#1. Oct. 25, 1954

Consider the translation group on the real line: $x \to x+a$. One can

easily see that a nec. and suff. condition for the operation

$\int_{-\infty}^{\infty} k(x,y)\, f(y)\, dy$ to commute with translations is that $k(x,y) = K(x-y)$

and then we have a convolution. Other operators are $\frac{d}{dx}$, etc. One gets

a ring of operators generated by $\frac{d}{dx}$ in terms of symbolic operations.

If h is the Fourier transformation of k, the operators are given by $h(i\frac{d}{dx})$.

Functions invariant under $x \to x+m$ are the exponentials $e^{2\pi i k x}$ and

series of these giving the Fourier series. For this case $\int_{-\infty}^{\infty} k(x-y) f(y)\, dy$

take the form $\int_0^1 \{\sum_{n=-\infty}^{\infty} k(x-y+n)\} f(y)\, dy$. If $k(x) = O\left(\frac{1}{|x|^{1+\varepsilon}}\right)$ for large $|x|$

then the series is uniformly bounded. Now the eigenfunctions of

$\lambda f(x) = \int_0^1 f(y) \sum_{n=-\infty}^{\infty} k(x-y+n)\, dy$

are still the exponential functions. The trace ~~of a~~ the eigenvalues can be

computed in two ways as $\sum_{-\infty}^{\infty} h(2\pi m)$ and as $\int_0^1 \sum_{-\infty}^{\infty} k(n)\, dx = \sum_{-\infty}^{\infty} k(n)$

and this is the Poisson ~~summation~~ summation formula. This works even tho

the kernel is not symmetric because it is normal.

Here a generalization of the Poisson formula will be made to more

general groups. This formula will yield information about the groups

not ~~generally~~ otherwise easily obtainable. Specialization will be made to

the hyperbolic group for the plane.

Another derivation of this formula depends on using $\sum_{-\infty}^{\infty} f(x+n)$.

This is periodic. Now determine the Fourier series. But the first

method based on the trace formula, will yield the better generaliza-

tion.

Let G be a group of isometries in a Riemannian space S. Actually (we need

a concept of distance and an invariant metric. Let

$ds^2 = \sum g_{ij}\, dx_i\, dx_j$ with infinitely differentiable functions g_{ij}. Elements

of G are denoted by m. G is to contain all true motions & is a

connected component. Let dy be the volume element $\sqrt{g}\,\frac{dx_1 \dots dx_n}{\sqrt{|g_{ij}|}}$.

Consider the operation $\int_S k(x,y) f(y)\, dy$. The

necessary & sufficient condition for invariance under the is that $k(x,y) = k(mx, my)$ for $m \in G$. The operator is on $f(x)$. Invariance means if $f(x)$ is replaced by $f(m^{-1}x)$ & then y is replaced by my, there is to be no change. Then we require

$$\int_S k(x,y)\, f(m^{-1}y)\, dy$$

& now $\int_S k(mx, my) f(y)\, dy$ should be $\int_S k(x,y) f(y)\, dy$.

Suppose h has the same invariance property as k. Applying first one & then the other, we get new kernels

$I_1 = \int_S k(x,z)\, h(z,y)\, dz$, $\quad I_2 = \int_S k(z,y)\, h(x,z)\, dz$.

We would like these to be equal. We now assume that for given x, y $\exists\, m \in G \ni my = x$. If this is the case, then the first integral becomes

$$\int_S k(z,x)\, h(mx, z)\, dz.$$

And if m can also be chosen so that $mx = y$, then the integral is

$$\int_S k(z,x)\, h(y,z)\, dz$$

Now k is symmetric under these conditions since

$$k(y,x) = k(mx, my) = k(x,y)$$

so that the integral is $\int_S k(x,z)\, h(z,y)\, dz$.

We can replace these assumptions by assuming the weaker condition that $\exists\, \mu \ni \mu G \mu^{-1} = G$ and $\exists\, m \in G \ni mx = \mu y$, $my = \mu x$. For now can take $\mu^2 \in G$

$$k(y,x) = k(\mu y, \mu x) = \tilde{k}(y,x).$$

~~$\int k(x,z)\, h(z,y)\, dz$~~

Now

$$I_2 = \int_S k(z, my)\, h(mx, z)\, dz = \int k(\mu z, my)\, h(mx, \mu z)$$

$$= \int_S k(\mu z, \mu x)\, h(\mu y, \mu z)\, dz = \int_S k(x,z)\, h(z,y)\, dz = I_1$$

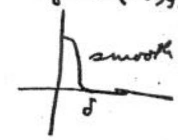
This applies so far to these integral operators.

If L_x is a linear operator generated by such k & differential operators of finite order, then $L_x k(x,y)$ is a point pair i.e. is invariant under $x \to mx$, $y \to my$. If $k_1(x,y) = L_x k(x,y)$ then since $L_{\mu y} k(\mu y, \mu x) = k_1(\mu y, \mu x)$ we have $L_{\mu y} k(x,y) = k_1(x,y)$

and $L_x \, k(x,y) = L_y \, k(x,y)$ so that $L_x^{(1)} L_x^{(2)} = L_x^{(2)} L_x^{(1)}$.

Thus, these operators commute.

$k_\delta = k(d(x,y))$



If $f(x) = \int_S k(x,y) F(y) \, dy$, consider the mean value

$$\frac{\int_S k_\delta(x,y) f(y)\, dy}{\int_S k_\delta(x,y)\, dy}$$ which is independent of x.

For smooth enough f, this tends to f & the partial derivatives tend to the partial derivatives of f. In this way, commutativity can be carried over to such f.

The functions k have a finite basis in the sense that that each k is a function of some $k_1, \ldots, k_l$ with $1 \leq l \leq n$.

#2. Nov. 2, 1954

We assumed the following:

G is a group of isometries in

For some μ, not necessarily in G, $\mu G \mu^{-1} = G$

For given x, y, $\exists\, m \in G \ni mx = \mu y$, say $y = \mu x$.

Riemannian space S $\quad ds^2 = \sum g_{ij}\, dx_i\, dx_j$.

Point pair: $k(mx, my) = k(x,y)$.

If f is a function and $f(x_0) \neq 0$, consider the group R_{x_0} of (rotations) transformations ρ which leave x_0 fixed. Then R_{x_0} is isomorphic to some subgroup of G. An element of volume $d\rho$ can be defined and we can consider $\int_{R_{x_0}} f(\rho x)\, d\rho = F(x)$ & this gives a function invariant under rotations about x_0. $F(\rho' x) = F(x)$. This can be made into a point-pair by defining $k(x,y) = F(mx)$ where m takes y into x_0. The function $F(x)$ is called the symmetrization of f about x_0.

From operators that are invariant about a point, one can obtain operators that are invariant everywhere by extending.

If two differential operators yield the same result when applied to $k(x_1, \ldots, x_l)$ & then evaluated at $(0, \cdots, 0)$, then these operators are identical. The eigenvalue of an eigenfunction is uniquely determined by the " " the fundamental eigenfunctions.

Suppose $\Gamma \subset G$ & Γ is a discontinuous group with elements M and fundamental domain D_Γ. Suppose that to each $M \in \Gamma$ there corresponds a unitary $r \times r$ matrix $\chi(M)$; i.e. $\chi(M)\bar{\chi}'(M) = I = E_n$ and $\chi(MN) = \chi(M)\chi(N)$. Then a representation of Γ. Let $F(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_r(x) \end{pmatrix}$ be such that $F(Mx) = \chi(M)F(x)$. Consider $\int_S k(x,y)\,F(y)\,dy$. The images of fundamental regions MD_Γ cover the space without overlap. Hence

$y \to My$

$$\int_S k(x,y)\,F(y)\,dy = \sum_{M\in\Gamma} \int_{MD_\Gamma} k(x,y)\,F(y)\,dy$$

$$= \sum_{M\in\Gamma} \int_{D_\Gamma} k(x, My)\,F(My)\,dy \qquad \text{by invariance}$$

$$= \sum_{M\in\Gamma} \int_{D_\Gamma} k(x, My)\,\chi(M)\,F(y)\,dy$$

$$= \int_{D_\Gamma} K(x,y)\,F(y)\,dy$$

$$K(x,y) = \sum_{M\in\Gamma} \chi(M)\,k(x, My)$$

$$\bar{K}'(y,x) = \sum_{M\in\Gamma} \bar{\chi}'(M)\,\bar{k}(y, Mx)$$

$$= \sum \bar{\chi}'(M)\,\bar{k}(M^{-1}y, x)$$

$$= \sum \chi(M^{-1})\,\bar{k}(M^{-1}y, x)$$

$$= \sum \chi(N)\,\bar{k}(Ny, x)$$

& this is of the same type as $K(x,y)$ with the point pair function $\bar{k}(y,x)$. Hence we have a normal operator.

If $K(x,y)$ is bounded & $\int_{D_\Gamma} dy$ is finite, then many statements can be made about the spectrum; e.g. it is discrete. If D_Γ is compact then $K(x,y)$ is bounded if $k(x,y)$ is bdd. & has compact support.

The eigenfunctions span the space. Then one could get the expansion in terms of the eigenfunctions. This is the analog of the Fourier expansion but is not very good since usually the eigenfunctions are hard to compute.

But by using the trace (σ) method ~~can~~ one computes the sum of the eigenvalues in two ways getting

$$\sum_i h(\lambda_1^{(i)}, \dots, \lambda_\ell^{(i)}) = \int_{D_\Gamma} \sigma(K(x,x))\, dx.$$

The functions on the left can be determined. Now the right side is

$$\sum_M \sigma(\chi(M)) \int_{D_\Gamma} k(x, Mx)\, dx.$$

Let ~~If~~ $\{M\}_\Gamma$ ~~be~~ the collection of NMN^{-1} with $N \in \Gamma$, ~~then this~~ ~~because, since~~ Then $NMN^{-1} = N_1 M N_1^{-1}$ implies $N_1^{-1} N M N^{-1} N_1 = M$ & so $N_1^{-1} N$ ~~comp~~ commutes with M. If Γ_M is the set of N which commute with M. Then the sum is

$$\sum_N \sigma \int_{D_\Gamma} k(N^{-1}x, MN^{-1}x)\, dx = \sum_N \sigma \int_{N^{-1}D_\Gamma} k(x, Mx)\, dx$$

$$= \sigma(\chi)\int_{D_{\Gamma_M}} k(x, Mx)\, dx$$

$$= \sum_{\{M\}_\Gamma} \sigma(\chi(M)) \underbrace{\mu(\{M\}_\Gamma)}_{\text{a meas. of a fundamental domain}} \; g(\{M\}_G)$$

a functional transform of k.

Thus

$$\sum_i h(\lambda_1^{(i)}, \dots, \lambda_\ell^{(i)}) = \sum_{\{M\}_\Gamma} \sigma(\chi(M))\, \mu(\{M\}_\Gamma)\, g(\{M\}_G)$$ trace formula.

#3. Nov. 9, 1954

Let Y be ~~the group of~~ any $n \times n$, symmetric, positive definite matrices with $ds^2 = \sigma(Y^{-1}dY \cdot Y^{-1}dY)$ with σ = trace. ~~The~~ Then

$$ds^2 = \sigma(Y^{-1/2} \cdot dY \cdot Y^{-1/2} \cdot Y^{-1/2}\, dY \cdot Y^{-1/2})$$

& this is positive definite. Let A be $n \times n$, real, non-singular. Let G be the group of transformations AYA' & these are isometries. Now if μ is the

$\cancel{dY_1 = A\, dY\, A^{-1}}$

$\cancel{Y_1^{-1} dY_1 = Y_1^{-1} A\, dY A^{-1}}$

isometry taking Y into Y^{-1} then $\mu G \mu^{-1} = G$. And for given Y, X we can find $m \in G$ such that $mX = \mu Y$ and $mY = \mu X$. For we can make ~~Y~~ the unit matrix E & X a diagonal matrix P by a transformation of the group. And now we can find m for E and $P = \operatorname{diag}(p_1, \dots, p_n)$. We can take $m = A$ $A = \operatorname{diag}\{p_1^{-1/2}, \dots, p_n^{-1/2}\}$. ~~so that~~

Consider $\sigma_k = \sigma((YY_1^{-1})^k)$, $k = 1, \dots, n$. This gives a point-pair invariant and the integral invariant operators would be of the type

$\int k(\sigma_1, \dots, \sigma_n)\, f(Y)\, dY$ $\qquad dY = \prod_{i \le j} dy_{ij} \cdot \dfrac{2^{-n(n-1)/4}}{|Y|^{(n+1)/2}}$

$$\text{Let } \frac{\partial}{\partial Y} = \left(\left(\varepsilon_{ij} \frac{\partial}{\partial y_{ij}}\right)\right) \quad \text{with } \varepsilon_{ij} = \begin{cases} \frac{1}{2} & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} = \frac{1+\delta_{ij}}{2}$$

Then $df = \sigma\left(\frac{\partial}{\partial Y} dY \,\middle|\, f\right)$. We can also extend the definition of $Y\frac{\partial}{\partial Y}$ as applying to functions f so as to have it apply to matrices F of functions f_{ij}. Consider $\sigma\left((Y\frac{\partial}{\partial Y})^k\right)$. The matrix $(Y\frac{\partial}{\partial Y})^k = Y\frac{\partial}{\partial Y} Y\frac{\partial}{\partial Y} \cdots$ if replaced by $Y^k \frac{\partial}{\partial Y}\frac{\partial}{\partial Y} \cdots \frac{\partial}{\partial Y}$ also gives an invariant rather simply related to the original.

Let $Y = ((y_{ij}))$ and Y_k be the matrix obtained by dropping the last $n-k$ rows & columns. Then

$|Y|^{s_n} |Y_n|^{s_{n-1}} \cdots |Y_1|^{s_0} = \varphi_{s_1, \dots, s_n}(Y)$ depends only on the eigenvalues.

Now $Y = RR'$ with triangular R & positive elements on the main diagonal. Then $Y_k = (r_1 \cdots r_k)^2$. Consider, with $Y_1 = R_1 R_1'$, $Y \to R_1 Y R_1'$

$$\int k\left(\sigma(YY_1^{-1})^k\right) \varphi_{s_1, \dots, s_n}(Y)\, dY$$

$$= \int k(\sigma(Y^k))\, \varphi_{s_1, \dots, s_n}(Y)\, dY \cdot \varphi_{s_1, \dots, s_n}(Y_1)$$

Consider now the special case $n=2$ with γ restricted to have determinant 1. Then if $z = x+iy$, $y>0$, $mz = \frac{az+b}{cz+d}$, $ad-bc=1$, $ds^2 = \frac{dx^2+dy^2}{y^2}$.
The transformation matrices $m=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ need not be considered to act only on points in the upper half plane. Consider pairs (z,φ) with real φ & $(z,\varphi) = (z, \varphi+2k\pi)$. Let

$$(z,\varphi) \to \{mz, \varphi - \arg(cz+d)\}$$

We also want the transformations yielding $(z,\varphi+\alpha)$, $(-\bar{z}, \varphi)$ & one gets a Riemannian space with

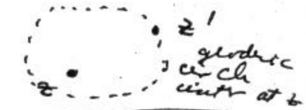
$$ds^2 = \text{combination of } \frac{dx^2+dy^2}{y^2} \ \& \ d\varphi - \frac{dx}{2y}$$

In addition to the Laplace operator, there is also the one $\frac{\partial}{\partial\varphi}$.
This corresponds to the theory of automorphic forms of dimension k where functions change by factor of $(cz+d)^k = |cz+d|^k e^{ik\arg(cz+d)}$
Forms of fractional dimension could also be treated.

