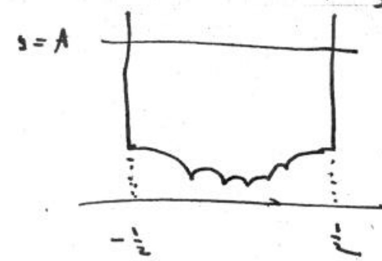
#16. March 8, 1955



We had

$$E(z,s) = y^s + \varphi(s) y^{1-s} + \Sigma' \ldots$$

$$\varphi(s) = \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,L^{(2)}(s), \qquad \varphi(s)\varphi(1-s) = 1, \quad |\varphi(\tfrac{1}{2}+ir)| = 1.$$

We define

$$\tilde{E}(z,s) = \begin{cases} E(z,s) & \text{for } y < A \\ E(z,s) - y^s - \varphi(s) y^{1-s} & \text{for } y \geq A \end{cases}$$

Let $s \neq s', 1-s'$. Consider

$$I = \iint_{D} \{ \tilde{E}(z,s)\,\Delta \tilde{E}(z,s') - \tilde{E}(z,s')\,\Delta \tilde{E}(z,s) \}\, dx\, dy.$$

Now $\{ y^2 \Delta + s(1-s) \} \tilde{E}(z,s) = 0$. Hence we can replace $\Delta \tilde{E}$ by $-\frac{s(1-s)}{y^2}\tilde{E}$. Hence

$$I = \{ s(1-s) - s'(1-s') \} \iint_{D} \tilde{E}(z,s)\tilde{E}(z,s')\,\frac{dx\,dy}{y^2}$$

$$= (s'-s)(s+s'-1) \iint_{D} \tilde{E}(z,s)\tilde{E}(z,s')\,\frac{dx\,dy}{y^2}$$

By Green's formula, taking account of discontinuity along $y = A$

$$I = \int_{\text{bdy.}} \left\{ \tilde{E}(z,s)\,\frac{\partial \tilde{E}}{\partial n}(z,s') - \tilde{E}(z,s)\,\frac{\partial \tilde{E}(z,s')}{\partial n} \right\} dS$$

Here $y = A$ = part of boundary in opposite directions. Most of boundary parts cancel. Hence we will get

$$I = \left(\{ y^s + y^{1-s}\varphi(s) \} \frac{d}{dy} \{ y^{s'} + \varphi(s') y^{1-s'} \} - \{ y^{s'} + \varphi(s') y^{1-s'} \} \frac{d}{dy} (y^s + \varphi(s) y^{1-s}) \right)_{y=A}$$

$$= \Big((s'-s)\left[y^{s+s'-1} - \varphi(s)\varphi(s') y^{1-s-s'} \right]$$
$$+ (1-s-s')\left[\varphi(s') y^{s-s'} - \varphi(s) y^{s'-s} \right] \Big)_{y=A}$$

$$I = (s'-s)\{ A^{s+s'-1} - \varphi(s)\varphi(s') A^{1-s-s'} \} + (s+s'-1)\{ \varphi(s) A^{s'-s} - \varphi(s') A^{s-s'} \}$$

$$= (s'-s)(s+s'-1) \iint_{D} \tilde{E}(z,s)\tilde{E}(z,s')\,\frac{dx\,dy}{y^2}$$

$$\therefore \quad \frac{A^{s+s'-1} - \varphi(s)\varphi(s') A^{1-s-s'}}{s+s'-1} + \frac{\varphi(s) A^{s'-s} - \varphi(s') A^{s-s'}}{s'-s} = \iint_{D} \tilde{E}(z,s)\tilde{E}(z,s')\,\frac{dx\,dy}{y^2}$$

Let $s = \sigma + ir$, $s' = \sigma - ir$, $r \neq 0$, $\sigma \neq \frac{1}{2}$. Then

$$\iint_{\mathcal{D}} |\tilde{E}(z, \sigma+ir)|^2 \frac{dx\,dy}{y^2} = \frac{A^{2\sigma-1} - |\varphi(\sigma+ir)|^2 A^{1-2\sigma}}{2\sigma-1}$$

$$+ \frac{\varphi(\sigma+ir) A^{2ir} - \overline{\varphi(\sigma+ir)} A^{-2ir}}{2ir}$$

With $r \neq 0$, let $\sigma \to \frac{1}{2}$. Since $|\varphi(\frac{1}{2}+ir)| = 1$, φ is regular on $\sigma = \frac{1}{2}$.

the first term has numerator

$$A^{2\sigma-1} - \varphi(\sigma+ir)\varphi(\sigma-ir) A^{1-2\sigma}$$

with derivative

$$A^{2\sigma-1}\log A \cdot 2 - \varphi(\sigma+ir)\varphi(\sigma-ir) A^{1-2\sigma} \log A(-2)$$

$$- A^{1-2\sigma}\{\varphi(\sigma+ir)\varphi'(\sigma-ir) + \varphi'(\sigma+ir)\varphi(\sigma-ir)\}$$

$$\to 2\log A + 2\log A - \{\varphi(\tfrac{1}{2}+ir)\frac{\varphi'(\frac{1}{2}-ir)}{\varphi(\frac{1}{2}-ir)} + \frac{\varphi'(\frac{1}{2}+ir)}{\varphi(\frac{1}{2}+ir)}\varphi(\tfrac{1}{2}-ir)\}$$

$$= 4\log A - \left\{\frac{\varphi'(\frac{1}{2}-ir)}{\varphi(\frac{1}{2}-ir)} + \frac{\varphi'(\frac{1}{2}+ir)}{\varphi(\frac{1}{2}+ir)}\right\}$$

$$= 4\log A - 2\,\mathcal{R}\,\frac{\varphi'(\frac{1}{2}+ir)}{\varphi(\frac{1}{2}+ir)}.$$

Hence the right side tends to

$$\frac{\cancel{4}^{2}\log A - \cancel{2}\,\mathcal{R}\,\frac{\varphi'}{\varphi}(\frac{1}{2}+ir)}{\cancel{2}} + \frac{\varphi(\frac{1}{2}+ir)A^{2ir} - \overline{\varphi(\frac{1}{2}+ir)}A^{-2ir}}{2ir}$$

$$= \lim_{\sigma\to\frac{1}{2}} \iint_{\mathcal{D}} |\tilde{E}(z, \sigma+ir)|^2 \frac{dx\,dy}{y^2}$$

Now, if $E(z,s)$ has a pole $s = \frac{1}{2}+ir$ with order $\nu > 0$ then $(s-\frac{1}{2}-ir)^\nu E(z,s)$ is regular at $\frac{1}{2}+ir$ & this has the same limit as $(s-\frac{1}{2}-ir)^\nu \tilde{E}(z,s) \to u(z) \neq 0$ which would be an eigen-function $\Rightarrow (y^2\Delta + \frac{1}{4}+r^2)u = 0$. But now $\iint |u|^2 \frac{dx\,dy}{y^2} = 0$ from the above since $\nu > 0$. Hence $u \equiv 0$ & this is a contradiction.

$\therefore$ No pole. Then we can replace the limit by $|\tilde{E}(z, \frac{1}{2}+ir)|^2$.

$\therefore$ No poles on $\sigma = \frac{1}{2}$ except possibly $s = \frac{1}{2}$.

Take $\frac{1}{2} < \sigma \leq 1$ and let $r \to 0$. Suppose $\varphi(s)$ is regular at σ. Now differentiating again we get the limit

$$\frac{A^{2\sigma-1} - \varphi^2(\sigma) A^{1-2\sigma}}{2\sigma-1} + 2\varphi(\sigma)\log A - \varphi'(\sigma)$$
$$= \lim_{r\to 0} \iint_{\mathcal{D}} |\tilde{E}(z, \sigma+ir)|^2 \frac{dx\,dy}{y^2}.$$

The left side ≥ 0. With a pole of order ν this can only happen if $\nu < 2$ & $\therefore$ $\nu = 1$. Then $(s-\sigma_0)\tilde{E}(z,s) \to u(z) \in L_2$ & an eigen function. Except for the modular group $\sim c\,y^{1-\sigma_0}$ one can take $A=1$ & this simplifies the above formula. No ~~zeros~~ poles for $\frac{1}{2} < s \leq 1$ except possibly a finite $\#$ of simple ones and one can prove that $s = \frac{1}{2}$ is also not a pole. On letting $s \to \frac{1}{2}$ in the above formula. To each pole at σ_0 one gets an eigenfunction $u \in L_2$ with $\iint |u|^2 \frac{dx\,dy}{y^2} = c^2$

where $(s-\sigma_0)\, E(z,s) \longrightarrow c\,y^{1-\sigma_0} + \sum \ldots$

In the general case, one considers

$$\iint_{\mathcal{D}} \tilde{E}_j(z,s,\chi)\, \overline{\tilde{E}_k(z,s',\chi)}\ \frac{dx\,dy}{y^2}$$

& cuts the fundamental region with several lines. With eigen ~~vectors~~ functions $E_1, E_2, \ldots, E_{\varkappa_1}$ &

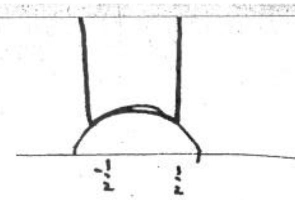
$$E(z,s,\chi) = \begin{pmatrix} E_1 \\ \vdots \\ E_{\varkappa_1} \end{pmatrix} = \Phi(s,\chi)\, E(z, 1-s, \chi)$$

One also uses $A_1, \ldots, A_{\varkappa_1}$ corresponding to $z_1, \ldots, z_{\varkappa_1}$. One gets

$$\iint_{\mathcal{D}} \tilde{E}(z, \sigma+ir,\chi)\, \overline{\tilde{E}(z,\sigma+ir,\chi)}'\ \frac{dx\,dy}{y^2}$$
$$= \frac{1}{2\sigma-1}\left\{ A^{2\sigma-1} E - \varphi(\sigma+ir,\chi)\,\overline{\varphi(\sigma+ir,\chi)}'\, A^{1-2\sigma}\right\}$$
$$+ \frac{\varphi(\sigma+ir,\chi) A^{2ir} - \overline{\varphi(\sigma+ir,\chi)}' A^{-2ir}}{2ir}$$

#17. March 13, 1955

Consider the modular group.



