ABSTRACTS OF LECTURES GIVEN BY A. SELBERG AT TEL AVIV UNIVERSITY, NOVEMBER 1991

1. Variations on a theme of Euler.

About 1740, Euler introduced the Beta-integral

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{1}$$

and in 1837, Jacobi, who had noticed the analogy between the gamma function $\Gamma(x)$ and the gaussian sum

$$\tau_\chi = \sum_{h(\mathrm{mod}\ .p)} \chi(h)e^{2\pi i \frac{h}{p}},$$

where p is a prime and χ a dirichlet character, gave the relation

$$\sum_{h(\mathrm{mod}\ .p)} \chi_1(h)\chi_2(1-h) = \frac{\tau_{\chi_1}\tau_{\chi_2}}{\tau_{\chi_1\chi_2}}, \tag{2}$$

valid if none of the characters χ_1, χ_2 and $\chi_1\chi_2$ is the principal character χ_0.

In 1941, I stated the formula

$$\int_0^1 \cdots \int_0^1 (t_1 \dots t_n)^{x-1}\big((1-t_1)\dots(1-t_n)\big)^{y-1}|\Delta(t)|^{2z}dt_1\dots dt_n = \tag{3}$$

$$= \prod_{\nu=1}^{n} \frac{\Gamma(1+\nu z)\quad \Gamma(x+(\nu-1)z)\quad \Gamma(y+(\nu-1)z)}{\Gamma(1+z)\quad \Gamma(x+y+(n+\nu-2)z)},$$

where $\Delta(t) = \prod_{1\le i<j\le n}(t_i - t_j)$.

Seeking a discrete analogue of (3) for general n, which would reduce to (2) for $n=1$, I was at the time only able to prove such an analogue for $n=2$ (and this with a proof that had no relation to my proof for (3)), the formula for the general case could only be conjectured to be:

Let P_n run over all polynomials

$$P_n(x) = x^n + a_1x^{n-1} + \cdots + a_n$$

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX modulo p, and let $D(P_n)$ be the discriminant of P_n, then

$$\sum_{P_n} \chi_1((-1)^n P_n(0))\chi_2(P_n(1))\chi_3\Psi(D(P_n)) =$$

$$= \prod_{\nu=1}^{n} \frac{\tau_{\chi_3^\nu}\ \tau_{\chi_1\chi_3^{\nu-1}}\ \tau_{\chi_2\chi_3^{\nu-1}}}{\tau_{\chi_3}\ \tau_{\chi_1\chi_2\chi_3^{n+\nu-2}}}, \tag{4}$$

where Ψ denotes the quadratic character mod.p and χ_1, χ_2 and χ_3 are characters such that none of the subscripts of the τ symbols on the righthand side is the principal character χ_0.

In the 1980's, this conjecture was generalized to galois fields by Ronald Evans, who also numerically verified it for certain small primes in the case $n = 3$.

Finally in 1990, Greg Anderson succeeded in proving the general case of (4), using a beautiful new idea which works both for the continuous and the discrete case.

Greg Anderson's proof was sketched as well as my own proof of (3) (published in 1944).

A number of other Beta-type integrals not found in the literature were given and some applications mentioned.

2. Conjectures and results concerning a general class of dirichlet series.

The class considered is: dirichlet series of the form

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

and such that $(s-1)^m F(s)$ is an integral function of finite order for some integer $m \geq 0$, and which satisfies a functional equation of the form

$$\Phi(s) = \overline{\Phi(1-\bar{s})},$$

where

$$\Phi(s) = \varepsilon Q^s \prod_{i=1}^{k} \Gamma(\lambda_i s + \mu_i)\ \ F(s),$$

and $|\varepsilon| = 1$, $Q > 0$, $\lambda_i > 0$ and $Re\mu_i \geq 0$.

In addition we assume that

$$\log F(s) = \sum \frac{b_n}{n^s},$$

where the $b_n = 0$ unless n is of the form $n = p^r$, where p is a prime and r a positive integer. We finally assume that

$$a_n = \mathcal{O}(n^\delta)$$

for any positive constant δ, and that

$$b_n = \mathcal{O}(n^\theta)$$

for some fixed $\theta < \frac{1}{2}$.

Conjectures:

(I) $\sum\limits_{p<x} \frac{|a_p|^2}{p} = n_F \log\log x + \mathcal{O}(1)$, where n_F is a positive integer.

(II) If F is "primitive", that is: F can not be written as a product of two functions both of which again satisfy our conditions, then

$$n_F = 1.$$

(III) If F and F' are distinct primitive functions, then

$$\sum_{p<x} \frac{a_p \overline{a'_p}}{p} = \mathcal{O}(1).$$

(IV) The Riemann hypothesis holds for $F(s)$.

