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1

General problem.

$$\mathcal{L}(s) = \sum_n \frac{a_n}{n^s} \; ; \quad a_n = O(n^{\delta}) \text{ for any } \delta > 0, \quad a_1 = 1$$

Eulerproduct

$$\log \mathcal{L}(s) = \sum \frac{b_n}{n^s} \; ; \quad b_n = 0 \text{ unless } n = p^{\alpha};\ \alpha > 0, \quad b_n = O(n^{\theta});\ \theta < \tfrac{1}{2}$$

$(s-1)^m \mathcal{L}(s)$ integral function

Functional equation. $|\varepsilon| = 1$

$$\Phi_{\mathcal{L}}(s) = \varepsilon Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) \; ; \quad \lambda_j > 0,\ \Re \mu_j \geq 0.$$

$$\Phi(s) = \overline{\Phi(1-\bar{s})} \; ; \quad \text{real for } s = \tfrac{1}{2} + it,\ t \text{ real.}$$

$\Lambda = \sum \lambda_j$

Several distinct $\mathcal{L}_j$; $j = 1, \dots n$, with same Γ factors

form

$$F(s) = \sum_{j=1}^{r} c_j \varepsilon_j Q_j^s \mathcal{L}_j(s) \; ; \quad c_j \neq 0, \text{ real.}$$

$$\prod_{j=1}^{m} \Gamma(\lambda_j s + \mu_j) F(s) \text{ real for } s = \tfrac{1}{2} + it.$$

Zeros of $F(s)$; apart from trivial zeros implied by functional equation. lie in some vertical strip $A \geq \sigma \geq -A$; number with im part between 0 and T

$$N(T,F) = \frac{\Lambda}{\pi} T(\log T + B) + O(\log T).$$

zeros with real part $\frac{1}{2}$, $N_0(T,F)$

conjectured $N_0(T,F) \sim N(T,F)$.

can be proved if a number of plausible conjectures are assumed.

What can we prove without any hypothesis

Clear we can only expect to do something where we can prove some real results for the single L's. This has only been possible in the case $\lambda = \frac{1}{2}$ and some cases for $\lambda = 1$.

We shall here look at the first case, this involve Dirichlets L functions. $L(s, \chi)$

Let χ be a primitive character mod q (we include case $q = 1$ and $\chi(n) = 1$ identically)

$$L(s, \chi) = \sum \frac{\chi(n)}{n^s}$$

integral function for $q \neq 1$; $(s-1)L$ integral for $q = 1$. Write $a = \dfrac{1 - \chi(-1)}{2}$

and
$$\Phi(s, \chi) = \varepsilon_\chi \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s, \chi), \ |\varepsilon_\chi| = 1$$

then
$$\Phi(s, \chi) = \overline{\Phi(1 - \bar{s}, \chi)}.$$

For simplicity we consider the case χ even with $a = 0$; χ odd is handled in the same way; n distinct χ_j;

Form
$$F(s) = \sum_1^n c_j \varepsilon_j q_j^{\frac{s}{2}} L(s, \chi_j); \ c_j \text{ real} \neq 0$$

or
$$\left(F(s) = \sum_1^n c_j \varepsilon_j (1 + q_j^{s - \frac{1}{2}}) L(s, \chi_j) \right)$$

Then $\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) F(s)$ is real for $s = \frac{1}{2} + it$; then

$$N(T, F) = \frac{T}{2\pi} (\log T + B) + O(\log T)$$

For the single $L(s,\chi)$ it has been proved that a pos. proportion of zeros have real part $\frac{1}{2}$. More precisely

$$N_0(T, L) \geq c\, T \log T \quad \text{for } T > Aq^2$$

where c and A are positive constants independent of q and χ.

For general combination some results implied in older literature. It follows from Hardy-Littlewood's work that

$$N_0(T, F) > c\, T \quad \text{for } T > T_0(F).$$

Recently A. A. Karatsuba has looked at combinations of the form

$$F(s) = \varepsilon L(s,\chi) + \bar{\varepsilon} L(s,\bar{\chi})$$

where χ is a complex character and obtained the result

(1994) $N_0(T, F) > T(\log T)^{\frac{1}{2}} e^{-c\sqrt{\log\log T}}$; $T > T_0$

where c is a pos constant. For more general linear combinations he has a much weaker and more complicated result.

