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DEPARTMENT OF MATHEMATICS

Elementary methods and the distribution of "generalized primes".

The problem I am going to speak of is the following:

- We suppose that we have an infinite sequence of real numbers > 1 tending to infinity

$$1 < p_1 \le p_2 \le p_3 \le \dots$$

these numbers we call "primes". From these "primes" we build up a sequence of "integers" in the way that we form all possible products of the form $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ with $\alpha_1 \geq 0$, $\alpha_2 \geq 0 \dots \alpha_r \geq 0$. The numbers that we get we order in a nondecreasing sequence

$$n_1 = 1 \le n_2 \le n_3 \le n_4 \dots$$

~~The problem~~ We denote by $N(x)$ the number of $n_i \le x$ and by $\pi(x)$ the number of $p_i \le x$; The problem is now ~~from a given~~ that asymptotic behaviour of the $N(x)$ all what we can ~~to deduce~~ about the behaviour of $\pi(x)$ as $x \to \infty$.

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This problem has been ~~treated~~ introduced first by A. Beurling in: Sur la lois asymptotiques de la distribution des nombres premiers generalises Acta Math. ((1938), and treated by him by analytical means. We shall treat this problem under the assumption that

$$(1) \qquad N(x) = hx + O\left(\frac{x}{(\log x)^{\alpha}}\right)$$

as $x \to \infty$ h and α being positive constants, and ~~$2 < \alpha <$~~ 3, by means of a method which is completely elementary in the technical sense of this word, and prove that if $\alpha > 2$ (1) implies that

$$(2) \qquad \pi(x) \sim \frac{x}{\log x}$$

as $x \to \infty$.

First let me remark that obviously all properties of the ordinary natural numbers, which depend only upon multiplicative properties, also hold true for our sequence of "integers". That means that concepts as divisor, divisibility, and also numerical functions as the Möbius function, can be extended with all the properties conserved to our sequence of "integers".

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In particular we define as usual the function

$$\lambda(n_i) = \begin{cases} \log p_i & \text{if } n_i \text{ is a power of the prime} \\ 0 & \text{otherwise.} \end{cases}$$

And as ~~usual~~ in the case of the ordinary we define $\psi(x) = \sum_{n_i \le x} \lambda(n_i)$

and (2) is then equivalent to

(3). $\psi(x) \sim x$ as $x \to \infty$.

To prove (3) we first have to establish some preliminary formulas.

$\psi(x) = O(x)$

(4) $$\sum_{n_i \le x} \frac{\lambda(n_i)}{n_i} = \log x + O(1)$$

(4) is proved by considering the relation

$$\sum_{n_i \le x} \log n_i = \sum_{n_i \le x} \sum_{d_i / n_i \le x} \lambda(d_i)$$

$$= \sum_{d_i \le x} \lambda(d_i)\, N\left(\frac{x}{d_i}\right)$$

From (4) ~~follows~~ $\psi(x) = O(x)$

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Besides (4) we need a deeper asymptotic formula, which can be written $2 < \alpha < 3$ in one of the two forms

$$(5) \quad \log x \cdot \psi(x) + \sum_{n_i \le x} \Lambda(n_i)\, \psi\left(\frac{x}{n_i}\right) = 2x \log x + O\left(x (\log x)^{\frac{3}{\alpha}}\right)$$

or

$$(5') \quad \sum_{n_i \le x} \Lambda(n_i) \log n_i + \sum_{n_i n_i' \le x} \Lambda(n_i) \Lambda(n_i') = 2x \log x + O\left(x (\log x)^{\frac{3}{\alpha}}\right).$$