the Eisenstein series is

$$E(z,s) = \frac{1}{2}\sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}}$$ with ~~$az+b$~~ the factor $\frac{1}{2}$ inserted so that we need no longer identify $\frac{az+b}{cz+d}$, $\frac{(-a)z+(-b)}{(-c)z+(-d)}$. Now

$$2\zeta(2s)\,E(z,s) = \sum_{\substack{(c,d)=1\\ n\geq 1}} \frac{y^s}{|cnz+dn|^{2s}} = \sum_{m,n}{}' \frac{y^s}{|mz+n|^{2s}}$$

$$= \sum{}' \frac{y^s}{\{(mx+n)^2+m^2y^2\}^s}$$ Epstein ζ-function functional equation

Pole at $s=1$ only on right side. Singularities of $E(z,s)$ correspond to zeros of $\zeta(2s)$.

$$E(z,s) = y^s + \Phi(s)\,y^{1-s} + \sum{}' e^{2\pi i x}\cdots$$

$$\Phi(s) = \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{c>0} \frac{1}{c^{2s}}\varphi(c)$$ Euler totient

$$= \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{c>0} \frac{1}{c^{2s-1}} \sum_{d|c} \frac{\mu(d)}{d}$$

$$= \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum\sum \frac{\mu(d)}{d^{2s}n^{2s-1}}$$

$$= \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,\frac{\zeta(2s-1)}{\zeta(2s)}$$

$$\frac{\pi^s}{\Gamma(s)}\sqrt{y}\int_0^\infty t^{s-\frac{1}{2}} e^{-\pi y(t+\frac{1}{t})}\frac{dt}{t} \sum_{c>0}\frac{1}{c^{2s}} \overbrace{\sum_{\substack{0<d<c\\ (c,d)=1}} e^{2\pi i d/c}}^{\mu(c)}$$

$$= \frac{\pi^s}{\Gamma(s)\,\zeta(2s)}\cdot\sqrt{y}\int_0^\infty t^{s-\frac{1}{2}} e^{-\pi y(t+\frac{1}{t})}\frac{dt}{t}$$

We had $\Phi(s)\Phi(1-s)=1$ $\therefore\ |\Phi(\frac{1}{2}+ir)|=1$. So

$$\left|\sqrt{\pi}\,\frac{\Gamma(ir)}{\Gamma(\frac{1}{2}+ir)}\,\frac{\zeta(2ir)}{\zeta(1+2ir)}\right| = 1$$

One can actually get the functional equation for $\zeta(s)$ from $\Phi(s)\Phi(1-s)=1$ although it is not obvious.

In the more general case

$$\sum_{|c|\leq x} (x-|c|)^{\alpha} \sum_{0\leq d\leq |c|} e^{2\pi i \frac{d}{c}} = \underbrace{\quad \cdots \quad}_{\text{main term due to poles.}} + o\left(\frac{x^{\alpha+1}}{\log^{\beta} x}\right)$$

for certain β

In the modular case

$$\sum_{n\leq x} (x-n)^{\alpha} \mu(n) = o\left(\frac{x^{\alpha+1}}{\log^{\beta} x}\right)$$

$$\sum_{n\leq x} \mu(n) = o\left(\frac{x}{\log^{\beta} x}\right).$$

Take the more general case with $\chi(M)=1$. Now $|\varphi(\frac{1}{2}+ir)|=1$. For ~~other s~~ $\mathcal{R}(s)\geq 1+\varepsilon$

$$\varphi(s) = \sqrt{\pi}\ \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{|c|>0} \frac{1}{|c|^{2s}} \sum_{0\leq d<|c|} 1$$

is unif. bounded & $\to 0$ as $|s|\to\infty$.

also

$$0 \leq \iint_{\mathcal{D}} |\tilde{E}(z,s)|^2 \frac{dx\,dy}{y^2} = \frac{A^{2\sigma-1} - |\varphi(\sigma+ir)|^2 A^{1-2\sigma}}{2\sigma-1}$$

$$+ \frac{\overline{\varphi(\sigma+ir)}\ A^{2ir} - \varphi(\sigma+ir)\ A^{-2ir}}{2ir}$$

taking $A=1$

$$= \frac{1-|\varphi(\sigma+ir)|^2}{2\sigma-1} + \frac{\overline{\varphi(\sigma+ir)} - \varphi(\sigma+ir)}{2ir}$$

Let $|r|\geq 1$.

$$\leq \frac{1-|\varphi(\sigma+ir)|^2}{2\sigma-1} + \frac{|\varphi(\sigma+ir)|}{|r|}$$

Hence, with $\sigma>\frac{1}{2}$

$$1-|\varphi|^2 + \frac{2\sigma-1}{r}|\varphi| \geq 0$$

$$|\varphi|^2 - \frac{2\sigma-1}{r}|\varphi| \leq 1$$

$$|\varphi| \leq \frac{\sigma-\frac{1}{2}}{|r|} + \sqrt{1+\left(\frac{\sigma-\frac{1}{2}}{|r|}\right)^2} < 1 + \frac{2\sigma-1}{|r|}.$$

$$|\varphi(\sigma+ir)| \leq 1 + \frac{2\sigma-1}{|r|}$$

& $\therefore$ a bound for $\varphi(s)$ for $\mathcal{R}\,s\geq \frac{1}{2}$.

If there are real poles between $\frac{1}{2}$ & 1 call them $\frac{1}{2}<\sigma_1, \ldots, \sigma_k \leq 1$. ~~Let~~ Consider $\varphi(s)\prod_{j=1}^{k} \frac{s-\sigma_j}{s-(1-\sigma_j)} = \varphi^*(s)$. Then

$|\varphi^*(\frac{1}{2}+ir)|=1$. And we finally get $|\varphi^*(s)|\le 1$ for $\Re(s)\ge 1/2$ since $\varphi^*(s)$ is regular for $\Re(s)>\frac{1}{2}$ & bounded for $\Re(s)>\frac{1}{2}$.

Now $z=\frac{s-1}{s}$ maps the half plane onto the unit circle.

If the zeros of the transformed functions are $\rho_1,\rho_2,\dots$ then $\prod|\rho_n|$ and $\sum\{1-|\rho_n|\}$ converge. This is because

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta \le \frac{1}{2}\log\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2\,d\theta$$

& Jensen's formula. Hence if $\frac{1}{2}+\beta+i\gamma$, $\beta>0$, are the zeros of $\varphi(s)$ (poles are $\frac{1}{2}-\beta+i\gamma$) not on the real line then $\sum\{1-|\rho_n|^2\}$ also converges &

$$|\rho|=\left|\frac{\frac{1}{2}+\beta+i\gamma-1}{\frac{1}{2}+\beta+i\gamma}\right|=\left|\frac{-\frac{1}{2}+\beta+i\gamma}{\frac{1}{2}+\beta+i\gamma}\right|$$

$$1-|\rho|^2=1-\frac{(\beta-\frac{1}{2})^2+\gamma^2}{(\beta+\frac{1}{2})^2+\gamma^2}=\frac{2\beta}{(\beta+\frac{1}{2})^2+\gamma^2}.$$

Also $\varphi(s)\ne 0$ for $\Re s\ge\Delta$ ∴ $\beta\le\Delta-\frac{1}{2}$. Hence

$\sum\limits_{\rho} \frac{\beta}{1+\gamma^2}$ converges. ∴ if γ is large, β is close to 0. Consider

with $\rho=\frac{1}{2}+\beta+i\gamma$

$$\prod_{\rho}\frac{s-\rho}{s-(1-\bar\rho)}$$

With $s=\sigma+ir$, $\rho=\frac{1}{2}+\beta+i\gamma$ then

$$\frac{s-\rho}{s-(1-\bar\rho)}=\frac{(\sigma-\frac{1}{2}-\beta)+i(r-\gamma)}{(\sigma-\frac{1}{2}+\beta)+i(r-\gamma)}=1-\frac{2\beta}{(\sigma-\frac{1}{2}+\beta)+i(r-\gamma)}$$

Arranging according to ρ & $\bar\rho$ we would get an absolutely convergent product! Then

$$\varphi^*(s)=e^{g(s)}\prod_{\rho}\frac{s-\rho}{s-(1-\bar\rho)}$$

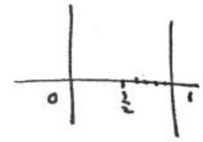
with $g(s)$ reg. for $\Re(s)>\frac{1}{2}$ & $\Re\, g(s)=0$ for $\Re(s)=\frac{1}{2}$.

With $\varphi(s)=\dfrac{\text{entire function of order}\le 4}{\text{" " " "}}$ $\Rightarrow$ $g(s)$ is a polynomial of degree ≤ 4

Actually, $g(s)$ is of the first degree, $g(s)=a(s-\frac{1}{2})$ with a real a. This can be established by examining certain directions and comparing the size of the function. Actually $\underline{a}$ can be computed. Actually $\sum\limits_{|\gamma|\le T}\beta=O(T\log T)$ & this is best possible. In the general case, one uses the determinant of the matrix.

$$\varphi(s,\chi) = \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,L^{(0)}(s,\chi)$$

$$\varphi(s,\chi)\,\varphi(1-s,\chi) = 1.$$



$\frac{1}{2} < \sigma_1, \ldots, \sigma_k \leq 1$.

Drop χ in ~~dependence~~ argument. Let

$$\varphi^*(s) = \varphi(s)\prod_{j=1}^{k}\frac{s-\sigma_j}{s-1+\sigma_j}. \quad \text{Then}$$

$|\varphi^*(s)| = 1$ for $s = \frac{1}{2} + ir$. And $\varphi^*(s)$ has no poles for $R(s) > \frac{1}{2}$. Then $|\varphi^*(s)| \leq 1$ for $R(s) \geq \frac{1}{2}$ ~~on writing $s = \sigma + ir$~~ all this we had previously. Also $\varphi^*(s)\varphi^*(1-s) = 1$.