I shall sketch a proof that for the general combination $F(s)$, we have

(1) $N_0(T, F) > c(n)\, T \log T$ for $T > T_0(F)$

and

(2) If $\omega(t) \to \infty$, then $F(\frac{1}{2}+it)$ has a zero in interval $(t, t+\frac{\omega(t)}{\log t})$ for almost all t.

$s = \frac{1}{2} + it;$ $t > 0$

$$\vartheta(t) = \arg \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)$$

$$X(t,\chi) = \varepsilon_\chi \, q^{\frac{it}{2}} e^{i\vartheta(t)} L(s,\chi)$$

"approx. funct. equ"

$$L(s,\chi) = \sum_{n < \sqrt{\frac{tq}{2\pi}}} \chi(n) n^{-s} + \varepsilon_\chi^{-2} q^{-it} e^{-2i\vartheta(t)} \sum_{n < \sqrt{\frac{tq}{2\pi}}} \bar{\chi}(n) n^{s}$$

$$+ O\left(\left(\frac{q}{t}\right)^{\frac{1}{4}}\right).$$

Write

$$\left(\zeta(s)\right)^{-\frac{1}{2}} = \sum \frac{\alpha_n}{n^s}, \; \alpha_1 = 1; \; \left(L(s,\chi)\right)^{-\frac{1}{2}} = \sum \frac{\alpha_n \chi(n)}{n^s}$$

and for $T \leq t \leq 2T$; $\xi = T^{\frac{1}{10}}$; $T > q^2$, write

$$\eta(s,\chi) = \sum_{n \leq \xi} \frac{\chi(n)\alpha_n}{n^s}\left(1 - \frac{\log n}{\log \xi}\right).$$ (for $\eta(\frac{1}{2}+it,\chi)$ write $\eta(t)$)

We consider for $\frac{1}{\log T} \leq H \leq \frac{\log\log T}{\log T}$
the three expressions.

$$I_\chi(t,H) = \int_t^{t+H} X(u,\chi) \left|\eta\left(\tfrac{1}{2}+iu,\chi\right)\right|^2 du$$

$$M_\chi(t,H) = \int_t^{t+H} L\left(\tfrac{1}{2}+iu,\chi\right)\eta^2\left(\tfrac{1}{2}+iu,\chi\right) du - H$$

$$J_\chi(t,H) = \int_t^{t+H} \left| X(u,\chi)\, \eta^2\left(\tfrac{1}{2}+iu,\chi\right)\right| du$$

Clear that if $J_\chi(t,H) > \left| I_\chi(t,H)\right|$

then $X(u, \chi)$ changes sign in $(t, t+H)$ and so has at least <u>one zero</u>.

Clearly

$$J_\chi(t, H) \geq H - |M_\chi(t, H)|;$$

thus if

$$|M_\chi(t, H)| + |I_\chi(t, H)| < H,$$

zero in $(t, t+H)$

Using approx funct. eqn. for $L(s, \chi)$

can show

$$(1) \quad \int_T^{2T} |I_\chi(t, H)|^2 dt = O\left(T \frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right),$$

$$(2) \quad \int_T^{2T} |M_\chi(t, H)|^2 dt = O\left(T \frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right),$$

and as we shall need much later:

$$(3) \quad \int_T^{2T} |L(\tfrac{1}{2}+iu, \chi)\, \eta^2(\tfrac{1}{2}+iu, \chi)|^2 du = O(T).$$

Here constants implied by O's are independent of χ and q.

We see now that

$$|I_\chi(t, H)| \leq \frac{H}{3}\ ; \ |M_\chi(t, H)| \leq \frac{H}{3}$$

except in a subset of $(T, 2T)$ of measure $O\left(\frac{T}{\sqrt{H \log T}}\right)$.