To prove (5') we start from the expression

$$\sum_{d_i / n_i} \mu(d_i) \log^2 \frac{x}{d_i} = \begin{cases} \log^2 x & \text{for } n_i = 1. \\ \log^2 x - \log^2 \frac{x}{p_i} & \text{if } n_i = p_i^r \\ 2 \log p_i \log p_i' & \text{if } n_i = p_i^r p_i'^{r'} \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\sum_{n_i \le x} \left\{ \sum_{d_i / n_i} \mu(d_i) \log^2 \frac{x}{d_i} \right\} = \log^2 x$$

$$+ \sum_{p_i^r \le x} \left(\log^2 x - \log^2 \frac{x}{p_i} \right) + \cancel{2} \sum_{p_i^r p_i'^{r'} \le x} \log p_i \log p_i'$$

$$= \log x \sum_{n_i \le x} \Lambda(n_i) + \sum_{n_i n_i' \le x} \Lambda(n_i) \Lambda(n_i') + O(x).$$

Secondly we have

4 reverse

$$2i \frac{1}{\sqrt{p}} \int_{|t|>p} \frac{e^{\pm i t \log p}\, dt}{t^{\theta}}$$

$n^{\frac{n}{2}}$ $t^{\frac{1}{2}} \log t$

$(|a_0|^2 + \cdots$ $(\sqrt{A_0})^{\frac{A_0}{2}} \cdots$

$$= \frac{1}{2i} \frac{1}{\sqrt{p}} \int_{|t|>p} \frac{e^{-it(\alpha \pm \log p)}\, dt}{t^{\theta}}$$

yes. thats $\boxed{\text{So}}$

$$\int \left| \sum \frac{e^{i\gamma\alpha}}{\gamma^{\theta}} \right|^2 d\alpha = \int |S(t)|^2 e^{\gamma t}$$

$$\int \frac{\log\log t}{t^{2\theta}}\, dt$$

divergent.

$p \log p$

$$\sum \frac{1}{|\gamma|^{2\theta}} = O(\cdots) \sum \frac{1}{\sqrt{p}\; p^{\theta}}$$

$\frac{1}{2}$ convergent.

$d\, S^2(t)$ $\int \frac{S(t)\, x^{i\gamma}}{\sqrt{t}}\, dt.$

$$\overline{\sqrt{p} \cdot p^{\theta} \left(1 + |\alpha \pm \log p| \right)}$$

$\int \frac{1}{t^{2\theta}}\, dt$ convergent.

$p^{\frac{1}{2}+\theta} \log p$. convergent

$t^{2\theta} = t \log t^{1+\epsilon}$

will be convergent.

$p^{\frac{1}{2}+\frac{1}{2}} (\log p)^{\frac{3}{2}+\frac{\epsilon}{2}}$

p

$\rho - \frac{1}{2} - \theta$

$$\sum_{\rho} \int \frac{u^{\rho-\frac{3}{2}}}{\log u} \left(\int_{|t|>u} \frac{e^{it(\alpha \pm \log u)}}{t^{\theta}}\, dt \right)$$

$\sum u^{\rho-1}$ $d\psi(u)$

$u^{\rho - \frac{1}{2} - \theta}$

$$\sum_{n_i\le x}\left\{\sum_{d_i\mid n_i}\mu(d_i)\log^2\frac{x}{d_i}\right\}=\sum_{d_i\le x}\mu(d_i)\log^2\frac{x}{d_i}\cdot$$

$$\sum_{\substack{d_i\mid n_i\\ n_i\le x}}1=\sum_{d_i\le x}\mu(d_i)\log^2\frac{x}{d_i}\;N\!\left(\frac{x}{d_i}\right)$$

$$=hx\sum_{d_i\le x}\frac{\mu(d_i)}{d_i}\log^2\frac{x}{d_i}+$$

$$O\left(x\sum_{d_i\le x}\frac{1}{d_i}\;\frac{\log^2\frac{x}{d_i}}{\left(1+\log\frac{x}{d_i}\right)^{\alpha}}\right)$$

$$=hx\sum_{d_i\le x}\frac{\mu(d_i)}{d_i}\log^2\frac{x}{d_i}+O\left(x\,(\log x)^{3-\alpha}\right).$$

Then we have to determine

$\sum_{d_i\le x}\frac{\mu(d_i)}{d_i}\log^2\frac{x}{d_i}$; to this end we need the formulas

