The problem we are going to consider in this lecture is the following:

Suppose that we have an infinite sequence of real numbers >1 tending to infinity

$$1 < p_1 \le p_2 \le p_3 \le \cdots \quad p_i \to \infty,$$

these numbers we call "primes". From the primes we build up a sequence of "integers" in the way that we form all possible products

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \quad ; \quad \alpha_i \ge 0$$

These numbers we order in a ~~in~~non de-creasing sequence

$$n_1 = 1 < n_2 \le n_3 \le \cdots, \quad n_i \to \infty$$

Now we denote by $N(x)$ the number of $n_i \le x$ and by $\pi(x)$ number of primes $p_i \le x$. Our problem is whether we from a given ~~law of~~ assymptotic behaviour of $N(x)$ as $x \to \infty$, can deduce anything about the assymptotic behaviour of $\pi(x)$ as $x \to \infty$.

This question was first considered A. Beurling in a paper in Acta Math. (1938). Sur la lois assymptotique de la distribution des nombres premiers generalisées, where

$$N(x) = A x + \cancel{\Theta(} \; o\left(\frac{x}{\log^2 x}\right)$$

$$\mu(n) = \{$$

$$\Lambda(n) = \{$$

$$\psi(x) = O(x) \ldots$$

$$x \int_1^x \frac{dt}{t \lg^2 \frac{x}{t}}$$

$$\int_0^{\lg x} \frac{du}{1+(\lg x - u)^2}$$

$$\sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

$$\sum_{n_i \le x} \log n_i = \sum_{n_i \le x} \sum_{d/n} \Lambda(d)$$

$$= \sum \Lambda(d) \; N\left(\frac{x}{d}\right) =$$

$$A x \sum \frac{\Lambda(d)}{d} + O\left(\cancel{\frac{x}{d}} \sum_{d \le x} \frac{\Lambda(d)}{d \log^2 \frac{x}{d}}\right)$$

$$\frac{\psi\left(\cancel{\frac{x}{2^r}}\right) 1.}{\cancel{2^r} \, 1+(\lg x - r)^2}$$

$$2^{k+1} \le \lg^2 \frac{x}{d} \le 2^k$$

$$\rho^{2k+1} \le \frac{x}{d} \le \rho^{2k}$$

he attacks it by rather deep analytical methods.

We shall consider this problem using only methods which are in the technical sense elementary.

under the assumption

(1) $N(x) = Ax + o\left(\frac{x}{\log^2 x}\right)$

where A ~~is~~ a pos. absolute constant and prove that (1) implies that

(2) $\pi(x) \approx \frac{x}{\log x}$

First let us remark that obviously all properties and concepts, connected with the ordinary natural properties, which depend only on the concept of multiplication ~~also hold for~~ can be extended to our generalized integers. This holds for concepts as divisors and divisibility, and numbertheoretical functions like Möbius function can be defined with all essential properties conserved.

Def.: $\mu(n) = \begin{cases} (-1)^\nu & \text{if } n \text{ is a product of } \nu \text{ diff primes} \\ 0 & \text{otherwise.} \end{cases}$

$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with } ~~\text{posi}~~\ k>0. \\ 0 & \text{otherwise.} \end{cases}$

As usual we define

$$\psi(x) = \sum_{n \le x} \Lambda(n) = x + R(x).$$

(2) is then equivalent to

$$(3) \qquad \psi(x) \sim x \quad \text{as } x \to \infty$$

or

$$(3') \qquad R(x) = o(x).$$

Preliminary formulas.

$$(A) \qquad \psi(x) = O(x)$$

$$(A') \qquad \sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

(A') stems from the relations

$$\sum_{n \le x} \log n = \sum_{n \le x} \sum_{d \mid n} \Lambda(d) = \sum_{d \le x} \Lambda(d)\, N\!\left(\frac{x}{d}\right)$$

Besides (A), we need a deeper asymptotic formula which can be written in one of the forms:

$$(B) \qquad \psi(x) \log x + \sum_{n \le x} \Lambda(n)\, \psi\!\left(\frac{x}{n}\right) = 2x \log x + o(x \log x)$$

$$(B') \qquad \sum_{n \le x} \Lambda(n) \log n + \sum_{mn \le x} \Lambda(m)\Lambda(n) = 2x \log x + o(x \log x)$$

