The problem we are going to consider in this lecture is the following:

Suppose that we have an infinite sequence of real numbers >1 tending to infinity

$$1 < p_1 \leq p_2 \leq p_3 \ldots \qquad p_i \to \infty$$

These numbers we call "primes". From the primes we build up a sequence of "integers" in the way that we form all possible products

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}\ ; \quad \alpha_i \geq 0$$

These numbers we order in an nondecreasing sequence

$$n_1 = 1 < n_2 \leq n_3 \leq \ldots \qquad n_i \to \infty.$$

Now we denote by $N(x)$ the number of $n_i \leq x$ and by $\pi(x)$ number of $p_i \leq x$. Our problem is from a given assymptotic behaviour of $N(x)$ as $x \to \infty$ to see what we can deduce about the behavior of $\pi(x)$ as $x \to \infty$.

This was first considered by A. Beurling in a paper in Acta Math (1938): Sur les lois assymptotique de la distribution des nombres premiers generalisées, where he attacks it by rather deep analytical methods.

We shall consider this problem under the assumption that:

[page 2 is missing, but see the earlier file]

To prove (3) in the form $R(x) = o(x)$, we first have to establish some preliminary formulas.

(4) $\psi(x) = O(x)$.

(4') $$\sum_{n_i \le x} \frac{\Lambda(n)}{n} = \log x + O(1)$$

(4') is proved by considering the relation

$$\overset{\sim N(x)\lg x \sim h x \lg x}{\sum_{n_i \le x} \lg n_i} = \sum_{n_i \le x} \sum_{d/n} \Lambda(d) = \sum_{d \le x} \Lambda(d)\, N\left(\frac{x}{d}\right)$$

Besides (4), we need a deeper assymptotic formula, which can be written in one of the two forms

(A) $$\psi(x)\lg x + \sum_{n \le x} \Lambda(n)\,\psi\left(\frac{x}{n}\right) = 2x\lg x + O\left(x(\lg x)^{3-\alpha}\right)$$

or

(A') $$\sum_{n \le x} \Lambda(n_i)\lg n_i + \sum_{nm \le x} \Lambda(n)\Lambda(m) = 2x\lg x + O\left(x(\lg x)^{3-\alpha}\right)$$

To prove (A') we start from the expression

$$\sum_{d_i/n_i} \mu(d_i)\log^2\frac{x}{d_i} = \begin{cases} \lg^2 x & \text{for } n = 1 \\ (2r-1)\lg^2 p \text{ if } & n = p^r \\ 2\lg p \lg p' \text{ if } & n = p^r p'^{r'} \quad p \ne p' \\ 0 \text{ otherwise} \end{cases}$$

Thus

$$\sum_{n \le x} \left\{ \sum_{d/n} \mu(d) \log^2 \frac{x}{d} \right\} =$$

$$= \log^2 x + \sum_{p^r \le x} \left(\log^2 x - \log^2 \frac{x}{p^r} \right) + $$

$$+ \sum_{\substack{p^r p'^{r'} \\ p \ne p'}} \log p \log p' = \log x \sum_{n \le x} \Lambda(n)$$

$$+ \sum_{n n' \le x} \Lambda(n) \Lambda(n') + O(x).$$

Secondly we have

$$\sum_{n \le x} \left\{ \sum_{d/n} \mu(d) \log^2 \frac{x}{d} \right\} =$$

$$= \sum_{d \le x} \mu(d) \log^2 \frac{x}{d} \cdot N\left(\frac{x}{d}\right)$$

$$= h x \sum_{d \le x} \frac{\mu(d)}{d} \log^2 \frac{x}{d} +$$

$$+ O\left(x \sum_{d \le x} \frac{1}{d} \frac{\log^2 \frac{x}{d}}{(1 + \log \frac{x}{d})^{\alpha}} \right)$$

$$= h x \sum_{d \le x} \frac{\mu(d)}{d} \log^2 \frac{x}{d} + O\left(x (\log x)^{3-\alpha} \right).$$

$$\sum_{d\le x} \frac{\mu(d)}{\cancel{d}} N\left(\frac{x}{d}\right) \log\frac{x}{d}.$$