(1) $$\sum_{n_i\le z}\frac{1}{n_i}=h\log z+c_1+O\left(\frac{1}{(\log z)^{\alpha-1}}\right)$$

and

(2) $$\sum_{n_i\le z}\frac{\log n_i}{n_i}=\frac{h}{2}\log^2 z+c_2\log z+c_3+O\left(\frac{1}{(\log z)^{\alpha-2}}\right)$$

further (3) $$\sum_{d_i\le z}\frac{\mu(d_i)}{d_i}=O(1).$$

From (1) and (2) we get

$$\log^2 z = \frac{2}{h}\sum_{n_i\le z}\frac{\log n_i}{n_i} + c_4\sum_{n_i\le z}\frac{1}{n_i} + c_5 + O\left(\frac{1}{(\log z)^{\alpha-2}}\right)$$

From this we get putting $z = \frac{x}{d_i}$

$$\sum_{d\le x}\frac{\mu(d)}{d}\log^2\frac{x}{d} = \frac{2}{h}\sum_{d\le x}\frac{\mu(d)}{d}\sum_{n\le \frac{x}{d}}\frac{\log n}{n} +$$

$$+ c_4\sum_{d\le x}\frac{\mu(d)}{d}\sum_{n\le \frac{x}{d}}\frac{1}{n} + c_5\sum_{d\le x}\frac{\mu(d)}{d} +$$

$$+ O\left(\sum_{d\le x}\frac{1}{d}\frac{1}{\left(1+\log\frac{x}{d}\right)^{\alpha-2}}\right)$$

$$= \frac{2}{h}\log x + O\left((\log x)^{3-\alpha}\right)$$

Inserting this we get

(B) $\log x\,\psi(x) + \sum_{n\le x}\Lambda(n)\psi\left(\frac{x}{n}\right) = 2x\log x + O\left(x(\log x)^{3-\alpha}\right)$

q.e.d.

From this we get

(B') $\phi(x) = \sum_{n\le x}\Lambda(n) + \sum_{nm\le x}\frac{\Lambda(n)\Lambda(m)}{\log nm} = 2x + O\left(x(\log x)^{2-\alpha}\right)$

and further applying B' to B we get

(B'') $\log x\,\psi(x) = \sum_{nm\le x}\frac{\Lambda(n)\Lambda(m)}{\log nm}\psi\left(\frac{x}{nm}\right) + O\left(x(\log x)^{3-\alpha}\right)$

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$$(7)\quad \lg x\, \psi(x) = \sum_{n_i n_i' \le x} \frac{\Lambda(n_i)\Lambda(n_i')}{\lg n_i n_i'}\, \psi\left(\frac{x}{n_i n_i'}\right) + O(x(\lg x)^{3-\alpha}).$$

Now we put

$$\psi(x) = x + R(x)$$

and get from (5) and (7) resp.

$$\log x \cdot R(x) = -\sum_{n_i \le x} \Lambda(n_i)\, R\left(\frac{x}{n_i}\right) + O(x(\lg x)^{3-\alpha})$$

and from

$$\lg x\, R(x) = \sum_{n_i n_i' \le x} \frac{\Lambda(n_i)\Lambda(n_i')}{\lg n_i n_i'}\, R\left(\frac{x}{n_i n_i'}\right) + O(x(\lg x)^{3-\alpha}).$$

or by combining

$$2 \lg x\, |R(x)| \le \sum_{n_i \le x} \Lambda(n_i) \left|R\left(\frac{x}{n_i}\right)\right| + \sum_{n n' \le x} \frac{\Lambda(n_i)\Lambda(n_i')}{\lg n_i n_i'} \left|R\left(\frac{x}{n_i n_i'}\right)\right|$$

$$+ O(x(\lg x)^{3-\alpha}) = \int_1^x \left|R\left(\frac{x}{t}\right)\right| d\phi(t) + O(x(\lg x)^{3-\alpha}).$$

or since by (6)

$$\phi(t) = 2t + q(t) \quad \text{where} \quad q(t) = O(t(\lg t)^{2-\alpha})$$

we get

$$2 \lg x\, |R(x)| \le 2\int_1^x \left|R\left(\frac{x}{t}\right)\right| dt + \int_1^x \left|R\left(\frac{x}{t}\right)\right| d|q(t)| + O(\ldots$$