We had a function $g(s)$. Now things could be made so that $g(s) + g(1-s) = 0$ or $2\pi i$. This means, since $g(s)$ is of degree ≤ 4 $g(s) = \alpha_1(s-\frac{1}{2})^3 + \beta_1(s-\frac{1}{2}) + \left\{\begin{matrix}0\\ \pi i\end{matrix}\right.$. Now $\alpha_1 + \beta_1$ are real by looking at what happens on $\sigma = \frac{1}{2}$. For large real s, $L^0(s,\chi)$ is approx. λc_0^{-2s}

so

$$\varphi^*(s) \sim \sqrt{\pi}\,\frac{\lambda c_0^{-2s}}{\sqrt{s}}.$$

Letting $s = \frac{1}{2} + \sigma$ we have

$$\cancel{\frac{s-1+\beta}{s-\beta} = \frac{1}{2}+\sigma - 1 + \beta}$$

$$\frac{s-\rho}{s-(1-\bar\rho)} = \frac{\sigma+\frac{1}{2}-(\frac{1}{2}+\beta+i\gamma)}{\sigma+\frac{1}{2}-1+(\frac{1}{2}+\beta-i\gamma)} = \frac{\sigma-\beta-i\gamma}{\sigma+\beta-i\gamma}$$

with the square of absolute value

$$\frac{(\sigma-\beta)^2+\gamma^2}{(\sigma+\beta)^2+\gamma^2} = 1 - \frac{4\sigma\beta}{(\sigma+\beta)^2+\gamma^2}$$

Consider ~~$|\gamma| < T$, $|\gamma| \geq T$~~ $|\gamma| \leq T$, $|\gamma| \geq T$. Then

$$\prod\left\{1-\frac{4\sigma\beta}{(\sigma+\beta)^2+\gamma^2}\right\} \geq \prod_{|\gamma|<T}\left\{1-\frac{4\beta\sigma}{(\sigma+\beta)^2}\right\}\prod_{|\gamma|\geq T}\left\{1-\frac{4\sigma\beta}{1+\gamma^2}\right\}$$

$$\geq \prod_{|\gamma|<T}\left\{1-\frac{4\beta}{\sigma}\right\}\prod_{|\gamma|\geq T}\left(1-\frac{4\sigma\beta}{1+\gamma^2}\right)$$

$$\geq \prod_{|\gamma|<T} e^{-2\frac{4\beta}{\sigma}}\prod_{|\gamma|\geq T} e^{-2\frac{4\sigma\beta}{1+\gamma^2}} \quad \text{large } \sigma$$

$$\geq \prod_{|\gamma|<T} e^{-\frac{A}{\sigma}} \cdot e^{-8\sigma\sum_{|\gamma|\geq T}\frac{\beta}{1+\gamma^2}} \to 0$$

$$\geq e^{-\frac{A}{\sigma}} T^{4+\frac{1}{2}} \cdot e^{-8\sigma \sum_{|\delta| \leq T} \frac{\beta}{1+\gamma^2}} > e^{-o(\sigma)}.$$

by suitable choice of T. and we get a contradiction.

If $\alpha \neq 0$ then $g(s) \sim \alpha\sigma^3$

$\therefore \alpha = 0$. Also we get $\beta = -2\log c_0$. Hence

$$\varphi^*(s) = \pm c_0^{1-2s} \prod \frac{s-\rho}{s-1+\bar{\rho}}$$

ρ are the zeros of $\varphi(s)$ not on the real line.

From this it follows that the logarithm of $\prod$ is $\sim \frac{3}{2}\log\sigma$. Then

$$\prod_{|\gamma| \leq \sigma} \left\{1 - \frac{4\sigma\beta}{(\sigma+\beta)^2+\gamma^2}\right\} > \frac{A}{\sigma}$$

$$\therefore \prod_{|\gamma| \leq \sigma} \left(1 - \frac{\beta}{\sigma}\right) > \frac{A}{\sigma}$$

$$\therefore \sum_{|\gamma| \leq T} \beta = O(T \log T)$$

Actually we can get $A\, T\log T + B\, T + O(\log T)$.

Consider

$\Re \frac{\varphi^{*\prime}(\frac{1}{2}+ir)}{\varphi^*(\frac{1}{2}+ir)}$ which arise from $\iint_{\mathcal{D}} |E(z, \tfrac{1}{2}+ir)|^2 \frac{dx\,dy}{y^2}$.

Now

$$\frac{\varphi^{*\prime}(s)}{\varphi^*(s)} = -2\log c_0 + \sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{s-1+\bar{\rho}}\right)$$

$$\frac{\varphi^{*\prime}(\frac{1}{2}+ir)}{\varphi^*(\frac{1}{2}+ir)} = -2\log c_0 + \sum_\rho \left(\frac{1}{\frac{1}{2}+ir-\frac{1}{2}-\beta-i\gamma} - \frac{1}{\frac{1}{2}+ir-1+\frac{1}{2}+\beta-i\gamma}\right)$$

$$= -2\log c_0 + \sum_\rho \left\{\frac{1}{-\beta+i(r-\gamma)} - \frac{1}{\beta+i(r-\gamma)}\right\}$$

$$\Re \frac{\varphi^{*\prime}(\frac{1}{2}+ir)}{\varphi^*(\frac{1}{2}+ir)} = -2\log c_0 + 2\sum_\rho \frac{-\beta}{\beta^2+(r-\gamma)^2}$$

Let

$$\omega(r) = -\Re \frac{\varphi^{*\prime}(\frac{1}{2}+ir)}{\varphi^*(\frac{1}{2}+ir)} = 2\log c_0 + 2\sum \frac{\beta}{\beta^2+(r-\gamma)^2}.$$

$r-\gamma = \beta u$

Now

$$\int_{-T}^{T} \omega(r)\,dr = 4T\log c_0 + 2\sum \beta \int_{-T}^{T} \frac{dr}{\beta^2+(r-\gamma)^2}$$

$$= \quad '' \quad + 2\sum\beta \int_{\frac{-T-\gamma}{\beta}}^{\frac{T-\gamma}{\beta}} \frac{\beta\,du}{\beta^2+u^2\beta^2}$$

$$= 4T \log c_0 + 2 \sum \int_{-\frac{T+\gamma}{\beta}}^{\frac{T-\gamma}{\beta}} \frac{du}{1+u^2}$$

$$\leqq 4T \log c_0 + 2 \sum_{|\gamma| \leqq 2T} \int_{-\infty}^{\infty} \frac{du}{1+u^2} + \sum_{|\gamma| \geqq 2T} \frac{8T\beta}{\beta^2+\gamma^2}$$

$$\leqq 4T \log c_0 + 2 \cdot \pi \sum_{|\gamma| \leqq 2T} 1 + 2 \cdot 8T \sum_{|\gamma| \geqq 2T} \frac{\beta}{1+\gamma^2}$$

$$= O(T) + 2\pi \sum_{|\gamma| \leqq 2T} 1 + O(T)$$

$$= O(T) + O(T^{4+\varepsilon}) = O(T^{4+\varepsilon})$$

Similar things can be done for the determinant of the matrix $\Phi(s, \chi)$ in the more general case.

We had

$$\iint_{D} |\tilde{E}(z, \tfrac{1}{2} + ir, \chi)|^2 \frac{dx\,dy}{y^2} = 2 \log A - \Re \frac{\varphi'}{\varphi}(\tfrac{1}{2} + ir, \chi)$$
$$+ \frac{\overline{\varphi(\tfrac{1}{2} + ir, \chi)}\, A^{2ir} - \varphi(\tfrac{1}{2} + ir, \chi) A^{-2ir}}{2ir}$$

The last term has ~~...~~ absolute value $\leqq \frac{1}{|r|}$. From this we can estimate $\tilde{E}(z, \tfrac{1}{2} + ir, \chi)$. To do this we will need to investigate

$$H(z, z') = \frac{1}{4\pi} \int_{-\infty}^{\infty} e(r)\, E(z, \tfrac{1}{2} + ir, \chi)\, \overline{E(z', \tfrac{1}{2} + ir, \chi)}\, dr.$$

This is to be subtracted from $K(z, z')$ the difference corresponding to the discrete spectrum.

#19. March 29, 1955

We had ~~for σ > ½~~ for $s = \sigma + ir$ if $\sigma > \frac{1}{2}$

$$\iint_{D} |\tilde{E}(z,s,\chi)|^2 \frac{dx\,dy}{y^2} = \frac{A^{2\sigma-1} - |\varphi(\sigma+ir,\chi)|^2 A^{1-2\sigma}}{2\sigma-1} + \frac{\overline{\varphi(\sigma+ir,\chi)}\,A^{2ir} - \frac{\varphi(\sigma+ir,\chi)}{A^{2ir}}}{2ir}$$

$$= 2\log A - 2\Re\frac{\varphi'}{\varphi}(\tfrac{1}{2}+ir,\chi) + \quad ''$$

if $\sigma = \frac{1}{2}$

$$E(z,s,\chi) = y^s + \varphi(s,\chi)\,y^{1-s} + \sum{}' \alpha_n(s,\chi)\, e^{2\pi i n x}\sqrt{y}\,K_{s-\frac{1}{2}}(2\pi|n|y)$$

with $K_{s-\frac{1}{2}}(y) = \int_0^\infty t^{s-\frac{1}{2}} e^{-y(t+\frac{1}{t})}\frac{dt}{t}$.

Let $s = \sigma + ir$, $r \geq 0$, $\frac{1}{2} < \sigma < \frac{3}{2}$. Consider, Schwarz' inequality:

$$\left|\int_a^\infty f(y)\,K_{s-\frac{1}{2}}(y)\,dy\right| \leq \left\{\int_a^\infty |f(y)|^2\, y\,dy \int_a^\infty |K_{s-\frac{1}{2}}(y)|^2\frac{dy}{y}\right\}^{\frac{1}{2}}$$

We seek an $f(y)$ with large $\left|\int_a^\infty f(y)K_{s-\frac{1}{2}}(y)\,dy\right|$ + small $\int_a^\infty |f(y)|^2 dy$. Now

$$\int_a^\infty f(y)K_{s-\frac{1}{2}}(y)\,dy = \int_a^\infty f(y)\,dy\int_0^\infty u^{s-\frac{1}{2}} e^{-y(u+\frac{1}{u})}\frac{du}{u} \qquad u = yt$$

$$= \int_a^\infty f(y)\,y^{s-\frac{1}{2}}\,dy\int_0^\infty t^{s-\frac{1}{2}} e^{-y^2t-\frac{1}{t}}\frac{dt}{t}$$

$$= \int_0^\infty t^{s-\frac{1}{2}} e^{-a^2t-\frac{1}{t}}\frac{dt}{t}\int_a^\infty f(y)\,y^{s-\frac{1}{2}} e^{-(y^2-a^2)t}\,dy$$