Thus for all t in $(T, 2T)$ except for subset of measure $O\left(\frac{T}{\sqrt{H \log T}}\right)$; $(t, t+H)$ contains a zero, choosing $H = \frac{\lambda}{\log T}$, with λ a large enough constant both statements made earlier follow for the single L-function

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To adapt this idea to the linear combination we need some general results about the value distribution of $\log | L(\frac{1}{2}+it, \chi)|$.

For $T > q^2$; k a positive integer and

$T^{\frac{1}{2k}} \leq x \leq T^{\frac{1}{k}}$, we can show

$$\int_T^{2T} \left| \log|L(\tfrac{1}{2}+it, \chi)| - R \sum_{p<x} \frac{\chi(p)}{p^{\frac{1}{2}+it}} \right|^{2k} dt =$$

$$= O\left(k^{4k} e^{Ak} T\right)$$

Constants implied by O independent of q and χ.

From this ~~we~~ can prove that

$$\frac{\log|L(\frac{1}{2}+it, \chi)|}{\sqrt{\pi \log\log t}}$$

has a normal Gaussian distribution more precisely, let $\mathcal{X}_{a,b}$ denote characteristic function of interval (a,b), then we have

$$\int_T^{2T} \chi_{a,b}\left(\frac{\log|L(s,\chi)|}{\sqrt{\pi \log\log T}}\right) dt = T\int_a^b e^{-\pi u^2} du + O\left(T \frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right)$$

(for our purposes here, weaker remainder

$$O(T(\log\log T)^{-\frac{1}{2}+\varepsilon}) \gg \varepsilon > 0$$

would do as well).

If we have two distinct L functions $L(s,\chi)$ and $L(s,\chi')$ then similar results hold for difference $\log|L(s,\chi)| - \log|L(s,\chi')|$ only here we have to divide by $\sqrt{2\pi \log\log t}$ to get the normal Gaussian distribution.

From this we see for instance that the set where

$$\left|\log|L(\tfrac{1}{2}+it,\chi)| - \log|L(\tfrac{1}{2}+it,\chi')|\right| \leq (\log\log T)^{2\delta} \quad ; \delta > 0$$

has measure

$$O\left(T(\log\log T)^{-\frac{1}{2}+2\delta}\right).$$

Thus for our n L functions $L(s,\chi_j)$ we have for $j \neq k$

$$\left|\log|L(\tfrac{1}{2}+it,\chi_j)| - \log|L(\tfrac{1}{2}+it,\chi_k)|\right| > (\log\log T)^{2\delta}$$

except in an exceptional subset of measure $O(T(\log\log T)^{-\frac{1}{2}+2\delta})$

Outside this exceptional set one single term is decisively dominant in linear comb. that forms $F(s)$.

Dominance is fairly stable over intervals of length H.

Define

$$\Delta_H(t,\chi) = \frac{1}{H}\int_t^{t+H} \log|L(\tfrac{1}{2}+iu,\chi)|\,du.$$

For $0 \le h \le H$, we can show for any fixed integer k

$$\int_T^{2T} \left(\Delta_H(t,\chi) - \log|L(\tfrac{1}{2}+i(t+h),\chi)|\right)^{2k} dt$$

$$= O\left(T\left(k^k e^{Ak} \log^k(e+H\log T) + k^{4k} e^{Ak}\right)\right)$$

Taking k large enough we see that except for an exceptional set of t of measure $O\left(\frac{T}{(\log\log T)^{2N}}\right)$ we have in $0 \le h \le H$ that

$$|\Delta_H(t,\chi_j) - \log|L(\tfrac{1}{2}+i(t+h),\chi_j)|| < (\log\log T)^{\delta}$$

except in a subset of $(t, t+H)$ of measure

$$O\left(\frac{H}{(\log\log T)^{N/2}}\right) \text{ ~~for~~ ..}$$

It is also now easy to show that we have for $j \ne k$,

$$|\Delta_H(t,\chi_j) - \Delta_H(t,\chi_k)| > \tfrac{1}{2}(\log\log T)^{2\delta}$$

outside of a set of measure at most

$O(T(\log T)^{-\frac{1}{2}+2\delta})$.