The geodesics are the circles orthogonal to the real axis including the lines $\perp$. The distance between z & z' is

$$\log \frac{|z-\bar{z}'| + |z-z'|}{|z-\bar{z}'| - |z-z'|}.$$

The point pairs are functions of this distance. Another point pair is $t(z,z') = \frac{|z-z'|^2}{yy'}$ which uniquely determines the distance & vice-versa.



The Laplacian is $y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ & is invariant & this is the only one. An independent proof can be given by Green's formula.

Area of circle is $\pi\, t(z,z')$.

y^s is transformed into $s(s-1)\, y^s$

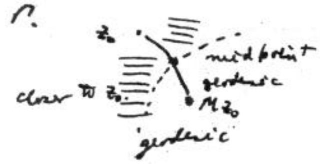
#4. Nov. 16

The hyperbolic plane

$z = x+iy$, $y>0$. $ds^2 = \frac{dx^2+dy^2}{y^2}$. The isometries are $\frac{az+b}{cz+d}$, $ad-bc=1$; or $\frac{a\bar{z}+b}{c\bar{z}+d}$, $ad-bc=-1$. The geodesics are the circles, including lines, orthogonal to the real axis. By a geodesic circle about a point is the locus of points having the same geodesic distance from the point. It is a Euclidean circle but with a different center. The circumference is $\pi(e^{\rho}-e^{-\rho}) = 2\pi\sinh\rho$ and the area is $\pi(e^{\rho}+e^{-\rho}-2) = 4\pi\sinh^2\frac{\rho}{2}$, where ρ is the geodesic radius.

The geodesic area of a geodesic triangle is $\pi-(\alpha+\beta+\gamma)$. For a polygon, the result is $(n-2)\pi-\sum\alpha$, summed over the interior angles. The area element is $dx\,dy/y^2$.

Let Γ be a discontinuous group. Let z_0 not be a fixed pt. of any transformation of Γ. For Γ is countable & ∴ the fixed points are countable. A fundamental domain D can be constructed as follows. We put $z \in D$ if z is closer to z_0 than to all ~~the~~ images of z_0 under Γ.



The intersection of all these half-planes form a polygon with a finite number of sides & its interior is a fundamental domain. It is a convex polygon containing z_0. The images of D cover the hyperbolic plane without overlap & without omission, except for the boundaries. There are a finite number of images D adjacent to D. If happens we break the original side in two & count it twice. It is possible by this & other means to make D have an even number of sides & these correspond to one another in pairs under Γ. The group Γ has a finite number of generators.

Let the number of sides be $2k$. The area $A(D) = (2k-2)\pi-\sum\alpha$.

Around a corner, the angles from Γ & its images, add up to 2π. A cycle of order $m>1$ is one in which some angle, & ∴ all, occur m times. Then $A(D) \equiv -2\pi\sum\frac{1}{m}$ (mod 2π). $\sum 1/m$ is determined by the elliptic transformations of the group; those which are ~~congruent~~ rotations about some point. Hence

$\frac{A(D)}{2\pi} \equiv -\sum\frac{1}{m} \bmod 1$. A number of these things hold not merely under the assumption of compact D but also if D has finite area.

Consider $W = Mz = \frac{az+b}{cz+d} \not\equiv z$, then $Mz = z$ has at most 2 solutions. Might have conjugate fixpoints $z_0, \bar{z}_0$, only 1 in the hyperbolic plane. Consider ~~$\frac{z-z_0}{z-\bar{z}_0}$ This leaves ...~~ Then. $\frac{w-z_0}{w-\bar{z}_0} = e^{i\varphi}\,\frac{z-z_0}{z-\bar{z}_0}$. Another possibility is two real fixed points ω and ω'. Then $\frac{w-\omega}{w-\omega'} = \rho\,\frac{z-\omega}{z-\omega'}$ with $\rho > 1$ & this is called hyperbolic. Another possibility is that there are 2 equal fixed points, $\omega \neq \infty$. Then $\frac{1}{w-\omega} = \frac{1}{z-\omega} + \lambda$; parabolic with λ of no importance since $W = \lambda w$, $\Omega = \lambda\omega$, $Z = \lambda z$ makes $\frac{1}{W-\Omega} = \frac{1}{Z-\Omega} + 1$. Groups with compact D have no parabolic transformations since these involve 0 angles.

Instead of the distance, we use $t(z,z') = \frac{|z-z'|^2}{yy'}$. The fundamental operation is the Laplacian $y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. This operation on y^s is $s(s-1)y^s = -s(1-s)y^s$ $\therefore$ $y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + s(1-s)y^s = 0$. Let H be the upper half-plane. The integral operators look like:

$$\iint_H k\left(\frac{|z-z'|^2}{yy'}\right) y'^s \frac{dx'dy'}{y'^2}$$

$$= \int_0^\infty \int_{-\infty}^{\infty} k\left(\frac{y}{y'} + \frac{y'}{y} - 2 + \frac{(x-x')^2}{yy'}\right) y'^s \frac{dx'}{y'^2}$$

$$= \int_0^\infty y'^{s-2} dy' \int_{-\infty}^{\infty} k\left(\frac{y}{y'} + \frac{y'}{y} - 2 + \frac{x'^2}{yy'}\right) dx'$$

$$= \sqrt{y}\int_0^\infty y'^{s-\frac{3}{2}} dy' \int_{-\infty}^{\infty} k\left(\frac{y}{y'} + \frac{y'}{y} - 2 + x'^2\right) dx'$$

Now if $\int_{-\infty}^{\infty} k(w + x^2)\, dx = Q(w)$ this becomes

$$\sqrt{y}\int_0^\infty Q\left(\frac{y}{y'} + \frac{y'}{y} - 2\right) y'^{s-3/2}\, dy'$$

Let $g(u) = Q(e^u + e^{-u} - 2)$. Then the integral is

$$\sqrt{y}\int_0^\infty g\left(\log\frac{y}{y'}\right) y'^{s-3/2}\, dy' = y^s \int_0^\infty g(\log y')\, y'^{s-3/2}\, dy'$$
$$= \varsigma y^s$$

Hence y^s is an eigenfunction. Let $s = \frac{1}{2} + ir$, $y' = e^u$. The integral $= C_s$ i.e.
$h(r) = \int_{-\infty}^{\infty} g(u) e^{iru}\, du$. $g(u)$ is even & $\therefore$ we have that $h(r)$ is even.
The process can be reversed beginning with $h(r)$. For we can get $g(u)$ by the Fourier inversion. Given $Q(w)$ we want $k(t)$ such that

$$2\int_0^\infty k(w+x^2)\, dx = Q(w)$$

Let $w + x^2 = t$. Then

$$\int_w^\infty \frac{k(t)}{\sqrt{t-w}}\, dt = Q(w).$$

This is an integral equation of the form of Abel & its solution is

$$k(t) = \frac{1}{\pi}\int_t^\infty \frac{d\,Q(w)}{\sqrt{w-t}}$$

This can be proved directly very easily by interchanging two integrals.
Hence $k(t)$ & $h(t)$ give each other if $h(r)$ is even.

Consider conditions needed on $k(t)$. We want $\sum_{M \in \Gamma} |k(z, Mz')|$ to be bounded. Let d_0 be the diameter of the fundamental region D. If MD is within ρ of z, then MD is within $\rho + d_0$ of D. If the # of image pts. in this circle is n_ρ, then

$n_\rho\, A(D) = O(e^\rho)$ & $n_\rho = O(e^\rho)$. Let $k(t) = O\left(\frac{1}{1+|t|^{1+\varepsilon}}\right)$ for some $\varepsilon > 0$. Now

$$t(z, z') = e^\rho + e^{-\rho} - 2 = O(e^\rho)$$

& $\therefore$ $k(t) = O\left(e^{-(1+\varepsilon)\rho}\right)$. Hence

$$\sum_M |k(z, Mz')| = \sum_{n=0}^\infty \sum_{n \le t(z, Mz') < n+1} \le \sum_{n=0}^\infty e^{-(1+\varepsilon)n} O(e^n)$$

$$= O\left(\sum_{n=0}^\infty e^{-\varepsilon n}\right) = O(1)$$

We now assume that $k(t) = O(|t|^{-1-\varepsilon})$ & hence that K is bounded.
Then the previous work is legitimate if $|\Re s - 1/2| \le \frac{1}{2} + \varepsilon$ or $-\varepsilon < \sigma \le 1 + \varepsilon$. Also $Q(w) = O\int_0^\infty \frac{dt}{(t^2+w)^{1+\varepsilon}} = O(w^{-\frac{1}{2}-\varepsilon})$ & bdd. near $w=0$.
And $g(u) = O\left(e^{-(\frac{1}{2}+\varepsilon)|u|}\right)$ & $\therefore$ $h(r)$ exists if $|\Im r| < \frac{1}{2}+\varepsilon$. A condition of the sort $h(r) = O(1/|r|^{2+\varepsilon})$ guarantees the reverse

$\int_w^\infty \frac{k(t)}{\sqrt{t-w}}\,dt = Q(w)$, $\quad Q(e^u + e^{-u} - 2) = g(u)$, $h(r) = \int_{-\infty}^{\infty} e^{iru} g(u)\,du$

$k(t) = -\frac{1}{\pi}\int_t^\infty \frac{dQ(w)}{\sqrt{w-t}}$ $\quad$ " $\quad$ $g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iru} h(r)\,dr.$

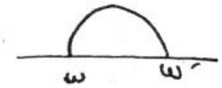
If $h(r)$ is even + regular for $|\Im(r)| < \frac{1}{2}+\varepsilon$ & $h(r) = O\left\{\frac{1}{(1+|r|)^{2+\varepsilon}}\right\}$

then $k(t) = O\left(\frac{1}{(1+t)^{1+\varepsilon}}\right)$.

Let M be a hyperbolic transformation. P is a primitive transformati if P can not be written as M^h with $h>1$. Suppose $M = P^k$ where P is primitive. ~~They~~ M & P have the same fixed points so that these can be taken to be 0 and ∞. Then they are of the kind $M(z) = \rho z$, $P(z) = \rho_0 z$. Then the group contains $\rho^m \rho_0^n z$. From this one can show that $\rho = \rho_0^\nu$ for positive integral ν.

If $MN = NM$ Then ~~these~~ M, N have the same fixed points. ~~If~~ For $M = NMN^{-1}$ & if the fixed points of ~~N~~ are ω, ω', those of M are $N\omega, N\omega'$. Actually ω must correspond to $N\omega$.

∴ The subgroup commuting with $M = P^k$ is the group generated by P.



For an elliptic transformation M we define a primitive rotation P as one having the smallest angle of rotation about the fixed pt. Again if $M = P^k$ then the subgroup commuting with M is the group generated by P.

By the norm of P, when written in the form ρz, $\rho > 1$, we mean ρ. Then $M = P^k$ has norm ρ^k.

$M_1 = NMN^{-1} \sim M$ & we can obtain equivalence classes of primitive transformation. P & P^{-1} represent the same class $\Leftrightarrow$ $I = QQ'$ with $Q^2 = E = Q'^2$ the identity.

Let $\chi(M)\bar{\chi}'(M) = E$ with M being $n \times n$ & this is a representation of the discontinuous group Γ. Let

$$F(z) = \begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix} \quad \text{with} \quad F(Mz) = \chi(M)\,F(z) \quad \text{for all } M \in \Gamma.$$

The hyperbolic plane $H = \sum_{M\in\Gamma} M(D)$ and

$$\iint_H k(z,z')F(z')\frac{dx'dy'}{y'^2} = \sum_M \iint_{M(D)} k(z,z')\,F(z')\frac{dx'dy'}{y'^2}$$

$$= \sum_M \chi(M) \iint_{\mathcal{D}} k(z, Mz')\ F(z')\ \frac{dx'\,dy'}{y'^2}$$

One form of the Poisson^sum formula

$$= \iint_{\mathcal{D}} K(z, z')\ F(z')\ \frac{dx'\,dy'}{y'^2}, \qquad K(z, z') = \sum_M \chi(M)\ k(z, Mz').$$

If $k(z, z')$ is real, then $K(z,z')$ is Hermitian. For

$$\bar{K}'(z', z) = \sum_M \bar{\chi}'(M)\ \bar{k}\overset{sym}{(z', Mz)} = \sum_M \chi(M^{-1})\ k(z, M^{-1}z') = \sum_M \chi(M) k(z, Mz')$$

$$= K(z, z').$$

In any case $K(z, z')$ is normal ~~since~~ even if $k(z, z')$ is not real.

Suppose $\iint_{\mathcal{D}} |F|^2 \frac{dx\,dy}{y^2} < \infty$. Then $\exists_n$ eigen vectors $F_1, F_2, \cdots$ & these will be complete. This was due to the fact that the kernels formed a complete set in L_2.

If $(y^2\Delta + \frac{1}{4} + r_i^2)F = 0$, $\pm r_i$ are called eigenvalues.

One gets $\sum_i h(r_i)\ \bar{F}_i'(z) F_i(z) = K(z, \bar{z}')$

Taking traces, $\sum_i h(r_i) = 2 \iint_{\mathcal{D}} \sigma\left(K(z, z)\right) \frac{dx\,dy}{y^2}$.