For this purpose we ~~start from the~~ need the following facts:

(i) $$\sum_{d|n} \mu(d) \log\frac{x}{d} = \begin{cases} \log x & \text{for } n = 1 \\ \Lambda(n) & \text{for other } n \end{cases}$$

(ii) $$\sum_{d|n} \mu(d) \log^2\frac{x}{d} = \begin{cases} \log^2 x & \text{for } n = 1 \\ \log^2 x - \log^2\frac{x}{p_i} & \text{for } n = p^r \\ 2\log p_i \log p_j' & \text{for } n = p_i^r p_j'^{r'} ;\ p \neq p' \\ 0 & \text{otherwise} \end{cases}$$

(iii). $$\sum_{n \le x} \frac{1}{n} = A \log\frac{x}{k} + O\left(\frac{1}{\log x}\right)$$

or

$$\log x = \frac{1}{A}\sum_{n \le kx} \frac{1}{n} + O\left(\frac{1}{\log x}\right)$$

where k is a pos. constant.

To prove (A), we consider the expression

$$\sum_{n \le x} \left\{ \sum_{d|n} \mu(d) \log^2\frac{x}{d} \right\} = \log x + \sum_{p^r \le x} \left(\log^2 x - \log^2\frac{x}{p}\right)$$

$$+ \sum_{\substack{p^r p'^{r'} \le x \\ p \neq p'}} \log p \log p' = \log x \sum_{n \le x} \Lambda(n) +$$

$$+ \sum_{nm \le x} \Lambda(n)\Lambda(m) + O(x);$$

Secondly we have

$$\sum_{m\le x}\left\{\sum_{d/m}\mu(d)\log^2\frac{x}{d}\right\} =$$

$$\sum_{d\le x}\mu(d)\log^2\frac{x}{d}\; N\left(\frac{x}{d}\right) =$$

$$= A\,x\sum_{d\le x}\frac{\mu(d)}{d}\log^2\frac{x}{d} +$$

$$x\sum_{d\le x}\frac{1}{d}\log^2\frac{x}{d}\cdot o\left(\frac{1}{\log^2\frac{x}{d}}\right)$$

$$= Ax\cdot S\cdot = o\left(x\sum_{d\le x}\frac{1}{d}\right) = o(x\log x).$$

~~Further by (iii)~~

$$\sum_{d\le x}\frac{\mu(d)}{d}\log^2\frac{x}{d} = \sum_{d\le kx}\frac{\mu(d)}{d}\log^2\frac{x}{d} + O(1)$$

further by (iii)

$$\log\frac{x}{d} = \frac{1}{A}\sum_{n\le k\frac{x}{d}}\frac{1}{n} + o\left(\frac{1}{\log\frac{x}{d}}\right)$$

Hence

$$\sum_{d\le kx}\frac{\mu(d)}{d}\log^2\frac{x}{d} = \frac{1}{A}\sum_{d\le kx}\frac{\mu(d)}{d}\log\frac{x}{d}\sum_{n\le\frac{kx}{d}}\frac{1}{n}$$

$$+ \sum_{d\le kx}\frac{\log\frac{x}{d}}{d}\, o\left(\frac{1}{\log\frac{x}{d}}\right) = o(\log x)$$

$$= \frac{1}{A}\sum_{dn\le kx} \frac{\mu(d)\log\frac{x}{d}}{dn} + o(\log x) =$$

$$= \frac{1}{A}\sum_{m\le kx}\frac{1}{m}\sum_{d|m}\mu(d)\log\frac{x}{d} + o(\log x)$$

by (i)

$$= \frac{1}{A}\Big(\log x + \sum_{m\le kx}\frac{\Lambda(m)}{m}\Big) + o(\log x) =$$

$$= \frac{2}{A}\log x + o(\log x)$$

Combining we get (B), from which (B') follows immediately. By partial summation

$$(B'')\quad \sum_{n\le x}\Lambda(n) + \sum_{nm\le x}\frac{\Lambda(n)\Lambda(m)}{\log nm} = 2x + o(x).$$

or

$$\psi(x) + \phi(x) = 2x + q(x)\,;\quad q(x) = o(x).$$

Now (B') gives when inserting (B'')

$$\cancel{\log x\,\psi(x) + \int_1^x \psi\Big(\frac{x}{t}\Big)\,d\psi(t) = 2x\log x + o(x\log x)}$$

$$(B''')\quad \log x\,\psi(x) - \sum_{nm\le x}\frac{\Lambda(n)\Lambda(m)}{\log nm}\,\psi\Big(\frac{x}{nm}\Big) = o(x\log x)$$

or introducing $\psi(x) = x + R(x)$

we get

(1) $$R(x)\lg x = -\sum_{n\le x}\lambda(n)\,R\left(\frac{x}{n}\right) + o(x\lg x)$$