7 reverse

$$\sum_{l-m \equiv v(g)} a_l a_m = 0$$

$$\begin{vmatrix} x_1 & x_2 \\ x_2 & x_1 \end{vmatrix}$$

$x_1^2 - x_2^2$.

$$\int |f(\log y)|^2 dy$$

$x^4 = x$ dy

$y = \log x$

$\sum_1^{q-1} |a_l|^2 = \underline{\underline{m}}$

at most $m \neq 0$.

$$\int_{-\infty}^{\infty} \left(S(t) + \frac{1}{\pi} \sum_{p \leq t^{\theta}} \cdots \right) \frac{x^{it}}{(1+t)^{\theta}} \, dt$$

$$\int |f(x)|^2 \frac{dx}{x}$$

$$\frac{1}{\log x} \int_{-\infty}^{\infty} \frac{x^{it}}{t^{\theta}} \, dS(t)$$

$a_{l_1} a_{l_2} \ldots a_{l_m}$

$$\sum_{\substack{i,j=1 \\ l_i - l_j \equiv m(g)}}^{m} a_{l_i} a_{l_j} = \underline{\underline{0}}.$$

$m - l = v$ or $g - g$

$l+$

$$\underline{\underline{\frac{1}{\log x} \sum \frac{x^{iy}}{y^{\theta}}}} -$$

$\log \frac{t}{2\pi}$

$\frac{t}{2\pi}$ $\log \frac{t}{2\pi e}$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{x^{it}}{t^{\theta}} \log \frac{t}{2\pi} \, dt$$

a_m $\sum a_l$

$(x_1 + x_2 + \cdot x_n) \prod_{i=1}^{n} (x_i - x_0 + 1)$

$$\sum_{m=1}^{v} a_m \{ a_{m+v} \}.$$

$t \log x = y$

$x^{it} = e^{it \log x}$

$(t^{\nu} - x^n)$

$t^1 - t^{\nu} x + t^{\nu} - \cdot +$

$$\frac{(\log x)^{1-\theta}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iy}}{y^{\theta}} \left(\log \frac{y}{2\pi} - \log x \right) dy$$

$$\begin{vmatrix} a_m & a_2 & \cdot & \cdot & a_n \\ a_2 & a_3 & \cdot & \cdot & a_1 \end{vmatrix}$$

$S\ (x_1^{\nu} - x^n)$

$t^1 + t^3$

$$\left(\log x^{\theta - \theta} \right) \left(c_1(\theta) - c_2(\theta) \log x \right).$$

$$(xp)^{it} - \left(\frac{x}{p} \right)^{-it}$$

$$\int_{|t| > p^{\frac{1}{\theta}}} \frac{x^{it} \sin t \log p}{t^{\theta} \sqrt{p}} \, dt$$

cyclic permutation.

even number of.

S_n

$x_1 - x^{\nu}$

$$\begin{vmatrix} x_1 - x_2, & x_2 - x_3 & \ldots & x_{n-1} - x_n \\ x_2 - x_3 & \ldots & & \\ 0 & 0 & 0 & \end{vmatrix} \begin{matrix} x_n \\ S_n \\ 1 \end{matrix}$$

$\frac{3}{\cdot}$.

3

$$\begin{vmatrix} x_1 & x_2 & \cdot & \cdot & x_n \\ x_2 & x_3 & & & x_1 \\ x_3 & x_4 & & & x_2 \\ x_n & x_1 & \cdot & \cdot & x_{n-1} \end{vmatrix} = 0$$

or $\boxed{\pm}$

prime numbers.