$\pm r$

If h is the square of a suitable function, then the formula is valid. Also if $h(r) = O\left(\frac{1}{(1+|r|)^{k+\epsilon}}\right)$ & also for the product of such functions since $h_1 h_2 = \left(\frac{h_1 + h_2}{2}\right)^2 - \left(\frac{h_1 - h_2}{2}\right)^2$. We simplify the integral getting

$$2 \sum_M \sigma(\chi(M)) \iint_{\mathcal{D}} k(z, Mz) \frac{dx\,dy}{y^2} = 2 \sum_{\{M\}_\Gamma} \sigma(\chi\{M\}_\Gamma) \sum_{M_0 \in \{M\}_\Gamma} \iint_{\mathcal{D}} k(z, M_0 z) \frac{dx\,dy}{y^2}$$

Now $M_0 = N^{-1} M N$ & so the integral becomes

$$\iint_{\mathcal{D}} k(z, N^{-1}MN z) \frac{dx\,dy}{y^2} = \iint_{\mathcal{D}} k(Nz, MNz) \frac{dx\,dy}{y^2}$$

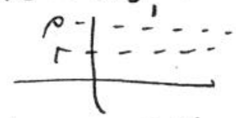
$$= \iint_{N\mathcal{D}} k(z, Mz) \frac{dx\,dy}{y^2}$$

Now M_0 is obtained when & only when N is replaced by TN with $TM = MT$. Hence letting N run through a system & if $D_M = \sum N\mathcal{D}$, then ~~on summing~~ D_M is a fundamental domain & the integral becomes

$$\sum_i h(r_i) = 2\sum_{\{M\}} \sigma(\chi\{M\}) \iint_{D_M} k(z, Mz)\,\frac{dx\,dy}{y^2}$$

$$= 2\eta k(0)\,\overset{\text{dist area}}{A(\mathcal{D})} + 2\sum_{\text{hyper.}} + 2\sum_{\text{ellip.}} + 2\sum_{\text{parab.}}$$

With $M = P^k$ we have $\{M\}_\Gamma = \{P\}_\Gamma^k$. We make a similarity trans. not necessarily by an element in Γ which makes the fixed pts 0 & ∞ & P take the form ρz with $\rho > 1$, $\rho = N(P)$, $Mz = \rho^k z$. Consider the fundamental domain $1 \leq y < \rho$



$$\int_1^\rho \int_{-\infty}^{\infty} k(z, \rho^k z)\,\frac{dx\,dy}{y^2} = \int_1^\rho \int_{-\infty}^{\infty} k\left(\frac{|z-\rho^k z|^2}{\rho^k y^2}\right)\frac{dx\,dy}{y^2}$$

$$= \int_1^\rho \int_{-\infty}^{\infty} k\left(\frac{(\rho^k-1)^2}{\rho^k}\left(1+\frac{x^2}{y^2}\right)\right)\frac{dx\,dy}{y^2}$$

$$= \int_1^\rho \int_{-\infty}^{\infty} k\left((\rho^{k/2}-\rho^{-k/2})^2(1+x^2)\right)\frac{dx\,dy}{y}$$

$$= \log\rho \int_{-\infty}^{\infty} k\left((\rho^{k/2}-\rho^{-k/2})^2(1+x^2)\right)dx$$

$$\frac{x'}{\rho^{k/2}-\rho^{-k/2}} = x^*$$

$$= \frac{\log\rho}{\rho^{k/2}-\rho^{-k/2}} \int_{-\infty}^{\infty} k\left\{(\rho^{k/2}-\rho^{-k/2})^2 + x^2\right\}dx$$

$$= \frac{\log\rho}{\rho^{k/2}-\rho^{-k/2}}\, Q\left((\rho^{k/2}-\rho^{-k/2})^2\right)$$

$$= \frac{\log\rho}{\rho^{k/2}-\rho^{-k/2}}\, Q\left(\rho^k - 2 + \rho^{-k}\right)$$

$$= \frac{1}{k}\underbrace{\frac{k\log\rho}{\rho^{k/2}-\rho^{-k/2}}}_{\text{depends only on class in } G.}\ \underset{\text{depends on } M \text{ only}}{g(k\log\rho)} = \frac{1}{k}\,\frac{k\log\rho/2}{\sinh\left(\frac{k\log\rho}{2}\right)}\, g(k\log\rho)$$

$n \times n$

Hence

$$\sum_i h(r_i) = \overset{?}{2}\eta k(0) A(\mathcal{D}) + 2\sum_{\{P\}}\sum_{k=1}^{\infty} \frac{\sigma(\chi^k\{P\})\log N\{P\}}{N(\{P\})^{k/2} - (N\{P\})^{-k/2}}\, g(k\log N\{P\}) + 2\sum_{\text{ellip}} + 2\sum_{\text{parab.}}$$

Now $k(0) = -\frac{1}{\pi}\int_0^\infty \frac{dQ(w)}{\sqrt{w}} = -\frac{1}{\pi}\int_0^\infty \frac{dg(u)}{e^{u/2}-e^{-u/2}}$ $\qquad e^u+e^{-u}-2=w$

$$= -\frac{1}{\pi}\int_0^\infty \frac{g'(u)}{e^{u/2}-e^{-u/2}}\,du = -\frac{1}{2\pi}\int_{-\infty}^\infty \frac{g'(u)}{e^{u/2}-e^{-u/2}}\,du$$

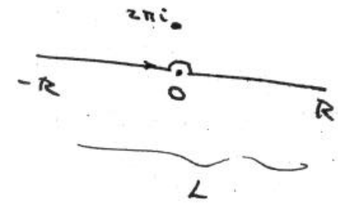
Hence ~~$g'(u) = \frac{1}{2\pi}\int_{-\infty}^\infty ir\,h(r)\,e^{iru}\,dr$ Also~~

$$g(u) = \frac{1}{\pi}\int_0^\infty h(r)\cos ru\,dr, \qquad g'(u) = -\frac{1}{\pi}\int_0^\infty r\,h(r)\sin ru\,dr$$

so

$$k(0) = \frac{1}{2\pi^2}\int_0^\infty r\,h(r)\,dr \int_{-\infty}^\infty \frac{\sin ru}{e^{u/2}-e^{-u/2}}\,du$$

Inside integral is calculated as follows.



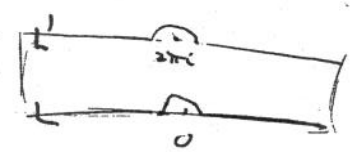
$\int_{-R}^{R}$:

$$\int_L = \frac{1}{2i}\int_L \frac{e^{iru}}{e^{u/2}-e^{-u/2}}\,du - \frac{1}{2i}\int_L \frac{e^{-iru}}{e^{u/2}-e^{-u/2}}\,du$$

$$= \quad '' \quad + \frac{1}{2i}\int_L$$

not same

$$= \frac{1}{2i}\left\{2\,I_r + 2\pi i\right\}$$



$I_r = \int_L = \int_{L'} - 2\pi i\cdot e^{-2\pi r} = I_r' - 2\pi i\,e^{-2\pi r}$ residue theorem

& also $I_r' = -e^{-2\pi r}\,I_r$ shift axis of integration

$$I_r = -e^{-2\pi r}\,I_r - 2\pi i\,e^{-2\pi r}$$

$$I_r = -\frac{2\pi i\,e^{-2\pi r}}{1+e^{-2\pi r}} = -\frac{2\pi i}{e^{2\pi r}+1}$$

$$\int_L = \frac{1}{2i}\,2\pi i\left\{1-\frac{2}{e^{2\pi r}+1}\right\} = \pi\,\frac{e^{2\pi r}-1}{e^{2\pi r}+1} = \pi\,\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}} = \pi\tanh \pi r$$

$$k(0) = \frac{1}{2\pi}\int_0^\infty h(r)\, r\, \frac{e^{\pi r} - e^{-\pi r}}{e^{\pi r} + e^{-\pi r}}\, dr$$

Letting $\frac{A(\mathcal{D})}{2\pi} = \mu_D$ we have $\mu_D + \sum \frac{1}{m}$ ~~is~~ an integer.

$$\cancel{\#} n\, k(0)\, A(\mathcal{D}) = \cancel{\#} n k(0)\, 2\pi \mu_D = \cancel{X} n \mu_D \int_{-\infty}^{\infty} h(r)\, r \frac{e^{\pi r} - e^{-\pi r}}{e^{\pi r} + e^{-\pi r}}\, dr.$$

Suppose $h(r) = e^{-r^2/R^2}$

$$\left| \int_{-\infty}^{\infty} \right| \le \int_0^\infty e^{-r^2/R^2}\, r\, dr = R^2$$

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-r^2/R^2 + iru}\, dr$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{r}{R} - \frac{iRu}{2}\right)^2} \cdot e^{-R^2u^2/4}\, dr$$

$$= \frac{1}{2\pi} e^{-R^2u^2/4}\, R \int_{-\infty}^{\infty} e^{-u^2}\, du$$

$$= \frac{1}{2\sqrt{\pi}}\, R e^{-R^2u^2/4}$$

Let $N(R)$ be the number of r such that $|r| \le R$. Then

$\frac{N(R)}{e} < n \mu_D R^2 + o(1)$. Hence $N(R) = O(R^2)$.

If $h(r) = O\left(\frac{1}{(1+r^2)^{1+\varepsilon}}\right)$, let $h_\varepsilon(r) = h(r)\, e^{-\varepsilon r^2}$; then

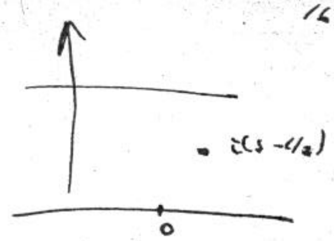
$h_\varepsilon(r) = O\left(\frac{1}{(1+r^2)^{2+\varepsilon}}\right)$ $\therefore$ $\sum h_\varepsilon(r)$ conv. unif. By taking the limit as $\varepsilon \to 0$ we also get the validity of the formula if

$h(r) = O\left(\frac{1}{(1+r^2)^{1+\varepsilon}}\right)$.

Let $h(r) = \frac{1}{(s-\frac{1}{2})^2 + r^2} - \frac{1}{(a-\frac{1}{2})^2 + r^2}$. $\quad a > 1$, $R(s) > 1$, ~~$R(a) > 1$~~, $R(s) < 0$

~~That~~ Now for $\mathcal{R}(s) > 1$

$$\cancel{\sum \left\{ \frac{1}{(s-\frac{1}{2})^2 + r^2} - \frac{1}{(a-\frac{1}{2})^2 + r^2} \right\}}$$



$$g(u) = g_s(u) - g_a(u) \quad \text{with}$$

$$g_s(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(s-1/2)^2+r^2}\, e^{iru}\, dr.$$

Let $u \geq 0$. Poles are at $r = \pm i\,(s-1/2)$, $r = i\,(s-1/2)$ if $\Re(s) > 1/2$. Move path of integration up & get

$$g_s(u) = \frac{1}{2\pi}\, \frac{e^{-u(s-\frac{1}{2})}}{2i\,(s-1/2)} \cdot 2\pi i = \frac{1}{2}\,\frac{e^{-u(s-1/2)}}{s-1/2}$$

& for $u \leq 0$ can use $|u|$ instead of u. Then

$$g(u) = \frac{1}{2}\left\{ \frac{e^{-|u|(s-1/2)}}{s-1/2} - \frac{e^{-|u|(a-1/2)}}{a-1/2} \right\}$$

We have $k \log N\{P\} \geq 0$. Hence if $N = N\{P\}$

$$2\sum_{\{P\}} \sum_{k=1}^{\infty} \frac{\sigma(\chi^k\{P\})\log N\{P\}}{(N\{P\})^{k/2} - (N\{P\})^{-k/2}}\, g(k \log N\{P\})$$

$$= \sum_{\{P\}} \sum_{k=1}^{\infty} \sigma(\chi^k\{P\}) \frac{\log N}{N^{k/2} - N^{-k/2}} \left\{ \frac{e^{-k\log N\cdot(s-1/2)}}{s-1/2} - \frac{e^{-k\log N\cdot(a-1/2)}}{a-1/2} \right\}$$

$$= \sum_{\{P\}} \sum_{k=1}^{\infty} \sigma(\chi^k\{P\}) \log N \left\{ \frac{1}{s-1/2}\,\frac{N^{-ks}}{1-N^{-k}} - \frac{1}{a-1/2}\,\frac{N^{-ka}}{1-N^{-k}} \right\}$$

$$= \frac{1}{s-1/2}\sum_{\{P\}} \sum_{k=1}^{\infty} \sigma(\chi^{(k)}\{P\})\, N^{-ks}\frac{\log N}{1-N^{-k}} - \frac{1}{a-1/2}\sum\sum \sigma\, N^{-ka}\frac{\log N}{1-N^{-k}}$$

$$= H(s) - H(a).$$

$$H(s) = \frac{1}{s-1/2} \sum_{\{P\}} \sum_{k=1}^{\infty} \sum_{\nu=0}^{\infty} \sigma(\;)\cdot \log N \; N^{-ks-k\nu}$$

$$= \frac{1}{s-1/2}\,\sigma \sum_{\{P\}} \sum_{\nu=0}^{\infty} \log N \cdot \sum_{k=1}^{\infty} N^{-(s+\nu)k}\, \chi\{P\}^k$$

$$= \frac{1}{s-1/2}\,\sigma\Big(\sum_{\{P\}} \sum_{\nu=0}^{\infty} \log N \cdot \frac{N^{-(s+\nu)}\chi\{P\}}{E - N^{-(s+\nu)}\chi\{P\}}\Big)$$

$$= \frac{1}{s-\frac{1}{2}}\, \sigma \sum_{\{P\}} \sum_{\nu=0}^{\infty} \frac{\log N \cdot \chi(\ )\, N^{-s-\nu}}{E - \chi(\ )\, N^{-s-\nu}}$$

Let $Z_\Gamma(s,\chi) = \prod_{\{P\}} \prod_{\nu=0}^{\infty} \det\{E - \chi\{P\}(N\{P\})^{-s-\nu}\}$

+ converges for $\Re(s) > 1$.

$\log \det A = \sigma(\log A)$ diagonal form

$\log \det \{\ \} = \sigma\left(\log\left(E - \chi\{P\}\, N\{P\}^{-s-\nu}\right)\right)$

logarithmic derivative is $\sigma\left(\frac{\log N(P) \cdot \chi(P) N(P)^{-s-\nu}}{E - \chi(P) N(P)^{-s-\nu}}\right)$. Then

$$H(s) = \frac{1}{s-1/2} \frac{Z'_\Gamma(s,\chi)}{Z_\Gamma(s,\chi)}.$$

$$\therefore \quad 2\sum_{P} \sum_{k=1}^{\infty} \cdots = \frac{1}{s-1/2} \frac{Z'_\Gamma}{Z_\Gamma}(s,\chi) - \frac{1}{a-1/2} \frac{Z'_\Gamma}{Z_\Gamma}(a,\chi)$$

We have to calculate

$$\int_{-\infty}^{\infty} r \frac{e^{\pi r} - e^{-\pi r}}{e^{\pi r} + e^{-\pi r}} \left\{ \frac{1}{(s-\frac{1}{2})^2 + r^2} - \frac{1}{(a-\frac{1}{2})^2 + r^2} \right\} dr$$

$e^{2\pi r} + 1 = 0$

$e^{2\pi r} = e^{\pi i + 2k\pi i}$

$r = \frac{\pi i + 2k\pi i}{2\pi}$

$= \frac{i}{2} + ki$

$$\frac{1}{(s-\frac{1}{2})^2 + r^2} = \frac{1}{2r} \left\{ \frac{1}{r + i(s-\frac{1}{2})} + \frac{1}{r - i(s-\frac{1}{2})} \right\}$$

$$\int_{-\infty}^{\infty} \frac{e^{\pi r} - e^{-\pi r}}{e^{\pi r} + e^{-\pi r}} \left\{ \frac{1}{r + i(s-1/2)} - \frac{1}{r + i(a-1/2)} \right\} dr$$

$$= \sum_{k=0}^{\infty} 2\pi i \frac{1}{\pi i} \left(\frac{1}{s+k} - \frac{1}{a+k} \right) = 2 \sum_{k=0}^{\infty} \left(\frac{1}{k+s} - \frac{1}{k+a} \right)$$

$$2n\mu_D \sum_{m=0}^{\infty} \left(\frac{1}{m+s} - \frac{1}{m+a} \right) = \frac{n\mu_D}{s-1/2} \sum_{m=0}^{\infty} \left(\frac{2s-1}{m+s} - \frac{2s-1}{m+a} \right) = -\frac{n\mu_D}{s-1/2} \sum_{m=0}^{\infty} \left\{ \frac{2m+1}{m+s} - 2 + \frac{2s-1}{m+a} \right.$$

$$\cancel{\;} = \frac{Z'}{Z}(s,\chi) - \frac{s-1/2}{a-1/2}\frac{Z'}{Z}(a,\chi) + n\mu_D \sum_{m=0}^{\infty}\left(\frac{2m+1}{m+s} - 2 + \frac{2s-1}{m+a}\right)$$

Integral coeff. for the residue.