The behaviour of the sum

$$\sum_{p<x} \frac{|a_p|^2}{p},$$

turns out to govern the value distribution of $\log F(s)$ on and very near the line $\sigma = \frac{1}{2}$ (we write $s = \sigma + it$). Several results concerning the value distribution were given, in particular the function

$$\frac{\log F(\frac{1}{2} + it)}{\sqrt{\pi n_F \log\log |t|}},$$

has a normal gaussian distribution in the complex plane. Also, if $F_1(s)$ and $F_2(s)$ have no common factor (unique factorization into primitive factors being an easy consequence of (I), (II) and (III)), then they are statistically independent on and very near the line $\sigma = \frac{1}{2}$.

Conclusions were drawn concerning the distributions of zeros of $F(s) - a$ where $a \neq 0$, as well as concerning the value-distribution and zeros of finite linear combinations

$$\sum_{i=1}^{n} c_i Q_i^s F_i(s)$$

of functions F_i which have the same products of Γ functions in their functional equations.

3. Some questions of equidistribution concerning discrete groups of motions of the hyperbolic plane.

Let H denote the upper half plane model of the hyperbolic plane

$$z = x + iy, \quad y > 0,$$

with metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

invariant under the group G of motions given by

$$z \longrightarrow \frac{az + b}{cz + d},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a real unimodular matrix. Let Γ be a discrete subgroup such that the Haarmeasure of $\Gamma \backslash G$ is finite.

We write

$$r(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & , & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & , & \cos\frac{\theta}{2} \end{pmatrix}$$

where $0 \leq \theta < 2\pi$, then the elements of G can be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm r(\theta_1) \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix} r(\theta_2),$$

and this representation is unique if $a^2 + b^2 + c^2 + d^2 > 2$ and $p > 1$. Using this representation for the elements of Γ, we investigate the behaviour of the sum

$$\sum_{p<x} e^{i(m\theta_1 + n\theta_2)},$$

where m and n are integers, and the summation is taken over the elements of Γ for which $p < x$. Estimations are proved which imply that as $x \to \infty$, $\left(\frac{\theta_1}{2\pi}, \frac{\theta_2}{2\pi}\right)$ are equidistributed in the unit square.

In particular, if there are no eigenfunctions of the Laplacian $y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ with eigenvalues between 0 and $\frac{1}{4}$, which are invariant under Γ and square integrable over $\Gamma \backslash H$, then

$$\sum_{p<x} e^{i(m\theta_1+n\theta_2)} = \varepsilon_{m,n} cx + \mathcal{O}(x^{\frac{2}{3}}),$$

where c is a constant and $\varepsilon_{m,n} = 1$ for $m = n = 0$ and $\varepsilon_{m,n} = 0$ otherwise.

If Γ has a maximal parabolic subgroup Γ_∞ generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we also look at the representation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm r(\theta) \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

with $p \geq 1$, which again is essentially unique. We look at the elements of Γ for which $p < x$ and $0 \leq \lambda < 1$. We can show that as $x \to \infty$ the points $(\frac{\theta}{2\pi}, \lambda)$ are again equidistributed in the unit square.

Under the same assumption about Γ we also look at the representation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{c} \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{pmatrix},$$

valid for all elements of Γ which are not in Γ_∞. We now consider the sum

$$\sum_{\substack{0<|c|<x \\ 0 \leq \frac{a}{c}, \frac{d}{c} < 1}} e^{2\pi i(\frac{ma+nd}{c})}.$$

Again, as $x \to \infty$, we get equidistribution of $(\frac{a}{c}, \frac{d}{c})$ in the unit square.

In particular, if we choose $m = 1$, $n = 0$ and assume that the Eisenstein series belonging to the cusp at ∞ has no poles on the segment between 1 and $\frac{1}{2}$ of the real axis, we get

$$\sum_{\substack{0<|c|<x \\ 0 \leq a < |c|}} (x - |c|) e^{2\pi i \frac{a}{c}} = o(x^2).$$

If we specialize to the modular group this relation becomes

$$\sum_{0<n<x} (x - n)\mu(n) = o(x^2),$$

a relation which long ago was established to be equivalent to the prime number theorem.

Similar questions may be formulated concerning discrete groups of motions in higher dimensional symmetric spares, but in general they are of course much more difficult to handle.

Dear Matilda, Here are some titles of Prof. Selberg's lectures.

The first two are for a more general audience :

1. VARIATIONS ON A THEME OF EULER .

2. SOME CONJECTURES AND RESULTS CONCERNING A GENERAL CLASS OF DIRICHLET SERIES.

3. ZETA- AND L- FUNCTIONS ASSOCIATED WITH DISCRETE GROUPS OF MOTIONS ON PRODUCTS OF HYPERBOLIC PLANES.

4. SOME QUESTIONS OF EQUIDISTRIBUTION ASSOCIATED WITH DISCRETE GROUPS OF MOTIONS OF THE HYPERBOLIC PLANE.

Princeton July 18,1991.

Lectures for Tel Aviv

1. Variations on a theme of Euler.

2. Some Conjectures and results concerning a general class of dirichlet series.

3. Zeta- and L-functions associated with discrete groups of motions on products of hyperbolic planes.

4. Some questions of equi-distribution associated with discrete groups of motions of the hyperbolic plane.

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1 and 2 for a more general audience, 1 particularly so.