Let $f(y) = y^{\frac{3}{2}-s}\, e^{ir\log(y^2-a^2)}$. Then we get

$$\int_0^\infty t^{s-\frac{1}{2}} e^{-a^2t-\frac{1}{t}}\frac{dt}{t}\int_a^\infty y\cdot e^{ir\log(y^2-a^2)-(y^2-a^2)t}\,dy$$

$$= \frac{1}{2}\int_0^\infty t^{s-\frac{1}{2}} e^{-a^2t-\frac{1}{t}}\frac{dt}{t}\int_0^\infty e^{ir\log u - tu}\,du \qquad \frac{\Gamma(1+ir)}{t^{1+ir}}$$

$$= \frac{1}{2}\Gamma(1+ir)\int_0^\infty t^{\sigma-\frac{1}{2}} e^{-a^2t-\frac{1}{t}}\frac{dt}{t}$$

$$= \frac{1}{2}\Gamma(1+ir)\,a^{\sigma-\frac{1}{2}}\int_0^\infty t^{\sigma-\frac{1}{2}} e^{-a(t+\frac{1}{t})}\frac{dt}{t}$$

$$\frac{1}{2}K_{\sigma-\frac{1}{2}}(a) = \frac{1}{2}\int_0^\infty t^{\sigma-\frac{1}{2}} e^{-a(t+\frac{1}{t})}\frac{dt}{t} \geq \int_1^\infty e^{-a(t+\frac{1}{t})}\frac{dt}{t} \geq \int_1^\infty e^{-a(t+1)}\frac{dt}{t}$$

$$\geq e^{-a}\int_1^\infty t^{-1}e^{-at}\,dt = e^{-a}\int_a^\infty t^{-1}e^{-t}\,dt$$

$$\cancel{\geq e^{-a}\int_a^2 t^{-1}e^{-t}dt \geq e^{-a}\int_a^2 t^{-1}e^{-2}dt = e^{-a-2}\log\frac{2}{a}} \qquad \cancel{a < 1}$$

$$\geq e^{-a}\int_a^\infty \tfrac{1}{2}e^{-t/2}\cdot e^{-t}\,dt \geq e^{-a}\tfrac{1}{2}\int_a^\infty e^{-\frac{3}{2}t}\,dt = e^{-a}\,\tfrac{1}{3}\,e^{-\frac{3}{2}a} \qquad a > 1$$

$$\left|\int_a^\infty f(y)\, K_{s-\frac12}(y)\,dy\right| \geq \frac13 a^{\frac12-\sigma}\, |\Gamma(1+ir)|\, e^{-\frac52 a}$$

$$\geq c_1 |\Gamma(1+ir)|\, e^{-3a}$$

Now $s\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}\, s$ or $\Gamma(1+s)\Gamma(1-s) = \frac{\pi s}{\sin \pi s}$. $s = ir$

$$|\Gamma(1+ir)|^2 = \frac{\pi i r}{\sin \pi i r} = \frac{\pi i r}{\frac{e^{-\pi r}-e^{\pi r}}{2i}} = \frac{-2\pi r}{e^{-\pi r}-e^{\pi r}} = \frac{2\pi r}{e^{\pi r}-e^{-\pi r}}$$

$$|\Gamma(1+ir)| = \sqrt{\frac{2\pi r}{e^{\pi r}-e^{-\pi r}}} \geq c_2\, e^{-\frac{\pi}{2}|r|}$$ and

$$\left|\int_a^\infty f(y)\, K_{s-\frac12}(y)\,dy\right| \geq c_1 c_2\, e^{-\frac{\pi}{2}|r|}\, e^{-3a} = c\, e^{-\frac{\pi}{2}|r| - 3a}$$

Now

$$|K_{s-\frac12}(y)| \leq \int_0^\infty t^{\sigma-\frac12} e^{-y(t+\frac1t)}\frac{dt}{t}$$

$$\leq \int_0^\infty t\, e^{-y(t+\frac1t)}\frac{dt}{t} = \int_0^\infty e^{-y(t+\frac1t)}\,dt$$

$$\leq \int_0^\infty \left(1+\frac{1}{t^2}\right) e^{-y(t+\frac1t)}\,dt$$

$$\leq 2\int_1^\infty e^{-y(t+\frac1t)}\,dt \leq 2\int_1^\infty e^{-yt}\,dt = \frac{2e^{-y}}{y}$$

and $|f(y)| = y^{\frac32-\sigma}$ so for $\Omega \geq 1$

$$\left|\int_\Omega^\infty f(y)\, K_{s-\frac12}(y)\,dy\right| \leq \int_\Omega^\infty y^{\frac32-\sigma}\cdot 2e^{-y} y^{-1}\,dy = 2\int_\Omega^\infty y^{\frac12-\sigma} e^{-y}\,dy$$

$$\leq 2\int_\Omega^\infty e^{-y}\,dy = 2e^{-\Omega}.$$ Let

$\Omega = 3a + \frac{\pi}{2}|r| + \omega$, large ω. Then

$$\left|\int_a^\Omega f(y)\, K_{s-\frac12}(y)\,dy\right| \geq \frac{c}{2}\, e^{-\frac{\pi}{2}|r| - 3a}$$

so

$$\frac{c}{2}\, e^{-\frac{\pi}{2}|r|-3a} \leq \int_a^\Omega f(y)\, K_{s-\frac12}(y)\,dy \leq \left\{\int_a^\Omega |f(y)|^2 y\,dy \int_a^\Omega |K_{s-\frac12}(y)|^2 \frac{dy}{y}\right\}^{1/2}$$

$$\leq \left\{\int_a^\Omega y^{4-2\sigma}\,dy \int_a^\infty |K_{s-\frac12}(y)|^2 \frac{dy}{y}\right\}^{1/2}$$

$$\leq \frac{\Omega^2}{2}\left\{\int_a^\infty |K_{s-\frac12}(y)|^2 \frac{dy}{y}\right\}^{1/2} \qquad \frac{1}{\sqrt{8}}$$

Hence

$$\int_a^\infty |K_{s-\frac12}(y)|^2 \frac{dy}{y} \geq c'\left(\frac{e^{-\frac{\pi}{2}|r|-3a}}{(3a+\frac{\pi}{2}|r|+\omega)^2}\right)^2 \geq c''\left(\frac{e^{-\frac{\pi}{2}|r|-3a}}{(\frac{\pi}{2}|r|+3a+\omega)^2}\right)^2$$

$$= c''\, \cancel{\frac{1}{x^2 e^x}} \geq \cancel{\frac{c'''}{e^x}\, e^{x/2}} \geq e^{-(\frac{\pi}{2}|r|+3a)(2+\epsilon)}$$

Hence

$$|\alpha_n(s,\chi)|^2 \int_\delta^\infty |K_{s-\frac{1}{2}}(\pi|n|y)|^2 \frac{dy}{y}$$

$$\geq |\alpha_n(s,\chi)|^2 \, e^{-(\frac{\pi}{2}|r| + 3\pi|n|\delta)(2+\varepsilon)}$$

We estimate $K_{s-\frac{1}{2}}(y)$ change path of integration to other line. Then

$$|K_{s-\frac{1}{2}}(y)| \leq \int_0^\infty e^{-(\frac{\pi}{2}-\varepsilon)r} \, t^{\sigma-\frac{1}{2}} \, e^{-y(t+\frac{1}{t})\sin\varepsilon} \frac{dt}{t}$$

$$= e^{-(\frac{\pi}{2}-\varepsilon)r} K_{\sigma-\frac{1}{2}}(y\sin\varepsilon)$$

$$\leq e^{-(\frac{\pi}{2}-\varepsilon)r} \cdot \frac{2e^{-y\sin\varepsilon}}{y\sin\varepsilon} \leq \frac{2e^{-(\frac{\pi}{2}-\varepsilon)r} \, e^{-y\frac{2}{\pi}\varepsilon}}{y\frac{2}{\pi}\varepsilon}$$

$$\leq \frac{\pi}{\varepsilon y} e^{-(\frac{\pi}{2}-\varepsilon)r - \frac{2}{\pi}\varepsilon y}$$

$$\leq \frac{4}{\varepsilon y} e^{-(\frac{\pi}{2}-\varepsilon)r - \frac{1}{2}\varepsilon y}$$

& in general

$$|K_{s-\frac{1}{2}}(y)| \leq \frac{4}{\varepsilon y} e^{-(\frac{\pi}{2}-\varepsilon)|r| - \frac{1}{2}\varepsilon y} \qquad \varepsilon > 0.$$

#20. ~~March~~ April 5, 1955

We had for $s = \sigma + ir$

$$\iint_D |E^{\sim}(z, s, \chi)|^2 \frac{dx\,dy}{y^2} = \begin{cases} 2\log A - 2\Re \frac{\varphi'}{\varphi}(\tfrac12 + ir) + \dfrac{\overline{\varphi(\frac12+ir)}A^{2ir} - \varphi(\frac12+ir)A^{-2ir}}{2ir} & \sigma = \tfrac12 \\ \dfrac{A^{2\sigma-1} - |\varphi(\sigma+ir)|^2 A^{1-2\sigma}}{2\sigma-1} + \dfrac{\overline{\varphi(\sigma+ir)}A^{2ir} - \varphi(\sigma+ir)A^{-2ir}}{2ir} & \sigma > \tfrac12 \end{cases}$$

Now the last term is bounded for s away from the poles on the real axis. The first term for $\sigma > \frac12$ is

$$\frac{(A^{2\sigma-1} - A^{1-2\sigma}) + A^{1-2\sigma}(1 - |\varphi(\sigma+ir)|^2)}{2\sigma-1}$$

with the first part $(\ )/2\sigma-1$ bounded for fixed A if $\frac12 < \sigma \le \frac32$. Hence, since we can take $A \ge 1$ everything depends on $\frac{1-|\varphi(\sigma+ir)|^2}{2\sigma-1}$.

i.e. the expression is $O(1) + O\left(\frac{1-|\varphi(\sigma+ir)|^2}{2\sigma-1}\right)$.