#7. ~~D~~ec. 7, 1954

If $\frac{A(D)}{2\pi} = \mu_D =$ integer (can be shown to be even)

$$\sum_r h(r) = \mu n \int_{-\infty}^{\infty} h(r)\,\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}\,r\,dr + 2\sum_{\{P\}}\sum_{k=1}^{\infty}\frac{\sigma(\chi^k(\{P\}))\log N\{P\}}{(N\{P\})^{k/2}-(N\{P\})^{-k/2}}$$

$$\cdot\, g(k\log N\{P\})$$

For $h(r) = \dfrac{1}{(s-\frac12)^2+r^2} - \dfrac{1}{(a-\frac12)^2+r^2}$ we obtained

$$-c(s-1/2)$$

$$\sum_r \left\{\frac{s-\frac12}{(s-\frac12)^2+r^2} - \frac{s-\frac12}{(a-\frac12)^2+r^2}\right\} + \frac{Z'}{Z}(s,\chi) - \frac{s-1/2}{a-\frac12}\frac{Z'}{Z}(a,\chi)$$

over all value

$$+ n\mu \sum_{k=0}^{\infty}\left(\frac{2k+1}{s+k} - 2 + \frac{2s-1}{k+a}\right) \qquad =$$

satisfying $(y^2\Delta + \frac14 + r^2)F = 0$ by $r \gtreqless 0$.

On integrating + using $'$ to indicate only those with $\Re(r) \geq 0$ + those with $\Re(r) = 0$ just once

$$\log Z(s,\chi) - \frac{c'}{2}(s-\tfrac12)^2 = \sum_{r\geq 0}{}' \left\{\log\frac{(s-\frac12)^2+r^2}{(a-\frac12)^2+r^2} - \frac{(s-\frac12)^2}{(a-\frac12)^2+r^2}\right\}$$

$+d(s-\frac12)$

$+d'\xi$

$$\cancel{+ \mu n \sum_{k=0}^{\infty}\left\{(2k+1)\log(1+\tfrac{s}{k}) - 2s + \frac{(s-\frac12)^2}{k+a}\right\}}$$

$$+ \mu n \sum_{k=0}^{\infty}{}^{*}\left\{(2k+1)\log(1+\tfrac{s}{k}) - \frac{2k+1}{k}s + \frac{2k+1}{2k^2}s^2\right\}$$

$\therefore$ on exponentiating

$$Z(s,\chi) = \prod{}' \frac{(s-\frac12)^2+r^2}{(a-\frac12)^2+r^2}\, e^{-\frac{(s-\frac12)^2}{(a-\frac12)^2+r^2}} \cdot \prod_{k=0}^{\infty}{}^{*}\left\{\left(1+\frac{s}{k}\right)^{2k+1} e^{-\frac{2k+1}{k}s + \frac{2k+1}{2k^2}s^2}\right\}^{n\mu}$$

$$\cdot\, e^{\alpha(s-1/2)^2 + \beta(s-\frac12)+\gamma}$$

$$= H(s)\cdot \frac{1}{G(s)^{n\mu}}\, e^{\alpha(s-1/2)^2+\beta(s-1/2)+\gamma}$$

with zero at $s = \frac12 \pm ir$; $s = -k$, multiplicity $(2k+1)n\mu$. This is an integral function of order 2

$$G(s)^{n\mu}\, Z(s,\chi) = e^{\alpha(s-1/2)^2+\beta(s-1/2)+\gamma}\, H(s).$$

$$G(1-s)^{n\mu}\, Z(1-s,\chi) = e^{\alpha(s-1/2)^2+\beta(\frac12-s)+\gamma}\, H(1-s)$$

$$= e^{\beta(1-2s)}\, G(s)^{n\mu} Z(s,\chi)$$

$$\mu n\int_{-\infty}^{\infty} r h(r) \underline{\qquad}\, dr = \mu n \frac{1}{i}\int_{-\infty}^{\infty}\left\{\frac{1}{s-\frac12-ir} - \frac{1}{s-\frac12-ir}\right\}\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}\, dr$$

Replacing s by $1-s$ yields a new term

$$-\mu n\, \cancel{2\pi i}\; 2\pi i\, \frac{e^{\pi i(s-1/2)} - e^{-\pi i(s-1/2)}}{e^{\pi i(s-1/2)} + e^{-\pi i(s-1/2)}}\,(s-\tfrac12)$$ a term

$$= 2\pi\mu n(s-\tfrac12)\tan\pi(s-\tfrac12)$$

The integral of this will replace the product $\frac{1}{G(s)}$.

We had $N(R) = O(R^2)$. By using the product we can get

$$N(R) = \mu n R^2 + O(R)$$

and

dimensional, principal char.

$$\sum_{N\{P\}\le x} \sigma(\chi\{P\}) \sim \frac{x}{\log x}$$ error term $O(x^{\frac23+\epsilon})$

Same sort of average of the kind

$$\sum_{N\{P\}\le x} \sigma(\chi\{P\})\log\frac{x}{N\{P\}}$$ .. .. $O(x^{\frac12+\epsilon})$

All this on the assumption of no elliptic transformations.

#8. Dec. 15, 1954

We now consider groups with elliptic transformations. There is one R, say, which is a rotation thru $2\pi/m$ and the others are of the form R^{k}, $k=1,\dots,m-1$. The contribution of the elliptic transformations is

$$S = 2\sum_{\{R\}}\sum_{\nu=1}^{m-1}\frac{1}{m}\sigma(\chi^{\nu}\{R\})\iint_{H} k(z, R^{\nu}z)\,\frac{dx\,dy}{y^2}$$

Taking the elliptic trans. to be a rotation about i of α we get $\dfrac{z\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{-z\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}$

$$\frac{\left|z - \dfrac{z\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{-z\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}\right|^2}{y\,\dfrac{y}{\left|-z\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right|^2}} = \sin^2\frac{\alpha}{2}\cdot\frac{|z^2+1|^2}{y^2}.$$

$= $ function of the distance ρ

geodesic circle

$= F(\rho)$

The integral is

$$\pi\int_0^\infty F(\rho)(e^{\rho}-e^{-\rho})^2\,d\rho$$

$$I = \pi\int_0^\infty k\left(\cancel{\sin}\,(e^{\rho}-e^{-\rho})^2\sin^2\frac{\alpha}{2}\right)\cdot(e^{\rho}-e^{-\rho})\,d\rho = \pi\int_0^\infty k\left(\beta^2(e^{\rho}-e^{-\rho})^2\right)(e^{\rho}-e^{-\rho})\,d\rho$$

where $\beta = \left|\sin\frac{\alpha}{2}\right|$. Let $t = \beta^2(e^{\rho}-e^{-\rho})^2$. $\therefore\ e^{\rho}-e^{-\rho} = \frac{\sqrt{t}}{\beta}$

and $e^{\rho}+e^{-\rho} = \frac{\sqrt{t+4\beta^2}}{\beta}$. And

$$(e^{\rho}-e^{-\rho})\,d\rho = d(e^{\rho}+e^{-\rho}) = \frac{1}{2\beta}\frac{dt}{\sqrt{t+4\beta^2}}.$$

Hence the integral is

$$I = \frac{\pi}{2\beta}\int_0^\infty k(t)\,\frac{dt}{\sqrt{t+4\beta^2}} = -\frac{1}{2\beta}\int_0^\infty dQ(w)\int_0^w\frac{dt}{\sqrt{(w-t)(t+4\beta^2)}}$$

Let $u^2 = \dfrac{w-t}{t+4\beta^2}$ and

$$\underline{I} = -\frac{1}{2\beta}\int_0^\infty dQ(w)\cdot 2\int_0^{\frac{\sqrt{w}}{2\beta}}\frac{du}{1+u^2} = -\frac{1}{\beta}\int_0^\infty \arctan\frac{\sqrt{w}}{2\beta}\,dQ(w)$$

$$= +\frac{1}{\beta}\int_0^\infty\frac{1}{1+\frac{w}{4\beta^2}}\,\frac{1}{2\beta}\,\frac{1}{2}\frac{1}{\sqrt{w}}\cdot Q(w)\,dw$$

$$= +\int_0^\infty\frac{Q(w)}{(w+4\beta^2)\sqrt{w}}\,dw \qquad w = e^{u}+e^{-u}-2$$

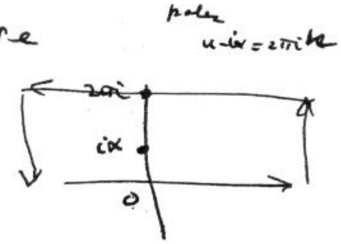
$$= \int_0^\infty\frac{g(u)}{e^{u}+e^{-u}-2\cos\alpha}\,\frac{e^{u}-e^{-u}}{e^{u/2}-e^{-u/2}}\,du = \frac{1}{2}\int_{-\infty}^{\infty}\frac{g(u)\,(e^{u/2}+e^{-u/2})\,du}{e^{u}+e^{-u}-2\cos\alpha}$$

Let $\zeta = e^{i\alpha}$. Then the coefficient of $g(u)$ is

$$\frac{e^{u/2}+e^{-u/2}}{e^{u}+e^{-u}-\zeta-\frac{1}{\zeta}} = \frac{\zeta}{\zeta-1}\left\{\frac{e^{u/2}}{e^{u}-\zeta}+\frac{e^{-u/2}}{e^{-u}-\zeta}\right\}$$

$$I = \frac{\zeta}{\zeta-1}\int_{-\infty}^{\infty}\frac{g(u)\,e^{u/2}}{e^{u}-\zeta}\,du = \cancel{\frac{\zeta}{\zeta-1}}\int_{-\infty}^{\infty}\frac{1}{2\pi}h(r)\,dr\left\{\frac{\zeta}{\zeta-1}\int_{-\infty}^{\infty}\frac{e^{(\frac{1}{2}+ir)u}}{e^{u}-\zeta}\,du\right\}$$

Inside integral J is evaluated by contour integration. We take $0\leq\alpha<2\pi$. Hence



$$J = 2\pi i\, e^{-i\frac{\alpha}{2}-\alpha r} - e^{-2\pi r}J$$

$$J = \frac{2\pi i\, e^{-i\frac{\alpha}{2}-\alpha r}}{1+e^{-2\pi r}}$$

$$\frac{\zeta}{\zeta-1}J = 2\pi i\cdot\frac{e^{i\alpha/2}}{e^{i\alpha}-1}\,\frac{e^{-\alpha r}}{1+e^{-2\pi r}} = \frac{2\pi i}{e^{i\alpha/2}-e^{-i\alpha/2}}\,\frac{e^{-\alpha r}}{1+e^{-2\pi r}} = \frac{\pi}{\sin\frac{\alpha}{2}}\,\frac{e^{-\alpha r}}{1+e^{-2\pi r}}$$

and

$$I = \frac{1}{2\sin\frac{\alpha}{2}}\int_{-\infty}^{\infty}\frac{e^{-\alpha r}}{1+e^{-2\pi r}}\,h(r)\,dr$$

Now $\alpha = 2\pi\nu/m$ with $\nu = 1,\dots,m-1$ and

$$S = \sum_{\{R\}}\sum_{\nu=1}^{m-1}\frac{\sigma(\chi^{\nu}\{R\})}{\cancel{2\pi}}\,\frac{1}{m\sin\frac{\pi\nu}{m}}\int_{-\infty}^{\infty}\frac{e^{-2\pi r\nu/m}}{1+e^{-2\pi r}}\,h(r)\,dr$$

This is valid with certain restrictions & some of these can now be removed.
The whole formula takes the shape:

$$\sum h(r) = \sigma(E)\,\frac{A(\mathcal{D})}{2\pi}\int_{-\infty}^{\infty}h(r)\,\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}\,r\,dr$$

$$+\sum_{\{R\}}\sum_{\nu=1}^{m-1}\frac{\sigma(\chi^{\nu}\{R\})}{m\sin\frac{\pi\nu}{m}}\int_{-\infty}^{\infty}h(r)\,\frac{e^{-2\pi r\nu/m}}{1+e^{-2\pi r}}\,dr$$

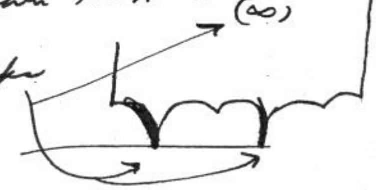
$$+\ 2 \sum_{\{P\}} \sum_{k=1}^{\infty} \frac{\sigma(\chi^k\{P\}) \log N\{P\}}{(N\{P\})^{k/2} - (N\{P\})^{-k/2}}\, g(k \log N\{P\}).$$

The # of eigenvalues $\leq R$ is $O(R^2)$ as can be proved as in the preceding case.

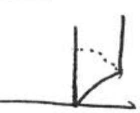
Now $\frac{A(\mathcal{D})}{2\pi} + \sum \frac{1}{m}$ is an integer. And indeed $\frac{A(\mathcal{D})}{4\pi} + \sum \frac{m-1}{2m}$ is an integer as will be shown later. This enables us to show that the zeros have integral multiplicity & the function is therefore analytic.

Consider the non-compact case but suppose $A(\mathcal{D}) = \iint_{\mathcal{D}} \frac{dx\,dy}{y^2} < \infty$ ~~has the boundary curve~~ is compact.

In this case also, a fundamental domain & only a finite no. of sides etc. The non-compactness can only arise when there are parabolic transformations which give rise to cusps.



It is convenient to put a given cusp at ∞.



~~The~~ We also arrange the fundamental domain so as to have inequivalent cusps which is not the case in the illustration to the right.

We have to investigate $K = \sum \chi(M)\, k(z, Mz')$.

~~In~~ Our earlier argument ~~at only and~~ would also apply to compact subsets of the fundamental region. Hence the kernel only has to be investigated when z, z' are near various cusps. We suppose ~~that~~ $\Im z > c$ so that z is in a neigh. of ∞.

