for $\mathcal{L}(s)\,\eta^2(s)$ is identical with that of

$$\left(1-\frac{1}{2\log\xi}\,\frac{\mathcal{L}'}{\mathcal{L}}(s)\right)^2 \quad \text{for } n \le \xi.$$

Subtracting from the expansion of $\mathcal{L}(s)\,\eta^2(s)-1$ the terms with $1<n\le\xi$, and handling

this part separately on the line $\sigma = \frac{1}{2}$, the remainder with the method just outlined we get easily for $H > \frac{2}{\log T}$

$$\int_T^{2T} |M(t,H)|^2 dt = O\left(T \frac{\log(H \log T)}{\log^2 T}\right)$$

a result only slightly worse (and much better than we actually need).