If we have no real pole then, with the poles $\rho = \beta + i\gamma$, $\beta < \frac12$

$$\varphi(s) = \pm\, c_0^{1-2s} \prod_\rho \frac{s-1+\bar\rho}{s-\rho}, \quad 2\sigma-1$$

$$|\varphi(\sigma+ir)|^2 = c_0^{2(1-2\sigma)} \prod_\rho \frac{(\sigma-1+\beta)^2 + (r-\gamma)^2}{(\sigma-\beta)^2 + (r-\gamma)^2}$$

$$= c_0^{-4(\sigma-\frac12)} \prod_\rho \left(1 - \frac{(2\sigma-1)(1-2\beta)}{(\sigma-\beta)^2 + (r-\gamma)^2}\right).$$

If $0 \le \vartheta_k \le 1$ then $1 - \prod \vartheta_k \le \sum(1-\vartheta_k)$. (induction) Hence

$$1 - |\varphi(\sigma+ir)|^2 \le 1 - c_0^{-4(\sigma-\frac12)} + (2\sigma-1)\sum_\rho \frac{1-2\beta}{(\sigma-\beta)^2 + (r-\gamma)^2}$$

$$\frac{1-|\varphi(\sigma+ir)|^2}{2\sigma-1} \le \frac{1 - c_0^{-2(2\sigma-1)}}{2\sigma-1} + \sum_\rho \frac{1-2\beta}{(\sigma-\beta)^2 + (r-\gamma)^2}$$

$$= O(1) + \sum_\rho \frac{1-2\beta}{(\frac12-\beta)^2 + (r-\gamma)^2}$$

$$= O(1) + 2\Re \frac{\varphi'}{\varphi}(\tfrac12 + ir)$$

If there are real poles then φ^* satisfies this inequality. And

$$\frac{1-|\varphi|^2}{2\sigma-1} = \frac{1-|\varphi^*|^2}{2\sigma-1} + \frac{|\varphi^*|^2-|\varphi|^2}{2\sigma-1}$$

The last term is uniformly bounded away from the real poles.

Hence the estimate for $\iint_{\mathcal{D}} |\tilde{E}(z,s,\chi)|^2 \frac{dx\,dy}{y^2}$ is

~~essentially~~ $O(\omega(r))$ with $\int_{-R}^{R} \omega(r)\,dr = O(R^{\alpha+\varepsilon})$,

and $\omega(r) = -\,\Re\frac{\varphi^{*\prime}(\frac{1}{2}+ir)}{\varphi^*(\frac{1}{2}+ir)}$; this holds away from the real poles.

By filling in with more fundamental regions we can prove that

$$\iint_{\substack{y\geq\delta\\ |x|\leq\frac{1}{2}}} |\tilde{E}(z,s,\chi)|^2 \frac{dx\,dy}{y^2} = O(\omega(r))$$

$$E(z,s,\chi) = y^s + \varphi(s)\,y^{1-s} + \sum{}' \alpha_n(s)\,e^{2\pi i n x}\sqrt{y}\,K_{s-\frac{1}{2}}(\pi|n|y)$$

Taking one term of the series

$$|\alpha_n(s)|^2 \iint_{\substack{y>\delta\\ |x|\leq\frac{1}{2}}} |K_{s-\frac{1}{2}}(\pi|n|y)|^2 \frac{dx\,dy}{y}$$

$$= |\alpha_n(s)|^2 \int_{\pi|n|\delta}^{\infty} |K_{s-\frac{1}{2}}(y)|^2 \frac{dy}{y} \qquad = O(\omega(r))$$

Hence

$$|\alpha_n(s)|^2 \cdot e^{-(\frac{\pi}{2}|r| + 3\pi|n|\delta)(2+\varepsilon)} = O(\omega(r))$$

$$|\alpha_n(s)|^2 = O\left(\omega(r)\, e^{(\frac{\pi}{2}|r| + \frac{3\pi}{\cancel{4}}|n|\delta)(2+\varepsilon)}\right)$$

$$|\alpha_n(s)| = O\left(\sqrt{\omega(r)}\, e^{(\frac{\pi}{2}|r| + \frac{3\pi}{\cancel{16}}|n|\delta)(1+\varepsilon)}\right)$$

but $E^*(z,s,\chi) = \sum{}' \alpha_n(s)\, e^{2\pi i n x}\sqrt{y}\,K_{s-\frac{1}{2}}(\pi|n|y)$. Then

$$E^*(z,s,\chi) = O\ \sum{}' \sqrt{\omega(r)}\, e^{(\frac{\pi}{2}|r| + \frac{3\pi}{16}|n|\delta)(1+\frac{\varepsilon}{2})} \sqrt{y}\ e^{-(\frac{\pi}{2}-\varepsilon')r - \frac{\varepsilon'}{2}\pi|n|y}\ \frac{1}{\pi|n|y}$$

$$= O\ \frac{\sqrt{\omega(r)}\, e^{(c+\varepsilon')|r|}}{\sqrt{y}} \sum{}' e^{10|n|\delta}\ \frac{1}{n}\cdot e^{-\frac{\pi}{2}|n|\varepsilon' y}$$

$$= O\sqrt{\omega(r)}\ \frac{e^{(\varepsilon+\varepsilon')r}}{\sqrt{y}}\ \Sigma'\ \cancel{e^{-\frac{\pi}{2}|n|\varepsilon' y}}\ e^{-|n|\left(\frac{\pi}{2}\varepsilon' y-10\delta\right)}$$

and we want $\frac{\pi}{2}\varepsilon' y - 10\delta > 0$ i.e. $y \geq \frac{20\delta}{\pi\varepsilon'}$. We take $y \geq a$ & then take $\delta \ni$ $10\delta \leq \frac{\pi}{4}\varepsilon' a \geq \frac{\pi}{4}\varepsilon' y$ so that

$$\frac{\pi}{2}\varepsilon' y - 10\delta \geq \frac{\pi}{2}\varepsilon' y - \frac{\pi}{4}\varepsilon' y = \frac{\pi}{4}\varepsilon' y$$

& then get

$$E^*(z,s,\chi) = O\sqrt{\omega(r)}\quad e^{(\varepsilon+\varepsilon')r}\ \frac{1}{\sqrt{y}}\ \Sigma'\ e^{-|n|\frac{\pi}{4}\varepsilon' y}$$

$$= O\sqrt{\omega(r)}\quad e^{(\varepsilon+\varepsilon')r}\ \frac{1}{\sqrt{y}}\ e^{-\frac{\pi}{4}\varepsilon' y}$$

$$= O\left(\sqrt{\omega(r)}\quad e^{2\varepsilon r}\ \frac{1}{\sqrt{y}}\ e^{-\frac{\varepsilon}{2}y}\right)$$

$$= O\left(\sqrt{\omega(r)}\ e^{2\varepsilon r - \frac{\varepsilon}{2}y}\ \cancel{\frac{1}{\sqrt{y}}}\right) \quad \text{for } y \geq a > 0.$$

We had some time ago

$$K(z,z') = \sqrt{yy'}\ g\left(\log \frac{y}{y'}\right).$$

Let

$$H(z,z') = \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\ E(z,\tfrac{1}{2}+ir,\chi)\ E(z',\tfrac{1}{2}-ir,\bar{\chi})\,dr$$

Suppose $h(r) = O\left(e^{-\delta|r|}\right)$ for some $\delta > 0$.

Then have to examine $\int_{-\infty}^{\infty} \omega(r)\ e^{4\varepsilon|r|}\ e^{-\delta|r|}\,dr$

& then $\int_{-\infty}^{\infty} \omega(r)\, e^{-\frac{\delta}{2}|r|}\,dr$. Break up into

$\sum \int_{2^k}^{2^{k+1}}$ having an estimate $\sum e^{-\frac{\delta}{2}2^k}\left(2^{k+1}\right)^{4+\varepsilon}$,

since $\int_{-R}^{R}\omega(r)\,dr = O(R^{4+\varepsilon})$ $\qquad O\ \text{"}\ \sum e^{-\frac{\delta}{2}2^k}\, 2^{5k}$ & we get convergence.

Last time –

$h(r)$ even, regular for $|\Im(r)| \leq \frac{1}{2}+\varepsilon$

$h(r) = O\left(e^{-\delta|r|}\right)$

$$H(z,z') = \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\, E(z,\tfrac{1}{2}+ir,\chi)\, \overline{E(z',\tfrac{1}{2}-ir,\bar\chi})\, dr$$

$$K(z,z') = \sum_{M\in\Gamma} \chi(M)\, k(z, Mz') \qquad k \text{ derived from } h$$

Then in the case of 1 cusp

$$K(z,z') - H(z,z') = O\left((yy')^{\alpha}\right) \quad \text{for some } \alpha < \tfrac{1}{2}$$

With $\varphi(s,\chi)$ having σ as the smallest pole $> \frac{1}{2}$, then we can take $\alpha = 1-\sigma$. Kernels of the above type commute among themselves, the spectrum is discrete, the eigenfunctions is square-integrable. We wrote

$$E(z,s,\chi) = y^{s} + \varphi(s,\chi)\, y^{1-s} + E^{*}(z,s,\chi)$$

& then

$$E^{*}(z,s,\chi) = O\left(\sqrt{\omega(r)}\; e^{2\varepsilon|r| - \frac{\varepsilon}{2}y}\right)$$

$\omega(r)$ even, >0, $\int_0^R \omega(r)\,dr = O(R^{\kappa+\varepsilon})$.

defined in A

$$\iint_{\mathcal{D}} |\tilde{E}(z,\tfrac{1}{2}+ir,\chi)|^2\, \frac{dx\,dy}{y^2}$$

$$= 2\log A - 2\Re\, \frac{\varphi'}{\varphi}(\tfrac{1}{2}+ir,\chi) + \cancel{\varphi(\tfrac{1}{2}+ir,\chi) A^{2ir}}$$

$$+ \frac{\overline{\varphi(\frac{1}{2}+ir,\chi)}\, A^{2ir} - \varphi(\frac{1}{2}+ir,\chi)\, A^{-2ir}}{2ir}$$

Now

$$\iint_{\substack{y\geq A\\ |x|\leq \frac{1}{2}}} |E^{*}(z,\tfrac{1}{2}+ir,\chi)|^2 \frac{dx\,dy}{y^2} = O\left(\omega(r)\, e^{\kappa\varepsilon|r|}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_A^{\infty} \frac{e^{-\varepsilon y}}{y^2}\, dx\,dy\right)$$

$$= O\left(\omega(r)\, e^{\kappa\varepsilon|r|}\, \frac{e^{-\varepsilon A}}{A^2}\right)$$

$$\therefore \iint_{\substack{\mathcal{D}\\ |y|<A}} |E^{(A)}(z,\tfrac{1}{2}+ir,\chi)|^2 \frac{dx\,dy}{y^2} = 2\log A - 2\Re\,\frac{\varphi'}{\varphi}(\tfrac{1}{2}+ir,\chi)$$

$$+ \frac{\overline{\varphi(\frac{1}{2}+ir,\chi)}\, A^{2ir} - \varphi(\frac{1}{2}+ir,\chi)\, A^{-2ir}}{2ir} + O\left(\frac{\omega(r)\, e^{\kappa\varepsilon|r| - \varepsilon A}}{A^2}\right)$$