$\underline{o(1)}$

$$A x \sum_{d\le x} \frac{\mu(d)}{d} \log\frac{x}{d} + O\left(x \sum \frac{\cancel{\log} \frac{x}{}}{d \log \frac{x}{d}}\right)$$

o $\quad o(x \log\log x).$ $\quad \int_1^x \frac{1}{t(1+\log\frac{x}{t})}\, dt$

$$O\, x \int_1^x \frac{1}{\cancel{t}}\, dt$$

$$\int_0^{\log x} \frac{du}{1+u}$$

$o.(\log\log x)$

$$\int_1^x o\left(\frac{x}{\cancel{t}}\right) dt$$

$\frac{x}{t} < K$ $\quad \varepsilon \quad O(x) + o$

$$x > t > \frac{x}{k}$$

$$\frac{1}{t\log t} \qquad o \int_1^\infty \frac{dt}{t\log^2 t}$$

$$\int_1^x \frac{1}{t}\, dN(t) = \frac{N(x)}{x} - \int_1^x \frac{N(t)}{t}\, dt$$

$$\sum_{d\le kx} \frac{\mu(d)}{d} \cancel{\log \frac{x}{d}}\; \underline{o(1)} \quad \text{as } d$$

$$\frac{x}{\cancel{d}} \cdot o\left(\frac{x}{\cancel{d}}\right)$$

$$\sum_{\substack{dn\le kx\\ dm\le kx}} \frac{\mu(d)\, d}{d^2 n m}$$

$\boxed{\text{4 reverse}}$

$$\sum_{m,n\le x} \frac{\sum_{d\mid(m,n)} \mu(d)\, d}{n m}.$$

$$\sum \frac{\mu(n)\,\varphi(n)}{n m}$$

Thus we have to determine.

$$S = \sum_{d\le x} \frac{\mu(d)}{d} \log^2\frac{x}{d}$$; to this end we need

the formulas

(1) $$\sum_{n\le z} \frac{\log n}{n} = \frac{h}{2}\log^2 z + c_1 \log z + c_2 + O\left(\frac{1}{(\log z)^{\alpha-2}}\right)$$

(2) $$\sum_{n\le z} \frac{1}{n} = h \log z + c_3 + O\left(\frac{1}{(\log z)^{\alpha-1}}\right)$$

(3) $$\sum_{d\le z} \frac{\mu(d)}{d} = O(1).$$

from (1) and (2) we get

$$\log^2 z = \frac{2}{h}\sum_{n\le z}\frac{\log n_i}{n_i} + c_4 \sum_{n\le z}\frac{1}{n} + c_5 + O\left(\frac{1}{(\log z)^{\alpha-2}}\right)$$

putting $z = \frac{x}{d}$ and inserting in exp. for S, we get

$$\sum_{d\le x}\frac{\mu(d)}{d}\log^2\frac{x}{d} = \frac{2}{h}\sum_{d\le x}\frac{\mu(d)}{d}\sum_{n\le\frac{x}{d}}\frac{\log n}{n}$$

$$+ c_4 \sum_{d\le x}\frac{\mu(d)}{d}\sum_{n\le\frac{x}{d}}\frac{h}{n} + c_5\sum_{d\le x}\frac{\mu(d)}{d}$$

$$+ O\left(\sum_{d\le x}\frac{1}{d}\,\frac{1}{(1+\log\frac{x}{d})^{\alpha-2}}\right) = \frac{2}{h}\log x + O((\log x)^{3-\alpha}).$$

Comparing above we get

$$(B) \qquad \log x\, \psi(x) + \sum_{n \le x} \Lambda(n)\, \psi\left(\frac{x}{n}\right) = 2x \log x + O\left(x (\log x)^{3-\alpha}\right).$$

From this we also get by partial summation

$$(B') \qquad \sum_{n \le x} \Lambda(n) + \sum_{nm \le x} \frac{\Lambda(n)\Lambda(m)}{\log n \log m} = 2x + O\left(x (\log x)^{2-\alpha}\right).$$

$$\underline{\psi(x) + \phi(x) = 2x + O\left(x (\log x)^{2-\alpha}\right).}$$

$$\log x\, \psi(x) - \sum_{n \le x} \Lambda(n)\, \phi\left(\frac{x}{n}\right) = O\left(x (\log x)^{3-\alpha}\right)$$

$$\sum_{n \le x} \Lambda(n)\left(\psi\left(\frac{x}{n}\right) - \phi\left(\frac{x}{n}\right)\right) = O\left(x (\log x)^{3-\alpha}\right)$$

$$2\psi(x) - 2x = \psi(x) - \phi(x) + o\left(\frac{x}{n}\right)$$

$$\sum_{n \le x} \Lambda(n)\, R\left(\frac{x}{n}\right) = O\left(x (\log x)^{3-\alpha}\right).$$

$$\psi(x) = 2x - \phi(x)$$

$$2\psi(x) = 2x + \left\{\psi(x) - \phi(x)\right\}$$

6 reverse

$$\int_{0}^{x} \psi\left(\frac{x}{t}\right) dt =$$

$$\sum_{n \le x} \lambda(n) \int_{0}^{\frac{x}{n}} dt .$$

$n \le \frac{x}{t}$

$t \le \frac{x}{n}$

$$x \sum_{n \le x} \frac{\lambda(n)}{n} - \psi(x) =$$

Comparing above we get

$$(B) \quad \log x \cdot \psi(x) + \sum_{n \le x} \Lambda(n)\, \psi\left(\frac{x}{n}\right) = 2x \log x + O(x(\log x)^{3-\alpha})$$