Thus we may divide $[T, 2T)$ into an exceptional set of measure $O(T(\log T)^{-\frac{1}{2}+2\delta})$ and n sets S_j such that in S_j

$$\Delta_H(t,\chi_j) > \Delta_H(t,\chi_k) + \frac{1}{2}(\log\log T)^{2\delta}$$

and in $(t, t+H)$; t in S_j

we have $t \le u \le t+H$

$$\log|L(\tfrac{1}{2}+iu, \chi_j)| \ge \log|L(\tfrac{1}{2}+iu,\chi_k)| + \frac{1}{3}(\log\log T)^{2\delta}$$

except for a bad subset of u of measure

$$O\left(\frac{H}{(\log\log T)^{\frac{N}{2}}}\right)$$

The measure of the S_j add up to

$T - O(T(\log\log T)^{-\frac{1}{2}+2\delta})$, can be shown

to have about equal measure (but not necessary)

~~In S_j we~~ We now look at the estimation

$$\int_T^{2T} |L(\tfrac{1}{2}+it,\chi_j)\,\eta^2(\tfrac{1}{2}+it,\chi_j)|^2\,dt = O(T)$$

see that we have

$$\int_t^{t+H} |L(\tfrac{1}{2}+iu,\chi_j)\,\eta^2(\tfrac{1}{2}+iu,\chi_j)|^2\,du \le H(\log\log T)^{\delta}$$

except for subset of $(T, 2T)$ of measure

$$O\left(\frac{T}{(\log\log T)^{\cancel{10}}}\right)$$

$\alpha = \frac{1}{\cancel{8}}$: exclude also these points from S_j -

Then in S_j' we have

$$\int_t^{t+H} | \quad |^2 du \leq H(\log\log T)^{\frac{1}{\cancel{2}}}$$

and thus for t in S_j' the bad subset can

contribute at most

$$O\left(H^{\frac{1}{2}} (\log\log T)^{\frac{1}{\cancel{4}}}\right) O\left(H^{\frac{1}{2}} (\log\log T)^{-\frac{N}{4}}\right)$$

$$= O\left(\frac{H}{(\log\log T)^{\frac{N}{5}}}\right)$$

to the integrals

$$I_{x_j}(t, H) \; ; \; M_x(t, H) \; ; \; J_x(t, H),$$

denoting ~~Calling~~ these integrals with the bad subset

excluded ~~by~~

$$I^*_{x_j}(t, H), \; M^*_{x_j}(t, H), \; J^*_{x_j}(t, H)$$

We see that we have a signchange for

of $X(t, x_j)$ in $(t, t+H)$ outside of the bad

subset if

$$J^*_{x_j}(t, H) > |I^*_{x_j}(t, H)|$$

which is equivalent to

$$H > |I_{\chi_j}(t,H)| + |M_{\chi_j}(t,H)| + O\left(\frac{H}{(\log\log T)^{\frac{N}{5}}}\right)$$

For T large enough this holds outside of a set with measure

$$O\left(\frac{T}{\sqrt{H\log T}}\right)$$ and therefore

in most of S_j' if

$$H = \frac{\lambda n^3}{\log T}$$

with λ a large enough constant.

This produces more than $\underbrace{}_{\text{of } \bar{n}^{-\frac{\Delta}{2}} \beta(\frac{\Delta}{2}) F(s)}$

$\frac{c}{n^3} T \log T$ sign changes in S_j' adding

up for all j' we get in all

$$> \frac{c}{n^2} T \log T \text{ sign changes.}$$

power statement 1. second follows taking $d = (\omega(t))^{\frac{1}{2}}$.

Dependence on n, sharpening estimations

$$O\left(T \frac{H^2}{(H\log T)^{\alpha}}\right) \quad \text{for any } \alpha$$

$$> \underline{c\, n^{-\frac{1}{\alpha}} T \log T} \quad ; \alpha = 1 ?$$

case $\Lambda = 1$. missing part.