$\mathcal{D}_A$

$$\iint_{D_A} H(z,z)\,\frac{dx\,dy}{y^2} = \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\int_{D_A}\left|E(z,\tfrac12+ir,\chi)\right|^2\frac{dx\,dy}{y^2}$$

$$= \frac{\log A}{2\pi}\int_{-\infty}^{\infty} h(r)\,dr - \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\,\Re\,\frac{\varphi'}{\varphi}(\tfrac12+ir,\chi)\,dr$$

$$+ \frac{1}{8\pi i}\int_{-\infty}^{\infty}\frac{h(r)}{r}\left\{\overline{\varphi(\tfrac12-ir,\chi)}\,A^{2ir} - \varphi(\tfrac12+ir,\chi)\,A^{-2ir}\right\}dr$$

$$+ O\Big(\underline{e^{-\varepsilon A}\int_{-\infty}^{\infty}|h(r)|\,\omega(r)\,e^{+\varepsilon|r|}\,dr}\Big)$$

A^2

Let $\varepsilon = \frac{\delta}{8}$ then integrand is $O(e^{-\frac{\delta}{2}|r|}\,\omega(r))$ & we get an estimate of $O(e^{-\frac{\delta}{8}A})$ for the last term. the next to the last integral is $O(\frac{1}{\log A})$ by partial integration. Hence

$$\iint_{D_A} H(z,z)\,\frac{dx\,dy}{y^2} = \frac{\log A}{2\pi}\int_{-\infty}^{\infty} h(r)\,dr - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\,\Re\,\frac{\varphi'}{\varphi}(\tfrac12+ir,\chi)\,dr$$

$$\underbrace{+\,O\left(\frac{1}{\log A}\right) + O\left(e^{-\frac{\delta}{8}A}\right)}_{=O\left(\frac{1}{\log A}\right)=o(1)\ \text{as}\ A\to\infty.}$$

$$k(z,z+n) = k\left(\left(\frac{|z-z-n|}{y}\right)^2\right) = k\left(\frac{n^2}{y^2}\right)$$

A

S_A

$u = n/y$

$$\iint_{S_A}\sum_{n=-\infty}^{\infty}{}' k(z,z+n)\,\frac{dx\,dy}{y^2} = \int_0^A\sum_{-\infty}^{\infty}{}' k\left(\frac{n^2}{y^2}\right)\frac{dy}{y^2} = \sum_{-\infty}^{\infty}{}'\int_0^A k\left(\frac{n^2}{y^2}\right)\frac{dy}{y^2}$$

$$= \sum_{-\infty}^{\infty}{}'\int_{\infty}^{n/A} k(u^2)\,\frac{-\frac{n}{u^2}}{n^2/u^2}\,du \qquad y=\frac{n}{u}$$

$$= 2\sum_{1}^{\infty}\frac{1}{n}\int_{\frac{n}{A}}^{\infty} k(u^2)\,du \qquad \frac{n}{A}\le u$$

$$= 2\int_{\frac{1}{A}}^{\infty} k(u^2)\,du\sum_{1\le n\le Au}\frac{1}{n} = 2\int_0^{\infty} k(u^2)\left\{\log Au + \gamma + O\left(\frac{1}{Au}\right)\right\}du$$

$$= 2\int_0^{\infty} k(u^2)\log u\,du + 2(\log A + \gamma)\int_0^{\infty} k(u^2)\,du$$

$$\underbrace{+\,O\left(\frac{1}{\sqrt{A}}\int_0^{\infty}\frac{|k(u^2)|}{\sqrt{u}}\,du\right)}_{O\left(\frac{1}{\sqrt{A}}\right)}$$

k is bdd. near 0

$k = O\left(\frac{1}{u^{2+\delta}}\right)$

$$2\int_0^\infty k(u^2)\,du = 2\int_0^\infty k(t)\,\tfrac{1}{2}\tfrac{1}{\sqrt{t}}\,dt = \tfrac{1}{4}\int_0^\infty \frac{k(t)}{\sqrt{t}}\,dt = g(0)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\,dr.$$

Hence the logarithmic term in A drops out. Letting $A \to \infty$ we now get

$$\iint_{\mathcal{D}} \{K(z,z) - H(z,z)\}\frac{dx\,dy}{y^2} = \ldots \qquad \text{(non parabolic terms)}$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\,\mathcal{R}\,\frac{\varphi'}{\varphi}(\tfrac{1}{2}+ir,\chi)\,dr$$

$$+ \cancel{\frac{\gamma}{2\pi}\int_{-\infty}^{\infty} h(r)\,dr} + \gamma g(0)$$

$$+ 2\int_0^\infty k(u^2)\log u\,du.$$

$$u^2 = t$$

$$\underline{I} = 2\int_0^\infty k(u^2)\log u\,du = \int_0^\infty k(t)\log t\cdot\tfrac{1}{2}\tfrac{1}{\sqrt{t}}\,dt = \tfrac{1}{2}\int_0^\infty k(t)\frac{\log t}{\sqrt{t}}\,dt$$

Now

$$\int_v^\infty \frac{k(t)}{\sqrt{t-v}}\,dt = Q(v) \qquad Q(e^u+e^{-u}-2) = g(u)$$

$$k(t) = -\frac{1}{\pi}\int_t^\infty \frac{dQ(w)}{\sqrt{w-t}}$$

so

$$I = -\frac{1}{2\pi}\int_0^\infty \frac{\log t}{\sqrt{t}}\,dt\int_t^\infty \frac{dQ(w)}{\sqrt{w-t}} \qquad 0 \le t \le w < \infty$$

$$= -\frac{1}{2\pi}\int_0^\infty dQ(w)\int_0^w \frac{\log t}{\sqrt{t(w-t)}}\,dt \qquad t = wu$$

$$= -\frac{1}{2\pi}\int_0^\infty dQ(w)\int_0^1 \frac{\log w + \log u}{\cancel{w}\sqrt{u(1-u)}}\,\cancel{w}\,du$$

$$\int_0^1 \frac{du}{\sqrt{u(1-u)}} = \int_0^1 u^{-1/2}(1-u)^{-1/2}\,du = B(\tfrac{1}{2},\tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = \pi$$

or $u = \sin^2\vartheta$ so $\displaystyle\int_0^{\frac{\pi}{2}} \frac{2\sin\vartheta\cos\vartheta\,d\vartheta}{\sin\vartheta\cos\vartheta} = \pi.$ Hence

$$I = -\frac{1}{2\pi}\int_0^\infty dQ(w)\left\{\pi\log w + \int_0^1 \frac{\log u}{\sqrt{u(1-u)}}\,du\right\}$$

Now $\displaystyle\int_0^1 u^{s-1}(1-u)^{\frac{1}{2}-1}\,du = \frac{\Gamma(s)\Gamma(\frac{1}{2})}{\Gamma(s+\frac{1}{2})} = \frac{\sqrt{\pi}\,\Gamma(s)}{\Gamma(s+\frac{1}{2})}$

Diff.

$$\int_0^1 u^{s-1}\log u \cdot (1-u)^{-\frac12}\,du = \sqrt{\pi}\;\frac{\Gamma(s+\frac12)\Gamma'(s) - \Gamma(s)\Gamma'(s+\frac12)}{\Gamma^2(s+\frac12)}$$

$$\int_0^1 u^{-\frac12}(1-u)^{-\frac12}\log u\,du = \sqrt{\pi}\left\{\frac{\Gamma'(\frac12) - \sqrt{\pi}\,\Gamma'(1)}{1}\right\}$$

$$= \sqrt{\pi}\left\{\Gamma'(\tfrac12) + \sqrt{\pi}\,\gamma\right\}$$

$$= \pi\left\{\frac{\Gamma'(\frac12)}{\Gamma(\frac12)} + \gamma\right\}$$

Now $\Gamma(s)\Gamma(s+\frac12) = A\,2^{1-2s}\,\Gamma(2s)$

$$\frac{\Gamma'(s)}{\Gamma(s)} + \frac{\Gamma'(s+\frac12)}{\Gamma(s+\frac12)} = -2\log 2 + 2\,\frac{\Gamma'(2s)}{\Gamma(2s)}$$

$s=\frac12$

$$\frac{\Gamma'(\frac12)}{\Gamma(\frac12)} + \frac{\Gamma'(1)}{\Gamma(1)} = -2\log 2 + \frac{2\,\Gamma'(1)}{\Gamma(1)}$$

$$\frac{\Gamma'(\frac12)}{\Gamma(\frac12)} = -2\log 2 + \frac{\Gamma'(1)}{\Gamma(1)} = -\gamma - 2\log 2$$

$$\int_0^1 \frac{\log u}{\sqrt{u(1-u)}}\,du = \pi(-\gamma - 2\log 2 + \gamma) = -2\pi\log 2.$$

Hence ~~But letting give~~ $\cancel{\pi(-2\log 2)}$.

$$I = -\frac{1}{2\pi}\int_0^\infty (\pi\log w - 2\pi\log 2)\,dQ(w)$$

$$= -\log 2\cdot g(0) - \frac12\int_0^\infty \log w\,.\,dQ(w) \qquad w = e^u + e^{-u} - 2 = (e^{\frac u2} - e^{-\frac u2})^2$$

$$= -g(0)\log 2 - \int_0^\infty \log(e^{\frac u2} - e^{-\frac u2})\,dg(u)$$

$$= -g(0)\log 2 - \int_0^\infty \log(e^{\frac u2} - e^{-\frac u2})\cdot g'(u)\,du$$

$$= -g(0)\log 2 - \frac12\int_0^\infty u\,g'(u)\,du - \int_0^\infty g'(u)\log(1-e^{-u})\,du$$

$$= -g(0)\log 2 - \frac12\left\{u\,g(u) - \int g(u)\,du\right\}_0^\infty - \int_0^\infty g'(u)\log(1-e^{-u})\,du$$

$$= -g(0)\log 2 + \frac14\int_{-\infty}^\infty g(u)\,du - \int_0^\infty g'(u)\log(1-e^{-u})\,du$$

$$= -g(0)\log 2 + \frac14 h(0) - \int_0^\infty g'(u)\log(1-e^{-u})\,du$$

Now

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^\infty h(r)\,e^{iru}\,dr$$

so

$$g'(u) = \frac{i}{2\pi}\int_{-\infty}^\infty h(r)\,e^{iru}\,r\,dr$$

$$\int_0^\infty g'(u)\log(1-e^{-u})\,du = \frac{i}{2\pi}\int_0^\infty \log(1-e^{-u})\,du \int_{-\infty}^{\infty} r\,h(r)\,e^{iru}\,dr$$

$$= \frac{i}{2\pi}\int_{-\infty}^{\infty} r\,h(r)\,dr \int_0^\infty e^{iru}\log(1-e^{-u})\,du .$$