Let S be the transformation $z \to z+1$. Then

$$K = \sum_{n=-\infty}^{\infty}{}' \chi^n(S)\, k(z, z+n) + k(z,z') + \sum_{M \in \Gamma}{}'' \chi(M)\, k(z, Mz')$$

i.e. $n \neq 0$ — i.e. those not in first Σ'.

$k\left(\frac{|z-z'|^2}{yy'}\right) = k\left(\frac{y}{y'} + \frac{y'}{y} - 2 + \frac{(x-x')^2}{yy'}\right)$

Consider $J = \sum_{M \in \Gamma}'' |k(z, Mz')|$ & we have assumed $k(z, z') = O\left(\frac{1}{\left\{\frac{y}{y'} + \frac{y'}{y} + \frac{(x-x')^2}{yy'}\right\}^{1+\varepsilon}}\right)$ if z' is in the fundamental region, then & suppose $\Im z \geq 1$. Now, $Mz' = \frac{az'+b}{cz'+d}$ and $y'_M = \Im Mz' = \frac{y'}{|cz'+d|^2} < 1$.

Hence if $k(t) = O\frac{1}{(2+t)^{1+\varepsilon}}$

$$k(z, Mz') = O\left\{\frac{y}{y'_M} + \frac{y'_M}{y} + \frac{(x-x'_M)^2}{yy'_M}\right\}^{-(1+\varepsilon)}$$

Now Mz' lies below the circles ... $\therefore$ $y'_M < c$. Hence

$$k(z, Mz') = O\left(\frac{y}{y'_M} + \frac{(x-x'_M)^2}{yy'_M}\right)^{-(1+\varepsilon)} = O\,\frac{y'^{\,1+\varepsilon}_M}{\left(y + \frac{(x-x'_M)^2}{y}\right)^{1+\varepsilon}}$$

Now take together all terms resulting from M by translations by n.

Then

$$\lambda = O\sum_M{}'' \, y'^{\,1+\varepsilon}_M \sum_{n=-\infty}^{\infty} \frac{1}{\left\{y + \frac{(x - x'_M + n)^2}{y}\right\}^{1+\varepsilon}}$$

$$= O\sum_M{}'' \, y'^{\,1+\varepsilon}_M \sum_{n=0}^{\infty} \frac{1}{\left(y + \frac{n^2}{y}\right)^{1+\varepsilon}}$$

$\begin{cases} n \leq y \\ n \geq y \end{cases}$ or by comparing with an integral

$$\frac{1}{y^{1+\varepsilon}} + \int_0^\infty \frac{du}{(y+u^2/y)^{1+\varepsilon}}$$

$$= O\left(\sum_M{}'' \, y'^{\,1+\varepsilon}_M \, y^{-\varepsilon}\right)$$

$$= O\left(y^{-\varepsilon}\sum_M{}'' \, y'^{\,1+\varepsilon}_M\right) = O\left(y^{-\varepsilon}\sum_M{}''\left\{\frac{y'}{|cz'+d|^2}\right\}^{1+\varepsilon}\right);$$

We will show later that this is $o(y^{-\varepsilon})$ as $z' \to$ cusp.

Hence the part $\sum_{M\in\Gamma}'' \chi(M)\, k(z, Mz') \to 0$

if one of the points tends to a cusp.

Among the parabolic transformations are $z \to z+\lambda$ & if λ is the smallest positive then is called the primitive transformation. Also there are the ones: $\frac{-\mu}{z-\xi} \to \frac{-\mu}{z-\xi} + \lambda$ & again take smallest positive λ.

If the cusps occur at $\xi_1, \dots, \xi_\kappa$ the transformation $z_j = -\frac{\mu_j}{z-\xi_j}$ the primitive one looks like $z_j \to z_j + 1$ if μ_j is taken suitably; if $\xi_j = \infty$, take $z_j = \mu_j z$. Then width of copy of ~~hyperbolic strip~~ fundamental domain is 1. Let the primitive transformation at ξ_j be S_j. If $|\chi(S_j) - E| \neq 0$ Then χ is said to be non-singular with respect to the cusp at ξ_j. " " " all cusps ξ_j.

We now assume that χ is non-singular " " ~~lecture by~~

~~We change the notation of the preceding~~

~~suppose $z^m \to \infty$. $\therefore$ take $y^m > c$. Now~~

$$\cancel{K(z, z') = \sum_{-\infty}^{\infty} \chi^n(S)\, k(z, S^n z') + \sum_{M \neq S^n}' \chi(M)\, k(z, Mz').}$$

~~Let $Mz = x_M + i y_M$.~~ If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $y_M = \frac{y}{|cz+d|^2}$.

We now consider the case of $\sum_{-\infty}^{\infty} \chi^n(S)\, k(z, z+n)$. By using a unitary transformation on K we can make $\chi(S)$ diagonal & $\therefore$ its diagonal elements have absolute value 1. If these elements are $e^{i\alpha_\nu}$ then ~~the elements of~~ the series look like

$$\sum_{n=-\infty}^{\infty} e^{in\alpha_\nu}\, k\left(\frac{(y-y')^2}{yy'} + \frac{(x-x'+n)^2}{yy'}\right).$$

We now assume that $k(t)$ is of bounded variation, $\int_0^\infty |dk(t)| < \infty$.

Suppose $e^{i\alpha_\nu} \neq 1$. Then

$$(1 - e^{i\alpha_\nu}) \Sigma = \Sigma\, e^{in\alpha_\nu} \left\{ k\left(\frac{y-y'}{yy'} + \frac{(x-x'-n)^2}{yy'}\right) - k\left(\quad\right)_{n \to (n-1)} \right\}$$

$$|1 - e^{i\alpha_\nu}|\, |\Sigma| \leq \Sigma\, |k(\quad) - k(\quad)|$$

$$\leq 2 \int_0^\infty |dk(t)| < \infty.$$

$\therefore \Sigma$ is bdd. If χ is non-singular with respect to a cusp then all $e^{i\alpha_\nu} \neq 1$ & $\therefore K$ is bdd.

If $e^{i\alpha_\nu} = 1$ then we have to deal with

$\Sigma = \sum_{-\infty}^{\infty} k(\quad)$. Now

$$\Sigma - \int_{-\infty}^{\infty} k\left(\frac{(y-y')^2}{yy'} + \frac{(x-x'-u)^2}{yy'}\right) du$$

$$= \Sigma - \sum_{n=-\infty}^{\infty} \int_n^{n+1} = O(1)$$

as can be seen. Hence

$$\Sigma = \int_{-\infty}^{\infty} k\left(\frac{(y-y')^2}{yy'} + \frac{u^2}{yy'}\right) du + O(1)$$

$$= \sqrt{yy'}\ Q\left(\frac{y}{y'} + \frac{y'}{y} - 2\right) du + O(1)$$ as can be proved

$$= \sqrt{yy'}\ g\left(\log \frac{y}{y'}\right) + O(1)$$

If $\frac{y}{y'} = O(1)$ then $g(\log \frac{y}{y'}) = O(1)$ & the first term $\to \infty$.

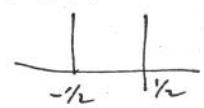
A sufficient condition for $\int_0^\infty |dk(t)| < \infty$ is that $h(r) = O\left(\frac{1}{|r|^{3+\varepsilon}}\right)$ & regular in $|\mathfrak{I}(r)| < \frac{1}{2} + \varepsilon$. This replaces the weaker $h(r) = O\left(\frac{1}{|r|^{2+\varepsilon}}\right)$.

$$\Sigma\, h(r) \# 2 \iint_{\mathcal{D}} \sigma(K(z,z))\frac{dx\,dy}{y^2}.$$

terms of elliptic & hyperbolic type treated before & the identity

- -

$$+ \quad \sigma\Big(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \chi^n(S)\ k(z, S^n z)\Big)$$



$$= \iint_{\substack{y>0 \\ |x|\le\frac{1}{2}}} \Sigma' \sigma(\chi^n(S))\ k(z, z+n)\ \frac{dx\,dy}{y^2}$$

$e^{i\alpha_\nu}$ $\quad \nu = 1, \dots, l. \qquad \sigma(\chi^n(S)) = \sum_{\nu=1}^{l} e^{in\alpha_\nu}$

$$\iint \Sigma' e^{in\alpha} k(z, z+n)\frac{dx\,dy}{y^2} = \iint \Sigma' e^{in\alpha} k\left(\frac{n^2}{y^2}\right)\frac{dx\,dy}{y^2}$$

$$= \int_0^\infty \Sigma' e^{in\alpha} k\left(\frac{n^2}{y^2}\right)\frac{dy}{y^2} = \lim_{\varepsilon\to 0+}\int_0^{1/\varepsilon}$$

$$= \lim_{\varepsilon\to 0} 2\sum_{n=1}^{\infty} \cos n\alpha \cdot \int_0^{1/\varepsilon} k\left(\frac{n^2}{y^2}\right)\frac{dy}{y^2}$$

$$= 2\lim_{\varepsilon\to 0}\sum_{n=1}^{\infty} \cos n\alpha \int_\varepsilon^\infty k(n^2u^2)\,du$$

$$= 2\lim_{\varepsilon\to 0}\sum_{n=1}^{\infty}\frac{\cos n\alpha}{n}\int_{n\varepsilon}^{\infty} k(u^2)\,du$$

$$= 2 \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} k(u^2) \sum_{n \le \frac{u}{\varepsilon}} \frac{\cos n\alpha}{n} \, du$$

$$\sum_{n \le x} \frac{\cos n\alpha}{n} = \log \left| \frac{1}{1 - e^{i\alpha}} \right| + O\left(\frac{1}{\sqrt{x}}\right) \qquad \text{or } O\left(\frac{1}{x}\right) \; ?$$

$$= 2 \log \frac{1}{|1 - e^{i\alpha}|} \int_0^{\infty} k(u^2)\, du + O\sqrt{\varepsilon} \int_0^{\infty} \frac{k(u^2)}{\sqrt{u}} \, du$$

On adding get

$$- \log \left| \prod (1 - e^{i\alpha_\nu}) \right| = - \log \left| \det(E - \chi(S)) \right| .$$

Letting $u^2 = t$ we get

$$- \log \left| \det(E - \chi(S)) \right| \underbrace{\int_0^{\infty} \frac{k(t)}{\sqrt{t}} \, dt}_{G(0) = g(0)} = - g(0) \log \left| \det(E - \chi(S)) \right|$$

$$2g(0) \log \left| \frac{1}{\det(E - \chi(S))} \right| \qquad \text{if } k(r) = O\left(\frac{1}{|r|^{6+\varepsilon}}\right)$$

& this condition can be replaced by $O\left(\frac{1}{|r|^{2+\varepsilon}}\right)$ as before. The new ~~terms~~ only very slightly alter the functional equation for the ~~Z~~-function.

We now assume that χ is singular for the first $\varkappa_1$ of the inequivalent $\xi_1, \dots, \xi_\varkappa$ on the real axis or ∞. We also keep the primitive transformation

$x_j + iy_j = z_j = -\frac{\lambda_j}{z-\xi_j}$ with λ_j fixed so that the primitive transformation amounts to translation by 1.

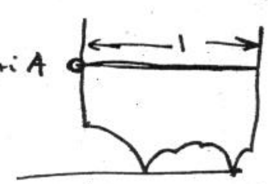
Let $\Re(s) > 1$. Let $\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in \Gamma$ correspond to ξ_j (?). ~~Consider~~ Then for some $\chi(S_j)$ will have some eigenvalue equal to 1. X we have $X\,\chi(S_j) = X$. Consider

$$E_{j,X}(z, s, \chi) = \sum_{\substack{M \in \Gamma \\ M\,(\text{mod } S_j)_{\text{left}}}} \frac{\bar{\chi}(M)\, y_j^s}{|c_j z_j + d_j|^{2s}}$$

which is an automorphic function having the desired properties.

$\{y^2\Delta + s(1-s)\}\, E_{j,X}(z,\chi) = 0$. It suffices to consider s real & χ identically 1 so that the series is

$$\sum_{\substack{M \in \Gamma \\ M\,(\text{mod } S_j)_{\text{left}}}} \frac{y_j^s}{|c_j z_j + d_j|^{2s}} \qquad s > 1.$$



Let $\tilde{\Sigma}$ be a partial sum of this series. Integrate this over a compact subset $\mathfrak{D}$ bounded away from the real axis & ∞. Then

$$\iint_{\mathfrak{D}} \tilde{\Sigma}\, \frac{dx_j\, dy_j}{y_j^2} \le \iint_{\mathfrak{D}} \Sigma\, \Im\big((Mz)_j\big)^s\, \frac{dx_j\, dy_j}{y_j^2} \le \iint_{\substack{|x_j| \le \frac{1}{2} \\ y_j \le A}} y_j^s\, \frac{dx_j\, dy_j}{y_j^2}$$

$$= \int_0^A y_j^{s-2}\, dy_j = \frac{A^{s-1}}{s-1} \quad \text{since } s > 1.$$

$\therefore$ Since $\tilde{\Sigma} \uparrow$ as the # of terms $\uparrow$, the series Σ converges almost everywhere to an integrable function. Consider a "circle" $C_\rho(z_0)$ with "center" z_0 and $\iint_{C_\rho(z_0)} y^s \frac{dx\,dy}{y^2} = A(\rho, s)\, y_0^s$ since it is an integral operator. And $A(\rho, s) > 0$. Likewise

$$\iint_{C_\rho(z_0)} \frac{y^s}{|cz+d|^{2s}} \frac{dx\,dy}{y^2} = A(\rho, s) \frac{y_0^s}{|cz_0+d|^{2s}}.$$

Taking $\mathfrak{D}$ to be a circle of this type we also get convergence at each point by combining these results; also uniform convergence.

Now assume χ is $\overset{|\log|}{1\times 1}$. Then

$$E_j(z,s,\chi) = \sum_{\substack{M \in \Gamma \\ M(\bmod S_j)\,\text{left}}} \frac{\overline{\chi}(M)\, y_j^s}{|c_j z_j + d|^{2s}}$$ Eisenstein series belonging to the cusp z_j.

.... Differ by a power of S_j on the left.