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or since

$$\int_1^x |R(\tfrac{x}{t})|\, dq(t) = q(x)|R(1)| - q(1)|R(x)|$$

$$- \int_1^x q(t)\, d|R(\tfrac{x}{t})| = O(x) + O\Big(\int_1^x \frac{t}{(\log t)^{\alpha-2}} |dR(\tfrac{x}{t})|\Big)$$

$$= O(x) + O\Big(-\int_1^x \frac{t}{(\log t)^{\alpha-2}}\, d\{\psi(\tfrac{x}{t}) + \tfrac{x}{t}\}\Big)$$

$$= O\Big(\int_1^x \frac{\psi(\frac{x}{t}) + \frac{x}{t}}{(\log t)^{\alpha-2}}\, dt\Big) = O(x(\log x)^{3-\alpha})$$

Hence

$$(8)\quad \cancel{x \log x}\, |R(x)| \le \frac{1}{\log x}\int_1^x |R(\tfrac{x}{t})|\, dt + O(x(\log x)^{2-\alpha})$$

$$= \frac{x}{\log x}\int_1^x \frac{|R(t)|}{t^2}\, dt + O(x(\log x)^{2-\alpha})$$

From (4) we get

$$(9)\quad \int_1^x \frac{R(t)}{t^2}\, dt = \int_1^x \frac{\psi(t) - t}{t^2}\, dt = \frac{\psi(x)}{x} + \int_1^x \frac{1}{t}\, d\psi(t)$$

$$- \log x = \sum_{m_i \le x} \frac{\lambda(m_i)}{m_i} - \log x + O(1) = O(1).$$

and from (8') if $t \le t'$

$$0 \le \psi(t') - \psi(t) \le 2(t' - t) + O(t'(\log t')^{2-\alpha})$$

8 reverse

$\prod_\nu (\alpha_\nu - \beta_\nu \theta)$ $\prod_\nu (\alpha_\nu^g - \beta_\nu^g) = \gamma g.$

$$|R(\theta)|^2 = A_0 + \sum_{\nu=1}^{g-1} A_\nu (\theta^\nu + \theta^{\nu-g})$$

$R(\theta) = \sqrt{A_0}\,\varepsilon$ ε .. $(\varepsilon_1\ \varepsilon_2\ \ldots\ \varepsilon_k)$ $g+1$

$$f(z)\cdot f\left(\tfrac{1}{z}\right) = A_0 + \sum_{\nu=1}^{g-1} A_\nu z^\nu (1 - z^{-g})$$

$\frac{R^2(\theta)}{A_0}$ all on unit circle $g A_0^2 + \sum 1 \cdot 1 = A_0 + 2\sum_\nu \frac{A_\nu^2 (g-1) A_0}{1 - z^g}$

$$\frac{z^g\left(f(z)\, f\left(\tfrac{1}{z}\right) - A_0\right)}{z^g - 1} = \frac{\sum_{\nu=1}^{g-1} A_\nu z^\nu}{z^{g-\nu}}$$

$\begin{vmatrix} x_1 & x_2 & \ldots & x_n \\ x_2 & x_3 & & x_n\ x_{n-1} \\ x_1 - x_2 & \ldots & & x_{n-1} - x_n \end{vmatrix}$

$\varepsilon_1\ \varepsilon_2 \ldots \varepsilon_n$

$z^{g-1} f(z)\, f\left(\tfrac{1}{z}\right)$

$A_\nu z^\nu - A_\nu z^{\nu-g}$ $x_{n-1}\ x_n \ldots x_m$

$a_0\, a_{g-1} = A_1$ $\alpha + \sqrt{1-\alpha^2}\, i$

$\frac{R^2(\theta)}{A}$ $A_\nu = -A$.. $a_0^2 + a_1^2 + \ldots$ $x_{n-1} - x_n\ \ x_n - x_1\ \ x_1 - x_2$

$A_\nu = -A_{g-\nu}$ $= A_{g-\nu} z^{-\nu}$ $\sum |A_\nu| \leq K$ if $(x_1 + x_2 + \ldots + x_n)$

$f(z)\, f\left(\tfrac{1}{z}\right) = A_0 + (z^g - 1)\frac{A_0}{2}$ $f(1) = A_0$

$x^2 + \alpha x + 1$

α is rational. $A_\nu z^\nu (1 - z^{-g}) - A_\nu z^{g-\nu}(1 - z^{-g})$

$a^g - b^g = c^g$ $A_\nu z^\nu (1 - z^{-g})(1 - z^{g-2\nu})$.