(1′) $$R(x)\lg x = \sum_{nm\le x}\frac{\lambda(n)\lambda(m)}{\lg(nm)}\,R\left(\frac{x}{nm}\right) + o(x\lg x)$$

$$2|R(x)|\lg x \le \sum_{n\le x}\lambda(n)\left|R\left(\frac{x}{n}\right)\right| + \sum_{nm\le x}\frac{\lambda(n)\lambda(m)}{\lg(nm)}\left|R\left(\frac{x}{nm}\right)\right| + o(x\lg x)$$

or

$$2|R(x)|\lg x \le \int_1^x\left|R\left(\frac{x}{t}\right)\right|\,d(\psi(t)+\phi(t)) + o(x\lg x)$$

$$2|R(x)|\log x \le 2\int_1^x\left|R\left(\frac{x}{t}\right)\right|dt + \int_1^x\left|R\left(\frac{x}{t}\right)\right|dq(t) + o(x\lg x)$$

or

(C) $$|R(x)| \le \frac{1}{\lg x}\int_1^x\left|R\left(\frac{x}{t}\right)\right|dt + o(x)$$

$$= \frac{x}{\log x}\int_1^x\frac{|R(t)|}{t^2}\,dt + o(x)$$

(c′) $$\int_1^x\frac{R(t)}{t^2}\,dt = O(1).$$

(c″) $$|R(t)-R(t')| \le |t-t'| + o(t+t').$$

Proof of ( ). 8.

1) $|R(x)| < K_1 x$ for $x \geq T$.

2) $|R(x)| < \sigma x$ for $x > x_0$

$$\left| \int_{x_1}^{x_2} \frac{R(t)}{t^2} dt \right| \leq K_2$$

Put $\rho = e^{\frac{4K_2}{\sigma}}$

if $R(t)$ changes sign at most once

(i) $$\int_{\rho^{\nu-1}}^{\rho^{\nu}} \frac{|R(t)|}{t^2} dt \leq 2K_2 = \frac{\sigma}{2} \log \rho$$

if $R(t)$ changes sign twice or more is a t_0 with $R(t_0) = 0$; $\underline{\rho^{\nu-1} > x_0}$

$$|R(t)| \leq |t - t_0| + o(t)$$

$$\leq \frac{\sigma}{2} t \quad \text{for } t_0 e^{-\delta} < t < t_0 e^{\delta}, \text{ where}$$

$$e^{\delta} = 1 + \frac{\sigma}{3}$$

$$\delta \geq K_4 \sigma.$$

Then $\rho^{\nu-1} > x_0$

(ii) $$\int_{\rho^{\nu-1}}^{\rho^{\nu}} \frac{|R(t)|}{t^2} dt \leq \sigma \lg \rho - \frac{\sigma}{2} \lg \delta =$$

$$\leq \sigma \lg \rho \left(1 - \frac{\lg \delta}{2 \lg \rho}\right) \leq \sigma \lg \rho (1 - K_5 \sigma^2).$$

Now

(C) gives that $\qquad \rho^\nu \le x_0$

$$|R(x)| \le \frac{x}{\log x}\left\{ \cancel{\int_1^{x_0} \frac{R(t)}{t^2}} \; \overline{\sum} \right.$$

$$\le \frac{x}{\log x}\left(K_6 + \sigma \log x\,(1 - K_5\sigma^2)\right) + o(x)$$

$$= \sigma(1 - K_5\sigma^2)\,x + o(x)$$

$$< \underline{\sigma(1 - K_7\sigma^2)}\,x \quad \text{for } x > X_1$$

$$\sigma_{i+1} = \sigma_i(1 - K_7\sigma_i^2). \quad ; \; \sigma_i \to 0 \text{ as } i \to \infty.$$