These will turn out to be linearly independent. Series is invariant under $x_j \to x_j + 1$. Hence there is a Fourier expansion

$$E_j(z,s,\chi) = \alpha_{jj}^{(0)}(y_j) + \sum_{n=-\infty}^{\infty}{}' \alpha_{jj}^{(n)}(y_j)\, e^{2\pi i n x_j}$$

with

$$\alpha_{jj}^{(n)}(y_j) = \int_{-\frac{1}{2}}^{1/2} E_j(z,s,\chi)\, e^{-2\pi i n x_j}\, dx_j .$$

Now $$E_j(z,s,\chi) = y_j^s + \sum_{c_j \neq 0} \sum_{0 \leq d_j < |c_j|} \frac{\chi_{c_j,d_j}}{|c_j|^{2s}} \sum_{m=-\infty}^{\infty} \frac{y_j^s}{|c_j z_j + \frac{d_j}{c_j} + m c_j|^{2s}}.$$

Hence

$$\alpha_{jj}^{(0)}(y_j) = y_j^s + \cdots .$$

We have to evaluate

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} \frac{y_j^s}{|z_j + \frac{d_j}{c_j} + m|^{2s}}\, dx_j = \sum_{m=-\infty}^{\infty} y_j^s \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx_j}{|z_j + \frac{d_j}{c_j} + m|^{2s}}$$

$$= \sum_{m=-\infty}^{\infty} \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \frac{y_j^s}{|z_j + \frac{d_j}{c_j}|^{2s}}\, dx_j$$

$$= \int_{-\infty}^{\infty} \frac{y_j^s}{|z_j + \frac{d_j}{c_j}|^{2s}}\, dx_j = \int_{-\infty}^{\infty} \frac{y_j^s}{|z_j|^{2s}}\, dx_j$$

$$= \int_{-\infty}^{\infty} \frac{y_j^s}{(x^2 + y_j^2)^s}\, dx = \int_{-\infty}^{\infty} y_j^s \frac{dx}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(x^2+y_j^2)t}\, dt$$

since ~~Also~~ $a^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at}\, dt$, $\Re(s) > 0$. Hence the integral is

$$\frac{y_j^s}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-y_j^2 t} \cancel{\int_{-\infty}^{\infty} e^{-x^2 t}\, dx} \sqrt{\frac{\pi}{t}}$$

$$= \sqrt{\pi}\, \frac{y_j^s}{\Gamma(s)} \int_0^{\infty} t^{s-\frac{3}{2}} e^{-y_j^2 t}\, dt = \sqrt{\pi}\, \frac{y_j^s}{\Gamma(s)} \frac{\Gamma(s-\frac{1}{2})}{y_j^{2s-1}}$$

$$= \sqrt{\pi}\, \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, y_j^{1-s}$$

$$\alpha_{jj}^{(0)}(y_j) = y_j^s + \sqrt{\pi}\, \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, y_j^{1-s} \sum_{c_j \neq 0} \sum_{0 \leq d_j < |c_j|} \frac{\overline{\chi}_{c_j,d_j}}{|c_j|^{2s}} .$$

If $n \neq 0$ the first term give 0 & we have to evaluate

$$\sum_{m=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{y_j^s\, e^{-2\pi i n x_j}}{|z_j + \frac{d_j}{c_j} + m|^{2s}}\, dx_j$$

$$= \sum_{-\infty}^{\infty} \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \frac{y_j^s\, e^{-2\pi i n x_j'}}{|z_j + \frac{d_j}{c_j}|^{2s}}\, dx_j \qquad x_j' = x_j + m \qquad x_j' = x_j + \frac{d_j'}{c_j}$$

$$= \int_{-\infty}^{\infty} \frac{y_j^s\, e^{-2\pi i n x_j}}{|z_j + \frac{d_j}{c_j}|^{2s}}\, dx_j = e^{2\pi i n \frac{d_j}{c_j}} \int_{-\infty}^{\infty} \frac{y_j^s\, e^{-2\pi i n x_j}}{|z_j|^{2s}}\, dx_j$$

We have to evaluate

$$\int_{-\infty}^{\infty} \frac{y^s e^{-2\pi i n x}}{(x^2+y^2)^s}\, dx = y^s \int_{-\infty}^{\infty} e^{-2\pi i n x} \frac{dx}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(x^2+y^2)t}\, dt$$

$$= \frac{y^s}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-y^2 t}\, dt \underbrace{\int_{-\infty}^{\infty} e^{-x^2 t - 2\pi i n x}\, dx}_{\sqrt{\frac{\pi}{t}}\, e^{-\frac{\pi^2 n^2}{t}}}$$

$$= \frac{y^s}{\Gamma(s)} \sqrt{\pi} \int_0^{\infty} t^{s-\frac{1}{2}} e^{-y^2 t - \frac{\pi^2 n^2}{t}} \frac{dt}{t} \qquad t = \frac{t'}{y} \pi |n|$$

$$= \frac{y^s}{\Gamma(s)} \sqrt{\pi} \left(\frac{\pi |n|}{y}\right)^{s-\frac{1}{2}} \int_0^{\infty} t'^{s-\frac{1}{2}} e^{-y\pi|n| t' - \frac{\pi^2 n^2}{\pi |n|} \frac{y}{t'}} \frac{dt'}{t'}$$

$$= \frac{|n|^{s-\frac{1}{2}} \pi^s \sqrt{y}}{\Gamma(s)} \int_0^{\infty} t^{s-\frac{1}{2}} e^{-\pi |n| y (t + \frac{1}{t})} \frac{dt}{t}$$

$$K_s(a) = \int_0^{\infty} t^s e^{-a(t+\frac{1}{t})} \frac{dt}{t}$$ a Bessel function, but not in any standard notation.

$$= O(e^{-(2-\varepsilon)a}) \text{ for large } a.$$

Then the series is

$$\frac{|n|^{s-\frac{1}{2}} \pi^s}{\Gamma(s)} \sqrt{y_j}\, e^{2\pi i n \frac{d_j}{c_j}} K_{s-\frac{1}{2}}(\pi |n| y_j) \quad \checkmark$$

and

$$\alpha_{jj}^{(n)}(y_j) = \frac{|n|^{s-\frac{1}{2}} \pi^s}{\Gamma(s)} \sqrt{y_j}\, K_{s-\frac{1}{2}}(\pi |n| y_j) \sum_{c_j \neq 0} \sum_{0 \leq d_j < |c_j|} \frac{\chi_{c_j, d_j}}{|c_j|^{2s}} e^{2\pi i n \frac{d_j}{c_j}}$$

$$= \frac{|n|^{s-\frac{1}{2}} \pi^s}{\Gamma(s)} \sqrt{y_j}\, K_{s-\frac{1}{2}}(\pi |n| y_j)\, L_{jj}^{(n)}(s, \chi)$$

where

$$L_{jj}^{(n)}(s, \chi) = \sum_{c_j \neq 0} \sum_{0 \leq d_j < |c_j|} \frac{\chi_{c_j, d_j}}{|c_j|^{2s}} e^{2\pi i n \frac{d_j}{c_j}}$$

From this we see that $E_j(z, s, \chi) = y_j^s + \pi \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} y_j^{1-s} \sum\sum +$ terms exponentially small for large y_j

Some of the lecture last time should have $\chi(M)$ replaced by $\bar\chi(M)$. For with $F(z) = \sum_{M\in\Gamma} \bar\chi(M)\varphi(Mz)$ we have

$$\chi(N)\,F(z) = \sum_{M\in\Gamma}\chi(N)\,\bar\chi(M)\,\varphi(Mz) = \sum_M \bar\chi(N^{-1}M^{*})\,\varphi(Mz)$$

$$= \sum \bar\chi(M)\,\varphi(MNz) = F(Nz)$$

In the case where the ~~transformation~~ matrix is not 1-1 we need $\overline{\chi(M)}'$ instead of $\bar\chi(M)$. Also the X belongs in the middle as $\bar\chi'(M)\,X\,\varphi(Mz)$.

We had

$$E_j(z, s, \chi) = \alpha_{jj}^{(0)}(y_j) + \sum_{n=-\infty}^{\infty}{}' \alpha_{jj}^{(n)}(y_j)\,e^{2\pi i n x_j}$$

$$\alpha_{jj}^{(0)}(y_j) = y_j^s + \sqrt{\pi}\,\frac{\Gamma(s-\frac12)}{\Gamma(s)}\,y_j^{1-s}\,L_{jj}^{(0)}(s,\chi)$$

$$L_{jj}^{(n)}(s,\chi) = \sum_{c_j\neq 0}\frac{1}{|c_j|^{2s}}\sum_{0\le d_j<|c_j|}\bar\chi_{c_j,d_j}\,e^{2\pi i n_j d_j/c_j}$$

$$\alpha_{jj}^{(n)}(y_j) = \frac{\pi^s|n|^{s-\frac12}}{\Gamma(s)}\sqrt{y_j}\;L_{jj}^{(n)}(s,\chi)\,K_{s-\frac12}(\pi|n|y_j), \quad n\neq 0$$

$$K_{s-\frac12}(\nu) = \int_0^\infty t^{s-\frac12}\,e^{-\nu(t+\frac1t)}\,\frac{dt}{t}$$

Consider now the behavior at another cusp. There are two cases depending on whether the function is singular there or not.

$$z_j = -\frac{\lambda_j}{z-\xi_j}, \qquad z_k = -\frac{\lambda_k}{z-\xi_k}$$

$\therefore$
$$z = \xi_j - \frac{\lambda_j}{z_j} \qquad z_k = -\frac{\lambda_k}{\xi_j-\xi_k-\frac{\lambda_j}{z_j}} = -\frac{\lambda_k z_j}{(\xi_j-\xi_k)z_j-\lambda_j}$$

$$= \frac{\alpha z_j}{\gamma z_j+\delta} \quad \text{with } \gamma\neq 0$$

also

$$\frac{y_j}{|c_j z+d_j|^2} = \frac{y_j}{|c_j^{(k)} z_j + d_j^{(k)}|^2} \quad \text{with } c_j^{(k)}\neq 0$$

Suppose $j\neq k \le \kappa_1$. We can determine the Fourier expansion. This works out as follows:

$$E_j(z,s,\chi) = \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,L^{(0)}_{jk}(s,\chi)\,y_k^{1-s} + \sum_{n=-\infty}^{\infty}{}' \ldots$$

if $j \neq k \leq \varkappa_1$.

Now there remains $\varkappa_1 < k \leq \varkappa$. Now $\chi(S_k) = e^{2\pi i\alpha} \neq 1$.

Then one gets

$$\sum_{n=-\infty}^{\infty} \alpha^{(n)}_{jk}(y_k)\, e^{2\pi i(n+\alpha)x_k}$$

with coeff. given by an expression similar to $\xi^{(n)}_{jj}$ with n replaced by $n+\alpha$. Since $n+\alpha \neq 0$, the $K_{s-\frac{1}{2}}(\pi|n+\alpha|y_k)$ tend exponentially to 0 as $y_k \to \infty$. Hence the Eisenstein series behaves very well.

If $j,k \leq \varkappa_1$ we have

$$L^{(0)}_{jk}(s,\chi) = L^{(0)}_{kj}(s,\bar\chi).$$

To prove this consider the fundamental domain

these cut-off circles (or cycles) correspond to [illegible] in the z_j plane

These are not to intersect each other.

We call the truncated domain $\bar{D}$.

Consider with $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$$\iint_{\bar D} \Big\{ E_j(z,s,\chi)\,y^2\Delta E_k(z,s,\bar\chi) - E_k(z,s,\bar\chi)\,y^2\Delta E_j(z,s,\chi) \Big\} \frac{dx\,dy}{y^2}$$

$= 0$ since the ~~integrands~~ operators are eigenfunctions with the factor $s(s-1)$

Use Green's formula to get

$$0 = \int_{\text{bdy of } \bar D} \Big\{ E_j \frac{\partial E_k}{\partial n} - E_k \frac{\partial E_j}{\partial n} \Big\}\, ds$$

On parts of bdy of $\bar D$ which are part of bdy of D, integral is 0 since the normal has opposite orientation on these + their symmetric parts. Hence

$$0 = \int_{B} \Big(E_j \frac{\partial}{\partial n} E_k - E_k \frac{\partial}{\partial n} E_j\Big)\, ds$$

where B is the set of arcs introduced. For the circles not corresponding to ξ_j, ξ_k the integral along line tends to 0 as the circle is shrunk to 0. Hence only the singles corresponding to ξ_j & ξ_k remain. The terms of the Fourier expansion are all small except the first so these ~~not~~ need not be considered.

Suppose $j = k$. We have

$$E_j(z,s,\chi) = y_j^s + \alpha_j(\chi)\, y_j^{1-s} + \cdots$$

$$E_j(z,s,\bar\chi) = y_j^s + \alpha_j(\bar\chi)\, y_j^{1-s} + \cdots$$

Integrating in the z_j plane along the line $y_j = A_j$ we have for $j=k$

$$o(1) = \int_{\substack{-1/2 \\ y_j = A_j}}^{1/2} = \left(E_j \frac{\partial E_k}{\partial y_k} - E_k \frac{\partial E_j}{\partial y_j} \right)_{y_j = A_j}$$

$$= \{ y_j^s + \alpha_j(\chi)\, y_j^{1-s} \} \{ s\, y_j^{s-1} + (1-s)\, \alpha_j(\bar\chi)\, y_j^{-s} \}$$

$$- \{ \qquad \} \{ \qquad \}$$

$$= (1-s)\{\alpha_j(\bar\chi) - \alpha_j(\chi)\} + s\{\alpha_j(\chi) - \alpha_j(\bar\chi)\}$$

$$= (1-2s)\{\alpha_j(\bar\chi) - \alpha_j(\chi)\}$$

Hence $\alpha_j(\bar\chi) - \alpha_j(\chi) = 0$ & so $L^{(0)}_{jk}(s,\chi) = L^{(0)}_{kj}(s,\bar\chi)$.

If $j \neq k$, the proof proceeds in a similar fashion.

We state the following:

<u>Lemma</u> Suppose $f(z)$ is given such that: ~~for $R(s) > 1$~~

(1) $\{y^2\Delta + s(1-s)\} f = 0$ for some s, $R(s) > 1$

(2) $f(Mz) = \chi(M)\, f(z)$ for $M \in \Gamma$

(3) $f(z) = O(e^{\varepsilon y_j})$ for $y_j \to \infty$ & every $\varepsilon > 0$, $1 \le j \le \kappa$

$z_j = -\frac{\lambda_j}{z - \xi_j} = x_j + i y_j$

then $f(z)$ is a linear combination of the Eisenstein series, $E_j(z,s,\chi)$, $1 \le j \le \kappa_1$.

Can replace (3) by some $\varepsilon < 2\pi$ if $1 \le j \le \kappa_1$; and for some $\varepsilon < 2\pi \min |n + \beta_j|$ if $\kappa_1 < j \le \kappa$.

#12. Feb. 8, 1955

We take the cusps $\xi_1, \cdots, \xi_\varkappa$ and suppose that χ is singular with respect to $\xi_1, \cdots, \xi_{\varkappa_1}$, $\varkappa_1 \leq \varkappa$.