$|a - b\theta|$ no.

$f(z)\, f\left(\tfrac{1}{z}\right) - A_0$ $|A_\nu| \leq A_0$ $\sim A_0 \times 2\sum A_\nu^2$

$f'(z)\, f\left(\tfrac{1}{z}\right) - \frac{1}{z^2} f(z)\, f'\left(\tfrac{1}{z}\right) = 0$ $\leq g$

$\frac{R(\theta^h)}{R(\theta)}$

$|A_\nu| \leq \sum_{l-m=\nu} |a_l|\,|a_m|$ $\left(|A_0| + 2\sum |A_\nu|\right)^2$

$= (|a_0| + |a_1| \ldots + |a_{g-1}|)^2$

$\sum_{l,m} \left(|a_l|^2 + \cancel{|a_m|^2}\right) \leq g \cdot A_0$

$\sum 2 a_i a_j = 0.$ $(a_0^2 + a_1 \ldots + a_{g-1})^2 =$

From which

$$(C''') \qquad |R(t') - R(t)| \leq |t'-t| + O\left(\frac{t+t'}{(\lg(t+t'))^{\alpha-2}} \right)$$

The proof that $R(x) = o(x)$ now goes as follows: We know that

$$|R(x)| < K_1 x \text{ for } x > 1.$$

Now suppose that for some pos. const. σ we have

$$|R(x)| < \sigma x \text{ for } x > x_0$$

From (C') in the form

$$\left| \int_{x_1}^{x_2} \frac{R(t)}{t^2}\, dt \right| = O(1)$$

it follows that if in the interval (x_1, x_2) the $R(t)$ changes its sign at most once then

$$(D) \qquad \int_{x_1}^{x_2} \frac{|R(t)|}{t^2}\, dt < K_2$$

on the other hand if $R(t)$ changes the sign more than once there is a point in the interval where $R(t)$ vanishes.

Now we divide the interval $(1, x)$ into intervals $(\rho^\nu, \rho^{\nu+1})$; $\nu = 0, 1, \ldots \left[\frac{\lg x}{\lg \rho} \right]$, and discuss

The integrals

$$J_\nu = \int_{\rho^\nu}^{\rho^{\nu+1}} \frac{|R(t)|}{t^2}\, dt$$

for $\rho^\nu > x_0$; then we have ~~if~~ if $R(t)$ changes sign at most once in $(\rho^\nu, \rho^{\nu+1})$, that

$$|J_\nu| < K_2 = \frac{\sigma}{2} \log \rho \quad \left(\text{if we take } \rho = e^{\frac{2K_2}{\sigma}}\right)$$

if $R(t)$ changes sign at ~~least~~ twice then there is a point $\rho^\nu \leq t_0^* \leq \rho^{\nu+1}$ when $R(t_0) = 0$, and so from C'' for

$$|R(t^*)| \leq |t - t_0^*| + O\left(\frac{t + t_0}{(\log t + t_0)^{\alpha-2}}\right).$$

or

$$\frac{|R(t)|}{t} \leq \left|1 - \frac{t_0}{t}\right| + O\left(\frac{1}{(\log t)^{\alpha-2}}\right)$$

~~if we~~ From this we easily see that there is an interval $(e^{-\delta} t_0, e^{\delta} t_0)$ which such that $e^\delta = 1 + \frac{\sigma}{3}$; $\delta > K_3 \sigma$ in this interval

$$\frac{|R(t)|}{t} \leq \frac{\sigma}{3} + O\left(\frac{1}{(\log t)^{\alpha-2}}\right) < \frac{\sigma}{2} \text{ for } x_0$$

sufficiently large. A part $(t_1, e^\delta t_1)$ of this is contained in $(\rho^\nu, \rho^{\nu+1})$.