~~Since $h(r)$ is even, $r\,h(r)$ is odd~~

$$\int_0^\infty e^{iru}\log(1-e^{-u})\,du = -\int_0^\infty e^{iru}\sum_{k=1}^{\infty}\frac{e^{-uk}}{k}\,du$$

$$= -\sum_{k=1}^{\infty}\frac{1}{k}\left\{\frac{e^{+u(ir-k)}}{ir-k}\right\}_0^\infty = \sum_{k=1}^{\infty}\frac{1}{k}\,\frac{1}{ir-k} = -\sum_1^\infty \frac{1}{k(k-ir)}$$

$$= \frac{1}{2}\left\{\sum_1^\infty \frac{1}{k(k+ir)} - \sum_1^\infty \frac{1}{k(k-ir)}\right\} - \frac{1}{2}\left(\sum_1^\infty \frac{1}{k(k+ir)} + \sum_1^\infty \frac{1}{k(k-ir)}\right)$$

$$= \frac{1}{2}\sum_1^\infty \frac{-2ir}{k(k^2+r^2)} - \frac{1}{2}\left(\qquad \right)$$

$$= -ir\sum_1^\infty \frac{1}{k(k^2+r^2)} - \frac{1}{2}\left(\qquad \right)$$

#23. April 26, 1955

We had

only parabolic terms

$$K(z,z') = \sum_{M \in \Gamma} \chi(M)\, k(z, Mz')$$

$$H(z,z') = \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r)\, E(z, \tfrac{1}{2}+ir, \chi)\, \overline{E(z', \tfrac{1}{2}+ir, \chi)}\, dr$$

Some things were not quite right in the earlier treatment so we go over it.

earlier had a factor of 2

actually real so do not need $\mathcal{R}$

$$\iint_{\mathcal{D}} |\tilde{E}(z, \tfrac{1}{2}+ir, \chi)|^2 \frac{dx\,dy}{y^2} = 2 \log A - \frac{\varphi'}{\varphi}(\tfrac{1}{2}+ir, \chi)$$

$$+ \frac{\varphi(\frac{1}{2}-ir) A^{2ir} - \varphi(\frac{1}{2}+ir) A^{-2ir}}{2ir}$$

$$+ O\left(\omega(r)\, e^{\frac{\delta}{2}r - \frac{\delta}{8}A}\right)$$

because $\varphi(s)\varphi(1-s) = 1$

$$\frac{\varphi'(s)}{\varphi(s)} = \frac{\varphi'(1-s)}{\varphi(1-s)}$$

$$\frac{\varphi'(\frac{1}{2}+ir)}{\varphi(\frac{1}{2}+ir)} = \frac{\varphi'(\frac{1}{2}-ir)}{\varphi(\frac{1}{2}-ir)} = \overline{\left\{\frac{\varphi'(\frac{1}{2}+ir)}{\varphi(\frac{1}{2}+ir)}\right\}}$$

$$\iint_{\mathcal{D}_A} H(z,z) \frac{dx\,dy}{y^2} = \frac{1}{2\pi} \log A \underbrace{\int_{-\infty}^{\infty} h(r)\, dr}_{= g(0) \log A} - \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{\varphi'}{\varphi}(\tfrac{1}{2}+ir)\, dr$$

$$+ \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{\varphi(\frac{1}{2}+ir) A^{2ir} - \varphi(\frac{1}{2}-ir) A^{-2ir}}{2ir}\, dr$$

$$+ O(e^{-\frac{\delta}{8}A})$$

The last integral J is O.K. except near $r=0$ & $\therefore$ $\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \to 0$ as $A \to \infty$. $\therefore$ value depends on $h(0)$ & $\varphi(\frac{1}{2})$. Consider then

$$I_\varepsilon = \frac{1}{4\pi} \int_{-\infty}^{\infty} h(0)\, \varphi(\tfrac{1}{2})\, e^{-\varepsilon r^2} \frac{A^{2ir} - A^{-2ir}}{2ir}\, dr.$$

Now

$$J - I_\varepsilon = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ \frac{h(r)\varphi(\frac{1}{2}+ir) - h(0)\varphi(\frac{1}{2}) e^{-\varepsilon r^2}}{2ir} A^{2ir} + \cdots \right\} dr$$

$\to 0$ as $A \to \infty$ by the ~~Cauchy~~ Riemann-Lebesgue theorem.

And, with conditional converge

$$I_0 = h(0) \frac{\varphi(\frac{1}{2})}{4\pi} \int_{-\infty}^{\infty} \frac{\sin(2r \log A)}{r}\, dr = \int_{-\infty}^{\infty} \frac{\sin u}{u}\, du = \pi$$

$$\therefore \quad J \to I_0 = \pi h(0) \frac{\varphi(\frac{1}{2})}{4\pi} = \frac{1}{4} h(0) \varphi(\tfrac{1}{2}), \qquad \varphi(\tfrac{1}{2}) = \pm 1.$$

$$\iint_{D_A} H(z,z)\,\frac{dx\,dy}{y^2} = g(0)\log A + \frac{1}{4}\,\varphi(\tfrac12)\,h(0)$$

$$\qquad - \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\,\frac{\varphi'}{\varphi}(\tfrac12 + ir)\,dr + o(1) \qquad \text{as } A\to\infty$$

~~Hence~~ And

$$\iint_{S_A} \sum{}' k(z, z+n)\,\frac{dx\,dy}{y^2} = (\log A + \underbrace{\gamma}_{\text{Euler constant}} - \log 2)\,g(0) + \frac14 h(0) - \int_0^\infty g'(u)\log(1-e^{-u})\,du$$

$$\qquad + o(1) \qquad \text{as } A\to\infty$$

Here

$$g'(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ir\,h(r)\,e^{iru}\,dr$$

So

$$L = -\int_0^\infty g'(u)\log(1-e^{-u})\,du = -\frac{1}{2\pi}\int_{-\infty}^{\infty} ir\,h(r)\,dr\int_0^\infty e^{iru}\log(1-e^{-u})\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} ir\,h(r)\sum_{k=1}^{\infty}\frac1k\,\frac{1}{k-ir}\,dr$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\sum_{k=1}^{\infty}\frac{ir}{k(k-ir)}\,dr$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\sum_{k=1}^{\infty}\left(\frac{1}{k-ir} - \frac1k\right)dr$$

Now

$$\frac{1}{\Gamma(z)} = e^{\overset{\text{Euler constant}}{\gamma} z}\, z\prod_{k=1}^{\infty}\left(1+\frac zk\right)e^{-\frac zk}$$

$$-\frac{\Gamma'(z)}{\Gamma(z)} = \frac1z + \gamma + \sum_{k=1}^{\infty}\left\{\frac{\frac1k}{1+\frac zk} - \frac1k\right\}$$

$$= \frac1z + \gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k+z} - \frac1k\right)$$

$$\Gamma(z+1) = z\Gamma(z)$$
$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac1z + \frac{\Gamma'(z)}{\Gamma(z)}$$

So

$$\sum_{k=1}^{\infty}\left(\frac{1}{k+z} - \frac1k\right) = -\gamma - \frac1z - \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{\Gamma'(z+1)}{\Gamma(z+1)}$$

$$L = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\left\{-\gamma - \frac{\Gamma'}{\Gamma}(1-ir)\right\}dr$$

& by evenness of $h(r)$ we get

$$L = -\gamma\, g(0) - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\,\frac{\Gamma'}{\Gamma}(1+ir)\,dr$$

$$\therefore \iint_{S_A} \Sigma' k(z, z+n) \frac{dx\,dy}{y^2} = (\log A - \log 2)\, g(0) + \tfrac{1}{4} h(0) - \tfrac{1}{2\pi}\int_{-\infty}^{\infty} h(r) \frac{\Gamma'}{\Gamma}(1+ir)\, dr + o(1)$$

$$\therefore \iint_{D} (K - H) \frac{dx\,dy}{y^2} = \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r) \frac{\varphi'}{\varphi}(\tfrac{1}{2}+ir)\, dr - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)\, dr$$

$$+ \tfrac{1}{4}(1 - \varphi(\tfrac{1}{2}))\, h(0) - g(0)\log 2$$

on letting $A \to \infty$.

(as $\frac{1}{2} - i\gamma$ for above!)

Letting the eigenvalues of the discrete spectrum be $\frac{1}{2} + i\gamma$ (even allowing γ to be purely imaginary). By the trace formula, (not Euler constant) without the factor 2

$$\Sigma_\gamma h(\gamma) - \tfrac{1}{2}\{1 - \varphi(\tfrac{1}{2})\} h(0) + \tfrac{1}{2\pi}\int_{-\infty}^{\infty} h(r) \frac{-\varphi'}{\varphi}(\tfrac{1}{2}+ir)\, dr$$

$$= \mu \int_{-\infty}^{\infty} r\, \frac{e^{\pi r} - e^{-\pi r}}{e^{\pi r} + e^{-\pi r}}\, h(r)\, dr - \frac{1}{\pi}\int_{-\infty}^{\infty} h(r) \frac{\Gamma'}{\Gamma}(1+ir)\, dr$$

(elliptic)

$$+ \cdots - g(0) \log 2$$

$$+ 2\left\{\sum_{\{P\}} \sum_{k=1}^{\infty} \frac{\chi^k\{P\} \log N\{P\}}{(N\{P\})^{\frac{k}{2}} - (N\{P\})^{-\frac{k}{2}}}\, g(k \log N\{P\})\right\}$$

In the general case have the matrix $\Phi(s, \chi)$ & logarithmic deriv. of determinant of $\Phi(s,\chi)$ in place of $\frac{\varphi'}{\varphi}(\tfrac{1}{2}+ir, \chi)$. Certain ~~factors~~ terms pick up a factor of κ in the general case.