To prove the lemma, we look at the Fourier series.

$$f(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x_j} g_n(y_j) \quad \text{if} \quad 1 \leq j \leq \varkappa_1$$

$$f(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i (n+\beta_j) x_j} g_n(y_j) \quad \text{if} \quad \varkappa_1+1 \leq j \leq \varkappa.$$

Condition (1) holds for the individual terms of the series. The operator is invariant under $z \to z_j$. Hence

$$\left\{ y_j^2 \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right) + s(1-s) \right\} g_0(y_j) = 0$$

$$x^2 y'' + a(1-a) y = 0$$

or

$$y_j^2 \, g_0''(y_j) + s(1-s) \, g_0(y_j) = 0$$

$\therefore$

$$g_0(y_j) = a \, y_j^s + b \, y_j^{1-s}$$

In the case $n \neq 0$ we have

$$y_j^2 \, g_n''(y_j) - 4\pi^2 n^2 y_j^2 g_n(y_j) + s(1-s) g_n(y_j) = 0$$

$$x^2 y'' - 4\pi^2 n^2 x^2 y + s(1-s) y = 0$$

$$x^2 y'' + (ax^2 + b) y = 0.$$

One solution, we already had, was

$$G_n(y_j) = \sqrt{y_j} \, K_{s-\frac{1}{2}}(\pi |n| y_j).$$

But this will not satisfy 3)

Hence we can find the other solution. Then which must be satisfied also by $g_n(y_j)$.

$$G: \quad y^2 g'' - 4\pi^2 n^2 y^2 g + s(1-s) g = 0$$

$$-g: \quad y^2 G'' - 4\pi^2 n^2 y^2 G + s(1-s) G = 0$$

$$\therefore \quad y^2 (G g'' - g G'') = 0 \qquad \therefore 0 = G g'' - g G'' = \frac{d}{dy}(G g' - G' g)$$

$\therefore$ $G g' - G' g = c$. If $c = 0$, $g = cG$. If $c \neq 0$, we have that $G(y) \to 0$ very rapidly, approx. $e^{-2\pi |n| y_j}$. The same for G'. With $c \neq 0$ either g or g' would have to be very large. If this $= g_1(3)$ would be violated; the same with g'. Hence just one solution cG satisfies (3).

Hence for $1 \leq j \leq \varkappa_1$ we must have

$$f(z) = a_j \, y_j^s + b_j \, y_j^{1-s} + \sum_{n=-\infty}^{\infty}{}' d_n^{(j)} e^{2\pi i n x_j} \sqrt{y_j} \, K_{s-\frac{1}{2}}(\pi |n| y_j)$$

& for $\varkappa_1 < j \leq \varkappa$ we have

$$f(z) = \sum_{n=-\infty}^{\infty} d_n^{(j)} e^{2\pi i (n+\beta_j) x_j} \sqrt{y_j} \, K_{s-\frac{1}{2}}(\pi |n+\beta_j| y_j)$$

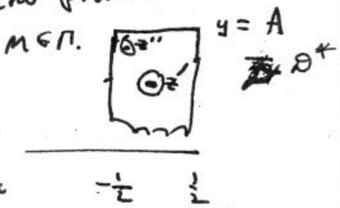
Now let $\tilde f(z) = f(z) - \sum_{j=1}^{n_1} a_j E_j(z,s,\chi)$. Then

$$\tilde f(z) = \sum_j \tilde c_j\, y_j^{1-s} + \sum_j \sum' \tilde a_n^{(j)} e^{2\pi i n x_j} \sqrt{y_j}\, K_{s-\frac12}(\pi |n| y_j) \qquad 1 \le j \le n_1$$

Hence $\iint_{\mathcal D} |\tilde f|^2 \frac{dx\,dy}{y^2} < \infty$. $\therefore$ $\tilde f$ satisfies 1) & 2) (& also (3)).

Then $\tilde f = 0$ since its eigen value is not a real pos. number. $\therefore$
$f(z) = \sum_{j=1}^{n_1} a_j E_j(z,s,\chi)$.

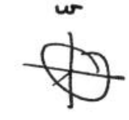
Consider a simplified situation for which we now prove the analytic continuation of $E_i(z,s,\chi)$. Take $\chi(M) = 1$ for all $M \in \Gamma$.



Then there exists Green's function

Let $z' \in \mathcal D$, inside. Then there exists $G(z,z')$ harmonic in z except for a logarithmic singularity at z' where 1) $G(z,z') - \log \frac{1}{|z-z'|}$ is regular near $z=z'$.

Let $w = e^{2\pi i z}$ (?) 2) $G(z,z') + 2\pi y$ is bounded as $y \to \infty$



i.e. is regular in w at $w=0$.

3) At corresponding sides of $\mathcal D$, G & $\frac{\partial}{\partial n} G$ should agree.

This is unique up to additive function of z'. It will turn out that $G(Mz, z') = G(z,z')$ for $M \in \Gamma$. If in (2) we require $G(z,z') + 2\pi y \to 0$ as $y \to \infty$ then $G(z,z')$ is uniquely specified.

Remove the discs about z', z'' & section $y > A$; call this $\mathcal D^*$.

Then
$$0 = \iint_{\mathcal D^*} \{ G(z,z'') \Delta G(z,z') - G(z,z') \Delta G(z,z'') \}\, dx\,dy$$

$$= \int_{\mathrm{bdy}\,\mathcal D^*} \left\{ G(z,z'') \frac{\partial G(z,z')}{\partial n} - G(z,z') \frac{\partial G(z,z'')}{\partial n} \right\} ds$$

We have 3 extra pieces of the boundary: O, O, —. The integrals along corresponding sides cancel. The integral along $y=A$ will $\to 0$ as $A \to \infty$ by 2) & its modification. Along the circle of radius ρ we can take $\frac{\partial}{\partial n}$ of $\log \frac{1}{|z-z'|}$ instead & get, around z''

$$O(\rho \log \tfrac{1}{\rho}) + G(z'', z') \cdot 2\pi + o(1).$$

The other one gives $-2\pi G(z', z'')$. Hence $0 = G(z'', z') - G(z', z'')$

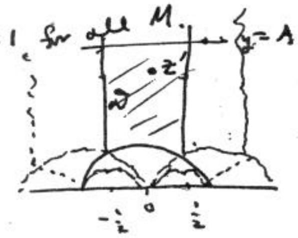
and $G(z', z'') = G(z'', z')$.

Hence $G(Mz, M'z') = G(z,z')$ for $M, M' \in \Gamma$.

If z' is on the boundary, then we have to require the same type of singularity for G at its corresponding point. At an elliptic pt. of order m we should use in place of 1) $\quad G(z,z') - m\log\frac{1}{|z-z'|}$ is reg.

#13. Feb. 15, 1955

The simple case is Γ the modular group, $\chi(M)=1$ for all M.



$$G(z,z') - \log\frac{1}{|z-z'|} \quad \text{is regular harmonic in } z \text{ at } z'.$$

$$G(z,z') + 2\pi y \to 0 \quad \text{as } y \to \infty$$

$$G(z,z') = G(z',z)$$

$$G(Mz, M'z') = G(z,z') \quad \text{for all } M, M' \in \Gamma.$$

Let $y = A$ be a line such that it lies above all circles. (Actually $A > 1$ will suffice.)

(a) Suppose $y, y' \leqq A$. ~~$M_1, \ldots, M_\nu$~~ Find the maps of the fundamental regions having at least one pt. (bdy. pt.?) in common with $\mathcal{D}$. There are only a finite #. Call the transformations (not including identity) $M_1, \ldots, M_\nu$. Then

$$G(z,z') - \log\frac{1}{|z-z'|} - \sum_{j=1}^{\nu} \log\frac{1}{|z - M_j z'|} \quad \text{is regular in}$$

the fundamental domain. Hence

$$G(z,z') = O\left(1 + \left|\log\frac{1}{|z-z'|}\right| + \sum_{j=1}^{\nu} \left|\log\frac{1}{|z-M_j z'|}\right|\right)$$

(b). Let $z, z' \in \mathcal{D}$, y or $y' > A$. Let $w = e^{2\pi i z}$, $w' = e^{2\pi i z'}$

Then $|w|$ or $|w'| < e^{-2\pi A}$. Consider

$$G(z,z') - \log\frac{1}{|e^{-2\pi i z} - e^{-2\pi i z'}|} = G(z,z') - \log\frac{1}{\left|\frac{1}{w} - \frac{1}{w'}\right|}$$

which is a harmonic function of w & even at $w=0$. Call it $H(w,w')$. Then $H(w,w') = O(|ww'|)$ near 0 since

$$H(w,w') = \sum \{a_{mn} w^m w'^n + b_{mn} \bar{w}^m w'^n + c_{mn} w^m \bar{w}'^n + d_{mn} \bar{w}^m \bar{w}'^n\}.$$

Then $$G(z,z') - \log\left|\frac{1}{e^{-2\pi i z} - e^{-2\pi i z'}}\right| = O\left(e^{-2\pi(y+y')}\right).$$

In the case of the modular group. There is a function $f(z) = w$

by the Riemann mapping theorem taking this into the upper half plane. Take the ~~upp~~ other half into the lower half plane. Then

$$G(z,z') = \log \frac{1}{|f(z) - f(z')|}$$

Actually, $f(z)$ is something like $J(z)$ (~~[illegible]~~).

$$\text{Then } \quad G(z,z') = \log \frac{1}{|e^{-2\pi i z} - e^{-2\pi i z'}|} + O\left(e^{-2\pi(y+y')}\right).$$

b_1) Let $y \geq y'$. Then $y > A$. Now

$$-\log |e^{-2\pi i z} - e^{-2\pi i z'}| = \log |e^{-2\pi i z}| \cdot |1 - e^{2\pi i(z - z')}|$$

$$\cancel{= e^{2\pi y} \{1 - e^{-2\pi(y-y')} \log(}$$

$$= -2\pi y + O\left\{e^{-2\pi(y-y')} \log\left(2 + \frac{1}{|y-y'|}\right)\right\}$$

Hence

$$G(z,z') = -2\pi y + O\left\{e^{-2\pi|y-y'|} \log\left(2 + \frac{1}{|y-y'|}\right)\right\}$$

<u>In general</u>

$$\therefore \quad G(z,z') = -2\pi \max\{y, y'\} + O\left\{e^{-2\pi|y-y'|} \log\left(2 + \frac{1}{|y-y'|}\right)\right\}.$$

Let $\widetilde{G}(z,z') = G(z,z') + 2\pi y'$ for $y' \geq A$, $= O\left(e^{-2\pi|y-y'|} \log\left(2 + \frac{1}{|y-y'|}\right)\right)$, $z, z' \in \mathcal{D}$

" $\widetilde{G}(z,z') = G(z,z') + 2\pi A$ for $y' \leq A$

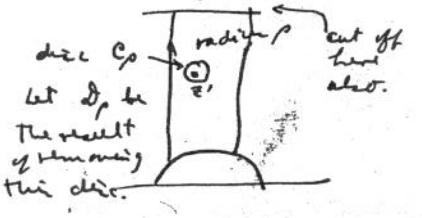
$\widetilde{G}$ is no longer symmetric.

This is the inversion of the Laplacian.

Suppose $v(z)$ is automorphic & regular in $\mathcal{D}$. Suppose v & its derivatives are $O(e^{(2\pi - \varepsilon)y})$ for some $\varepsilon > 0$, as $y \to \infty$.

Let $u(z) = y^2 \Delta v(z)$. Consider

$$\iint_{\mathcal{D}} \widetilde{G}(z,z') \, u(z') \frac{dx'dy'}{y'^2}$$



$$= \lim_{\rho \to 0} \iint_{D_\rho} \{\widetilde{G}(z,z') \Delta_{z'} v(z') - v(z') \Delta_{z'} \widetilde{G}(z,z')\} \, dx'dy'$$ Green's formula

$$= \lim_{\rho \to 0} \int_{\mathrm{bdy}\, D_\rho} \left\{\widetilde{G}(z,z') \frac{\partial}{\partial n'} v(z') - v(z') \frac{\partial}{\partial n'} \widetilde{G}(z,z')\right\} ds'$$

The parts of the boundary of $\mathcal{D}_0$ which are also parts of the boundary of $\mathcal{D}$ cancel. Hence we get, taking account of $y = A$

$$\lim_{\rho \to 0} \int_{C_\rho} \left\{ \tilde{G}(z, z') \frac{\partial}{\partial n'} v(z') - v(z') \frac{\partial}{\partial n'} \tilde{G}(z, z') \right\} ds'$$

$\rho \log \frac{1}{\rho}$ gives 0

$$+2\pi \int_{-1/2}^{1/2} v(x' + iA)\, dx'$$

$$= \cancel{-2\pi\, v(z)}$$

$$= -2\pi\, v(z) + 2\pi \int_{-1/2}^{1/2} v(x' + iA)\, dx'$$

Thus the integral

$$-\frac{1}{2\pi} \iint_{\mathcal{D}} \tilde{G}(z, z')\, u(z') \frac{dx'\,dy'}{y'^2} = v(z) - \int_{-\frac{1}{2}}^{+\frac{1}{2}} v(\alpha + iA)\, d\alpha$$

and this integral therefore gives an inversion of the Laplacian.

And also

$$y^2 \Delta \frac{-1}{2\pi} \iint_{\mathcal{D}} \tilde{G}(z, z')\, u(z') \frac{dx'\,dy'}{y'^2} = u(z').$$

Next time we consider the integral equation

$$u(z, \lambda) = 1 + \frac{\lambda}{2\pi} \iint_{\mathcal{D}} \tilde{G}(z, z')\, u(z', \lambda) \frac{dx'\,dy'}{y'^2}$$

and the solution is related to the Eisenstein series with $\lambda = s(1-s)$.

Consider the integral equation

$$u(z,\lambda) = 1 + \frac{\lambda}{2\pi} \iint_{\mathcal{D}} \widetilde{G}(z,z')\, u(z',\lambda) \frac{dx'\,dy'}{y'^2}$$ Fredholm, 3rd type

The standard theory is for

$$f(x,\lambda) = g(x) + \lambda \int_a^b K(x,y) f(y,\lambda)\, dy$$ with $K(x,y) \in L_2$ on $[a,b]\times[a,b]$, $g(x) \in L_2[a,b]$.

Then $f(x,\lambda) = D(x,\lambda)/D(\lambda)$ with the Fredholm determinants & Numerator & denom. are entire functions of order ≤ 2, $O(e^{|\lambda|^{2+\varepsilon}})$.
this is meromorphic in λ. And if $D(\lambda) \neq 0$ then $f(x,\lambda) \in L_2[a,b]$.
That we have a different metric is of no consequence. In our case $\widetilde{G}(z,z') \notin L_2$ so the standard theory is not applicable. For

$$\iint_{\mathcal{D}} \iint_{\mathcal{D}} |\widetilde{G}(z,z')|^2 d\omega\, d\omega' = \infty$$

The kernel is ~~un~~symmetric & this raises the possibility of multiplying by a factor so that the new kernel is in L_2.
(will not work for symmetric kernel.)