From this by part. summation

$$(B') \quad \sum_{n \le x} \Lambda(n) + \sum_{nm \le x} \frac{\Lambda(n)\Lambda(m)}{\log nm} = 2x + O(x(\log x)^{2-\alpha})$$

or $\quad \psi(x) + \phi(x) = 2x + q(x) \quad ; \quad q(x) = o(x).$

$$\log x \cdot \psi(x) + \int_1^x \psi\left(\frac{x}{t}\right) d\psi(t) = 2x \log x + o(x \log x)$$

$$\log x \cdot \psi(x) - \int_1^x \psi\left(\frac{x}{t}\right) d\phi(t) = o(x \log x)$$

or $\quad \psi(x) = x + R(x).$

$$\log x \; R(x) = - \int_1^x R\left(\frac{x}{t}\right) d\psi(t) + o(x \log x)$$

$$\log x \; R(x) = \int_1^x R\left(\frac{x}{t}\right) d\phi(t) + o(x \log x)$$

$$2R(x) \log x = \int_1^x R\left(\frac{x}{t}\right) d\big(\phi(t) - \psi(t)\big) + o(x \log x$$

$$2|R(x)| \log x \le \int_1^x \left|R\left(\frac{x}{t}\right)\right| d\big(\phi(t) + \psi(t)\big) + o(x \log x)$$

$$\le 2\int_1^x \left|R\left(\frac{x}{t}\right)\right| dt + \int_1^x \left|R\left(\frac{x}{t}\right)\right| dq(t) \,..$$

or

$$\int_1^x \left|R\left(\tfrac{x}{t}\right)\right| dq(t) = \left[R\left(\tfrac{x}{t}\right) q(t)\right]_1^x$$

$$\underbrace{- \int_1^x q(t)\, d\left|R\left(\tfrac{x}{t}\right)\right|}\; \le\; -\int_1^x o(t)\, d\left(\tfrac{x}{t} + \psi\left(\tfrac{x}{t}\right)\right)$$

$$= o(x \log x).$$

(C) $|R(x)| \le \dfrac{1}{\log x} \displaystyle\int_1^x \frac{|R(t)|}{t^2}\, dt + o(1).$

(C') $\displaystyle\int_1^x \frac{R(t)}{t^2}\, dt = O(1).$ $\left|\displaystyle\int_{x_1}^{x_2} \frac{R(t)}{t^2} dt\right| \le K_2$

(C'') $\left|R(t) - R(t')\right| \le |t - t'| + o(t + t').$

$R(t_1) - R$ t'

$|R(x)| < K_1 x$ for $x \ge 1$

$|R(x)| < \alpha x$ for $x > x_0$ $\rho = e^{\frac{2K_2}{\alpha}}$

(i) $\displaystyle\int_{\rho^{\nu-1}}^{\rho^\nu} \frac{|R(t)|}{t^2}\, dt \le K_2 = \frac{\alpha}{2}\log\rho$ for ρ suff. large.

(ii) $\displaystyle\int_{\rho^{\nu-1}}^{\rho^\nu} \frac{|R(t)|}{t^2}\, dt \le \alpha \log\rho - \frac{\alpha}{2}\delta$

$R(t_1) = 0$

$|R(t)| \le |t - t_1| + O(t+t_1)$

$\le \alpha \log\rho \left(1 - \dfrac{\delta}{2\log\rho}\right)$ $\left|\dfrac{R(t)}{t}\right| \le \left|1 - \dfrac{t_1}{t}\right| + O(1)$

$\le \dfrac{\alpha}{2}$ $\le \underline{\alpha \log\rho\left(1 - K_3\alpha^2\right)}$ $\le \dfrac{\alpha}{2}$

$\alpha_1 = \alpha(1 - K_5\alpha^2)\ldots$ for $te^{-\delta} \le t \le te^{\delta}$; $e^{\delta} = 1 + \frac{\alpha}{3}$, $\delta \ge K_4\alpha$