~~This holds for~~ This holds for $h(r)$ when ~~$h(r) = O(e$~~ ~~$|\mathfrak{I}(r)| \le$~~ $h(r) = O(e^{-\delta|r|})$ in for $|\mathfrak{I}(r)| \le \frac{1}{2} + \varepsilon$. We want to extend this to more general $h(r)$ & for this need an estimate for the number of zeros. This depends on the new integral $\int_{-\infty}^{\infty} h(r) \frac{-\varphi'}{\varphi}(\tfrac{1}{2}+ir)\, dr$. Now

$$\varphi(s) = C_0^{1-2s} \prod_\rho \frac{s - 1 + \bar\rho}{s - \rho}$$

ρ are poles, $\rho = \frac{1}{2} - \beta + i\gamma$ with $\beta \ge 0$ except for a finite #

$$\frac{\varphi'(s)}{\varphi(s)} = -2\log C_0 + \sum_\rho \left\{\frac{1}{s-1+\bar\rho} - \frac{1}{s-\rho}\right\}. \text{ Then}$$

$$\frac{\varphi'}{\varphi}(\tfrac{1}{2}+ir) = -2\log C_0 + \sum_\rho \left\{ \frac{1}{-\beta+i(r-\lambda)} - \frac{1}{\beta+i(r-\lambda)} \right\}$$

$$= -2\log C_0 + \sum_\rho \frac{-\beta-i(r-\lambda)-\beta+i(r-\lambda)}{\beta^2+(r-\lambda)^2}$$

$$= -2\log C_0 + \sum_\rho \frac{-2\beta}{\beta^2+(r-\lambda)^2}$$

$C_0 \geqq 1$

$$-\frac{\varphi'}{\varphi}(\tfrac{1}{2}+ir) = 2\log C_0 + 2\sum_\rho \frac{\beta}{\beta^2+(r-\lambda)^2}.$$

Let $h(r) = e^{-r^2/R^2}$ with R large. Then $\int_{-\infty}^{\infty}\cdots = O(R^2)$

The result is that $\sum_\gamma h(\gamma) + \cancel{O(R^2)}\ \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{-\varphi'}{\varphi}\,dr = O(R^2)$

$$\therefore \quad \sum_\gamma h(\gamma) \geqq \sum_{|\gamma|\leqq R} \frac{1}{e} = \frac{1}{e}N(R). \text{ So}$$

$$\frac{1}{e}N(R) + \frac{1}{e}\int_{-R}^{R}\frac{-\varphi'}{\varphi}(\tfrac{1}{2}+ir)\,dr = O(R^2)$$

$$\therefore \ N(R) = O(R^2) \ \text{ \& } \int_{-R}^{R}\frac{-\varphi'}{\varphi}(\tfrac{1}{2}+ir)\,dr = O(R^2).$$

If $h(r) = O\left(\frac{1}{r^{2+\varepsilon}}\right)$ the right side of the trace formula is meaningful. Now use in place of such an $h(r)$ the function $h_\varepsilon(r) = h(r)e^{-\varepsilon r^2}$ & on letting $\varepsilon\to 0$ get the result for general even $h(r) = O\left(\frac{1}{r^{2+\varepsilon}}\right)$

We had

$$Z_\Gamma^{*}(s,\chi) = \prod_{\{P\}}\prod_{\nu=0}^{\infty}\left(1-\chi\{P\}(N\{P\})^{-s-\nu}\right)$$

& took

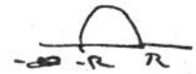
$$h(r) = \frac{1}{(s-\frac{1}{2})^2+r^2} - \frac{1}{(a-\frac{1}{2})^2+r^2}, \quad a>1, \quad \Re(s)>1$$

$$= O\left(\frac{1}{r^4}\right)$$

The new term $\int_{-\infty}^{\infty} h(r)\frac{-\varphi'}{\varphi}\,dr$

gives a new contribution. This depends on

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(s-\frac{1}{2})^2+r^2}\;\frac{2\beta}{\beta^2+(r-\lambda)^2}\,dr$$



which, by contour integration, can be evaluated. We get

$$\frac{1}{2}\sum_{\rho}\left(\frac{1}{s-\frac{1}{2}}\left\{\frac{1}{s-\rho}+\frac{1}{s-\bar\rho}\right\}-\frac{1}{a-\frac{1}{2}}\left\{\frac{1}{a-\rho}+\frac{1}{a-\bar\rho}\right\}\right)$$

The terms $2\sum_{\{P\}}\sum_{k=1}^{\infty}\cdots$ gave rise to $\frac{1}{s-\frac{1}{2}}\frac{Z'}{Z}(s,\chi)$.

The zeros of $\varphi(s)$ are not symmetric about $\frac{1}{2}$ so there is not a real symmetry for $Z(s)$. But it turns out that

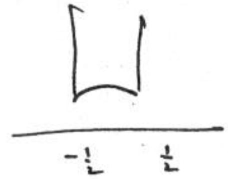
$$\frac{Z(1-s)}{Z(s)} = \varphi(s)\,\Psi(s)$$

where $\Psi(s)$ has zeros at integers & at half integers.

For the modular group one uses $\frac{Z(s)}{\zeta(2s)}$ which has a nice functional equation.

Consider the full modular group Γ ~~so that~~, let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $f(Mz) = f(z)$ invariant under Γ. Let

$$f^*(z) = \begin{cases} f(z) & \text{for } z \in \mathcal{D} \\ 0 & \text{for } z \notin \mathcal{D} \end{cases}$$



Then $f(z) = \sum_{M \in \Gamma} f^*(Mz)$.

Now let $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = n > 0$ & $M_n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Consider

$$\sum_{M_n} f^*(M_n z) = T_n f(z)$$

which is automorphic. These are actually the Hecke operators. They commute & $T_m T_n = T_{mn}$ for $(m, n) = 1$.

We had $K(z, z') = \sum_{M \in \Gamma} k(z, Mz')$. Now we use $\sum_{M_n} k(z, M_n z')$, & get commuting operators, symmetric ones, etc.

Equivalently, we ~~the~~ could have dealt with unitary $r \times r$ matrices. $F(Mz) = \chi(M) F(z)$. (?) with $\mathfrak{X}$ an r-dimensional space,

$\varphi(z, \mathfrak{X}) = \varphi(Mz, \chi(M)\mathfrak{X})$ with $\mathfrak{X}$

& the diff. equation $y^2 \Delta \varphi + (\frac{1}{4} + r^2)\varphi = 0$.

Automorphic forms satisfy $f(z) = \chi(M)(cz+d)^{-k} f(Mz)$; k is an integer although it might only be restricted to real values. If $z' = Mz$ then $y' = \frac{y}{(cz+d)^2} \cdot = \frac{y}{(cz+d)(c\bar{z}+d)}$ & $\therefore (cz+d)(c\bar{z}+d)\, y' = y$.

Then letting $\mu(z) = y^{\frac{k}{2}} f(z)$ we have

$$\mu(z) = \chi(M) \left(\frac{c\bar{z}+d}{cz+d}\right)^{\frac{k}{2}} \mu(Mz)$$

↑ absolute value 1.

For G: $z \to mt$, $t \to t - \arg(cz+d) + \alpha$. For μ: $z \to -\bar{z}$, $t \to -t$ universal covering group

We can establish that with $p = (z, t)$, $q = (z', t')$ we can make $\left\{ \begin{matrix} p \to \mu q \\ q \to \mu p \end{matrix} \right\}$ with same μ. As an invariant we have $\frac{|z - \bar{z}'|^2}{y y'}$ as before.

~~and~~ Now $\arg(z - \bar{z}') \to \arg\left(\frac{az+b}{cz+d} - \frac{a\bar{z}'+b}{c\bar{z}'+d}\right) = \arg \frac{z - \bar{z}'}{(cz+d)(c\bar{z}'+d)}$

$= \arg(z-\bar z') - \arg(cz+d) + \arg(cz'+d)$.

And $t-t' \to t-t' - \arg(cz+d) + \arg(cz'+d)$. Hence another invariant is $t-t' - \arg(z-\bar z')$. Under μ this last changes sign. Hence we should deal with

$$k\left(\frac{|z-z'|^2}{yy'},\ t-t' - \arg(z-\bar z')\right).$$

Then $\frac{dx^2+dy^2}{y^2}$ is an invariant form & so is $dt - \frac{dx}{2y}$.

This will not give a symmetric space in the sense of Cartan ~~as~~ on using $\frac{dx^2+dy^2}{y^2} + k\left(dt - \frac{dx}{2y}\right)^2$ with $k>0$. One gets $\frac{\partial}{\partial t}$ & something else an invariant operators.

The eigenfunction would have to look like

$$\varphi(z,t) = f(z)\, e^{i\alpha t}$$

& if $\chi(M)=1$ & k is an integer would get $\varphi(z,t) = e^{ikt} f(z)$. From ~~see~~ $k(z,z')$ one forms $K(z,z')$ & the rest of the theory. Not much more difficult than before.

Consider, with $\chi(M)=1$,

$$f(z,\bar z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}, \frac{a\bar z+b}{c\bar z+d}\right)$$

$$\frac{\partial f}{\partial \bar z}(z,\bar z) = (cz+d)^{-k}(c\bar z+d)^{-2}\frac{\partial f}{\partial \bar z}(Mz, M\bar z).$$

& if

$$y^2\frac{\partial f}{\partial \bar z} = f^*(z)$$

then

$$f^*(z) = (cz+d)^{2-k} f^*(Mz)$$

Can do a similar thing to get $(c\bar z+d)^{k-2}$ $\therefore$ only k's between -1 & 1 need be looked at. & by conjugation need only look at $0 \ldots 1$.

Let $K(z,z') = \sum \frac{1}{(cz+d)^k (Mz-\bar z')^k}$ & consider

$$f(z) = \lambda \iint_{\mathcal{D}} y'^k K(z,z') f(z') \frac{dx'dy'}{y'^2}.$$

Can also use the ~~Hecke~~ ~~forms~~ operators by using M_n in place of M. One could get the Ramanujan $\tau(n)$ but not a better estimate than $O(n^6)$.

Dpif

Positive definite symmetric matrices $Y > 0$; $AYA' = G$
$|Y| = 1$.

u: $Y \to Y^{-1}$ Take 3×3. Two fundamental operators.

with $|A| = 1$ & $\mathfrak{X}$ a vector

$$\sum' \frac{1}{(\mathfrak{X}' Y \mathfrak{X})^{s} (Z' Y^{-1} Z)^{s'}}$$ are eigen-functions, New type of zeta function

$\mathfrak{X}' Z = 0$

~~with~~ Dirichlet series of 2 variables. There are Functional equations.

Also a discrete spectrum. One considers

$$k\left(\sigma(AYA'Y_1^{-1}),\ \sigma(AYA'Y_1^{-1})^2\right).$$

$N(\alpha) - q N(\beta)$

$u^2 - A v^2 - q x^2 + q r z^2$

72 reverse side