(a) If $z, z' \in D$; $y, y' \leq A$ then

$$\widetilde{G}(z,z') = O\left(1 + |\log|z-z'|| + \sum_{j=1}^{\nu} |\log|z - M_j z'||\right)$$

(b) If $y \geq y'$ and $y \geq A$ then

$$\widetilde{G}(z,z') = 2\pi(y'-y) + O\left\{e^{-2\pi|y-y'|} \log\left(2 + \frac{1}{|y-y'|}\right)\right\}$$

(c) If $y' \geq y$ and $y' \geq A$ then

$$\widetilde{G}(z,z') = O\left\{e^{-2\pi|y-y'|} \log\left(2 + \frac{1}{|y-y'|}\right)\right\}$$

The difficulty in getting $\widetilde{G} \in L_2$ lies in the term $2\pi(y'-y)$ in (b).
We write the equation in the form

$$\underbrace{e^{-\varepsilon y} u(z,\lambda)}_{U(z,\lambda)} = e^{-\varepsilon y} + \frac{\lambda}{2\pi} \iint_{\mathcal{D}} \underbrace{e^{-\varepsilon(y-y')} \widetilde{G}(z,z')}_{\widetilde{G}_1(z,z')} \underbrace{e^{-\varepsilon y'} u(z',\lambda)}_{U(z',\lambda)} \frac{dx'\,dy'}{y'^2}$$

Now (a) holds for $\widetilde{G}_1(z,z')$. And for (b) we have

$$\widetilde{G}_1(z,z') = O\left(e^{-\varepsilon|y-y'|} \left\{|y-y'| + \log\left(2 + \frac{1}{|y-y'|}\right)\right\}\right).$$

Also, for (c) we have

$$\tilde{G}_\gamma(z,z') = O\left\{e^{-(2\pi-\varepsilon)|y-y'|}\log\left(2+\frac{1}{|y-y'|}\right)\right\}$$

Now for $0<\varepsilon<2\pi$ we have $\tilde{G}_\varepsilon(z,z')\in L_2$. Hence we have a solution

$$e^{-\varepsilon y}u(z,\lambda) = \frac{D(z,\lambda;\varepsilon)}{D(\lambda;\varepsilon)} \in L_2 \quad \text{if } D(\lambda;\varepsilon)\neq 0$$

Now $u(z,\lambda)$ does not depend on ε alone because $y^2\Delta v(z)=u(z)$. Also if $D(\lambda;\varepsilon)\neq 0$

$y^2\Delta u(z,\lambda)+\lambda u(z,\lambda)=0.$

$u(Mz,\lambda)=u(z,\lambda)$

Now $\iint_{\mathcal{D}} e^{-2\varepsilon y}|u(z,\lambda)|^2\frac{dx\,dy}{y^2}<\infty$ since $\in L_2$.

Hence for most z, $u(z,\lambda)=O(e^{\varepsilon y})$. By a type of previous argument we get $u(z,\lambda)=O(e^{\varepsilon y})$ for each $\varepsilon>0$. Also

$$\int_{-1/2}^{1/2} u(A+ix,\lambda)\,dx = 1.$$

For a connection with the Eisenstein series, let $\lambda=s(1-s)$. Suppose $\Re(s)>1$. For the modular case, there is just one cusp & hence one Eisenstein series

$$E(z,s) = \sum\frac{y^s}{|cz+d|^{2s}}$$

By a previous lemma

$$u(z,s(1-s)) = a(s)\,E(z,s).$$

By the Fourier expansion, the right side is

$$a(s)\left\{y^s+\sqrt{\pi}\,\frac{\Gamma(s-1/2)}{\Gamma(s)}\,\zeta^{(0)}(s)\,y^{1-s}\right\} + \cdots$$

& its integral is

$$a(s)\left\{A^s+\sqrt{\pi}\,\frac{\Gamma(s-1/2)}{\Gamma(s)}\,\zeta^{(0)}(s)\,A^{1-s}\right\}=1=\int_{-\frac12}^{\frac12}u(A+ix,\lambda)\,dx$$

which gives $a(s)$. Now $u(z,s(1-s))$ is meromorphic with numerator & denominator of entire of order at most 4 in s.

And

$$u(z,s(1-s)) = a(s)\,y^s+\beta(s)\,y^{1-s}$$

with $a(s)$, $\beta(s)$ meromorphic & order at most 4. Hence

$E(z,s)$ is a meromorphic function of order 4 at most in s with denominator free of z. This gives the analytic continuation of $E(z,s)$.

With $\Re(s_0)>1$, suppose $1-s_0$ is not a pole, $\Re(1-s_0)<0$. Since $\{y^2\Delta + s(1-s)\}E(z,s)=0$ has a symmetric operator, we have that $E(z,s)$ has a functional equation. This follows also, more simply from $u(z, s(1-s)) = a(s)E(z,s)$. Then, $E(z,1-s) = \varphi(s)\,E(z,s)$. Using the Fourier expansion, or equivalently the formula for $a(s)$, we have on setting

$$\alpha_0(s) = \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,\zeta^{\alpha}(s)$$

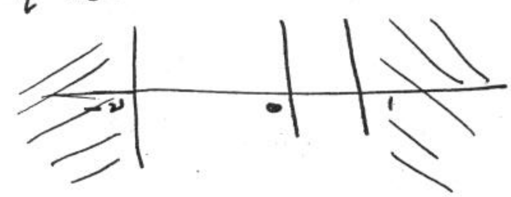
~~so~~ that

$$y^{1-s}\cancel{+\alpha_0(1-s)\,y^{s}} \quad \varphi(s) = \alpha_0(1-s)$$

$$\alpha_0(s)\,\alpha_0(1-s) = 1\,,$$

Other terms of the Fourier expansion give more functional equations. Using the definition of $\alpha_0(s)$ we can state

That $\alpha_0(s)$ is free of singularities for $\Re(s)<-\nu$ as well as for $\Re(s)>1$. In the modular case $\zeta^{\alpha}(s) = \zeta(2s-1)/\zeta(2s)$



Now the more general case where there are more than 1 cusp & $\chi(M)$ is not necessarily $\equiv 1$.

Let the cusps be $\xi_1, \ldots, \xi_\varkappa$ and $\chi(M)=1$ for all $M\in\Gamma$. We can still get the symmetric Green's function and there is some arbitrariness. Put a logarithmic singularity at ξ_1 say. Then if $z'\to\xi_j$ with $j\neq 1$, then $G(z,z')\to$ harmonic functions $=g_j(z)$. They are approx. $O(y)$. Together there are $\varkappa-1$ of them.

with $g_1(z)=1$ we have $\varkappa$ linearly independent harmonic functions $g_1(z),\ldots,g_\varkappa(z)$

We can get non-symmetric

$$\tilde{G}(z,z') = G(z,z') + \sum_{j=1}^{\varkappa}\omega_j(z')\,g_j(z)$$

with $\omega_j(z')$ having cont. deriv. up to 2nd order with some possible exception curves. Moreover these are to be determined so that $\widetilde{G}(z,z') \to 0$ rapidly if z' tends to a cusp.

For $j = 1, \cdots, n$ we consider

$$u_j(z,\lambda) = g_j(z) + \frac{\lambda}{2\pi}\iint_D \widetilde{G}(z,z')\, u_j(z,\lambda)\, \frac{dx'\,dy'}{y'^2}.$$

One uses a similar device as before. The solutions $u_j(z,\lambda)$ will be linearly independent & satisfy $(y^2\Delta + \lambda)u = 0$, are automorphic. Then each u_j is a linear combination of Eisenstein series for $\lambda = s(1-s)$. One gets similar results.

#15. <u>March 1, 1955</u>

If we are going to have a Green's function with normal derivative 0 on boundary of discs, we need at least 2 pts of sing. Because otherwise $\int \frac{\partial u}{\partial n}\, ds = 0$ but the conjugate function v has $\int dv = 2\pi$ since it contains the part accounting for the variation of the argument. For n points we proceed 2 at a time getting principal parts

$$\log\frac{1}{|z-z_0|} - \log\frac{1}{|z-z_1|}$$

$$\log\frac{1}{|z-z_1|} - \log\frac{1}{|z-z_2|}$$

$$\cdots$$

$$\cdots$$

$$\log\frac{1}{|z-z_{n-1}|} - \log\frac{1}{|z-z_n|}$$

these cancel except the first & last and get $\log\frac{1}{|z-z_0|} - \log\frac{1}{|z-z_n|}$

We deal with the case in which $\chi(M)$ is not necessarily 1.

We obtain a Green's function $G(z,z',\chi)$ such that

$$(?) \quad G(Mz, Nz', \chi) = \chi(M)\,\overline{\chi(N)}\;\overline{G(z',z,\chi)}$$

$$G(Mz,z') = \chi(M)\, G(z,z')$$

We define

$$\widetilde{G}(z,z',\chi) = G(z,z',\chi) - \sum_{j=1}^{n_1} g_j(z)\,\omega_j(z',\chi)$$

$$\omega_j(Mz',\chi) = \overline{\chi(M)}\,\omega_j(z',\chi) \qquad \text{cont. deriv., etc.}$$

Consider ~~let~~ $u_j(z,\lambda,\chi) = g_j(z) + \frac{\lambda}{2\pi} \iint_{\mathcal{D}} \tilde{G}(z,z',\chi) \overset{E}{u_j}(z',\lambda,\chi) \frac{dx'dy'}{y'^2}$

The other theory goes through & the function u_j can be linearly expressed in terms of the Eisenstein series. If χ is $r \times r$, the Eisenstein series is a vector $r \times 1$; G is a matrix, but we do not go into this.

An alternative approach obviating the need for the Green's function theory, is as follows. Consider

$$\left(y^2 \Delta + s(1-s)\right) k\left(\frac{|z-z'|^2}{yy'}\right) = 0$$

We make one solution $k(t)$ be bounded at $t=0$ & the other will have a logarithmic singularity at 0. The second one is dominated by
$\mathcal{O}\left(\frac{1}{|t|^{\mathcal{R}(s)}}\right) = O\left(\frac{1}{|t|}\right)^{1+\varepsilon}$ for $\mathcal{R}(s) \geqq 1+\varepsilon$.

We then construct

$$K_{s_0}(z,z',\chi) = \sum \chi(M)\, k(z, Mz')$$

with logarithmic singularities when $z = Mz'$. We would consider

$$\tilde{K}_{s_0}(z,z',\chi) = K_{s_0}(z,z',\chi) - \sum_{j=1}^{\varkappa_1} E_j(z,s_0,\chi)\, \omega_j(z')$$

And we would investigate

$$u_j(z,\lambda) = E_j(z,s_0,\chi) + \frac{\lambda}{2\pi} \iint_{\mathcal{D}} \tilde{K}(z,z',\chi)\, u_j(z',\lambda) \frac{dx'dy'}{y'^2}$$

The Fourier expansion of $K_{s_0}(z,z',\chi)$ would give some trouble but this can be overcome. In the special case we had

$$E(z,s) = \varphi(s) E(z,1-s)$$

$$\varphi(s)\varphi(1-s) = 1$$

$$\varphi(s) = \sqrt{\pi}\, \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} L^{(0)}(s)$$

Now we have

$$E_j(z,s,\chi) = \delta_{jk}\, y_k^{\,s} + \varphi_{jk}(s,\chi)\, y_k^{\,1-s} + \dots$$

$$\varphi_{jk}(s,\chi) = \sqrt{\pi}\, \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} L^{(0)}_{jk}(s,\chi).$$

Let
$$E(z,s,\chi) = \begin{pmatrix} E_1(z,s,\chi) \\ \vdots \\ E_{\varkappa_1}(z,s,\chi) \end{pmatrix}$$

Then $E(z,s,\chi) = \underset{\substack{\varkappa_1\times\varkappa_1 \\ \text{matrix}}}{\Phi(s,\chi)}\, E(z,1-s,\chi)$

Then

$$\Phi(s,\chi) = \|\varphi_{jk}(s,\chi)\|$$

$$\Phi(s,\chi)\,\Phi(1-s,\chi) = E = \text{unit matrix } \varkappa_1\times\varkappa_1$$

Consider with $\chi = 1$

$\varphi(s)\,\varphi(1-s) = 1$, $\quad s = \frac{1}{2} + ir$

$\varphi(\frac{1}{2}+ir)\,\varphi(\frac{1}{2}-ir) = 1$

$\varphi(\frac{1}{2}+ir)\,\overline{\varphi(\frac{1}{2}+ir)} = 1$

or $|\varphi(\frac{1}{2}+ir)|^2 = 1$, $\quad |\varphi(\frac{1}{2}+ir)| = 1.$

In the matrix case

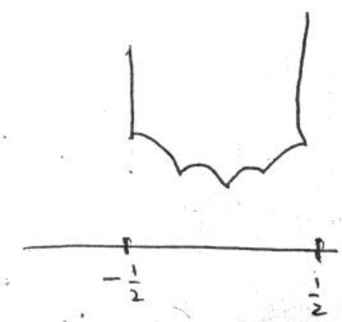
$\varphi_{jk}(s,\chi) = \varphi_{kj}(s,\bar\chi)$. $\quad$ so $\Phi(s,\chi) = \{\Phi(s,\bar\chi)\}'$

and $\quad \Phi(\frac{1}{2}+ir,\chi)\,\Phi(\frac{1}{2}-ir,\chi) = E$

$\Phi(\frac{1}{2}+ir,\chi)\,\Phi(\frac{1}{2}-ir,\bar\chi)' = E \quad \checkmark$

$\Phi(\frac{1}{2}+ir,\chi)\,\overline{\Phi(\frac{1}{2}+ir,\chi)}' = E \quad \checkmark$

$\therefore\ \Phi(\frac{1}{2}+ir,\chi)$ is unitary. i.e. $\Phi(s,\chi)$ is unitary for $s = \frac{1}{2}+ir$.



Consider again $\chi(M) = 1$.

Theorem: Now $E(z,s)$ is regular for ~~complex~~ nonreal s and $R(s) > \frac{1}{2}$. We already had this for all s with $R(s) > 1$. Now

$$E(z,s) = y^s + \varphi(s)\,y^{1-s} + \sum_{n=-\infty}^{\infty}{}' e^{2\pi i n x}\ldots$$

Let ν be the smallest # such that

$\lim_{s\to s_0} (s-s_0)^\nu E(z,s) = u(z)$ is not identically 0 in z. Then

$$u(z) = c_0 y^{1-s_0} + \sum_{n=-\infty}^{\infty}{}' c_n e^{2\pi i n x} \sqrt{y} \int_0^\infty t^{s_0-\frac{1}{2}} e^{-\pi|n|y(t+\frac{1}{t})} \frac{dt}{t}$$

then $u(Mz) = u(z)$

$$\{y^2\Delta + s_0(1-s_0)\} u(z) = 0.$$

with $\mathcal{R}(s_0) > 1/2$, $\mathcal{R}(1-s_0) < 1/2$ & $\therefore$ $u \in L_2$ since

$\iint_{\mathcal{D}} |u|^2 \frac{dx\,dy}{y^2} < \infty$. For the series is exponentially small

and $\int |y^{2(1-s_0)}| \frac{dy}{y^2} = \int y^{-2\mathcal{R}(s_0)} dy < \infty$. We then

get $E(z,s)$ is regular for non-real $\mathcal{R}(s) > 1/2$.

The general case goes in a similar way.

If $N(T)$ is the number of poles for $|\mathcal{I}(s)| \leqq T$ then

$$A T \log T \leqq N(T) \leqq A_1 T^2$$

& this will be in $-\Delta \leqq \mathcal{R}(s) \leqq \frac{1}{2}$ for a certain Δ.

In the modular group $N(T) \sim A\, T \log T$.