B bounded homogeneous complex domain points $z = (z_1, \dots, z_n)$. G transitive group of complex analytic mappings $z \to gz$.

Automorphy factor $\rho_g(z)$, defined on (B, G) and such that

(1) $\rho_{g_1 g_2}(z) = \rho_{g_1}(g_2 z)\,\rho_{g_2}(z)$,

assume $\rho_g(z)$ holomorphic and $\neq 0$ in B
$\rho_g(z)$ bounded on compact subset of (B, G)
assume differentiable in G.

$\frac{f(gz)}{f(z)}$ where f holomorphic $\neq 0$ in B is trivial automorphy factor.

Can show that if B is irreducible that is not a direct product of domains fulfilling same conditions in lower dimension then

(2) $\rho_g(z) = \left(J_g(z)\right)^2 \frac{f(gz)}{f(z)}$

where
$J_g(z)$ is jacobian $\left| \frac{\partial (gz)_i}{\partial z_j} \right|$.

same conclusion if in (1) we only require it to hold up to a factor ε_{g_1, g_2} of abs. value 1. So can restrict ourselves to factors that are powers of jacobian of mapping. If B is reducible $\rho_g(z)$ is product of powers of the jacobians of the mappings of the irreducible factors.

Let $k(z, \bar{\zeta})$ be Bergmann kernel function of B and let $d\omega_z = k(z,\bar{z}) \prod_{i=1}^{n} dx_i dy_i$.
We look at Hilbert space H_r

$$(f, f)_r = \int_B \frac{|f(z)|^2}{(k(z,\bar{z}))^r} d\omega_z < \infty,$$

when $r > 0$ and assume it is not empty which is case of integral exists for $f = 1$, that is for $r > r_0$ where $0 < r_0 < 1$. If we write

$$T_g^r f(z) = (J_g(z))^r f(gz)$$

then

$$(f, h)_r = (T_g^r f, T_g^r h).$$

Produce complete orthonormal system $f_i(z)$, form

$$k_r(z, \bar{\zeta}) = \sum f_i(z) \overline{f_i(\zeta)}$$

(convergence proof along Bergmanns lines)
get that

$$\frac{k_r(z,\bar{\zeta})}{(k(z,\bar{\zeta}))^r}$$

is invariant for $z \to gz$, $\zeta \to g\zeta$, since B homogeneous get

$$k_r(z,\bar{\zeta}) = c(r) (k(z,\bar{\zeta}))^r.$$

Get

$$(3) \quad f(z) = c(r) \int_B \left(\frac{k(z,\bar{\xi})}{k(\xi,\bar{\xi})} \right)^r f(\xi)\, dw_\xi ,$$

for f in H_r, easy to see that formula holds also if f not in H_r as long as integral exists if we take abs. values. In all cases $c(r)$ is a polynomial of degree n (and r_0 is largest pos. zero of $c(r)$).

Let Γ be discrete subgroup of G, with compact fundamental domain $\mathcal{D}$ ~~in~~ B. Form of weight r, function $\overbrace{\text{in } B}^{\text{meromorphic}}$, and such that

$$(4) \quad \varepsilon_\gamma \left(J_\gamma(z) \right)^r f(\gamma z) = f(z)$$

for elements γ of Γ where $|\varepsilon_\gamma| = 1$, say $\varepsilon_\gamma (J_\gamma(z))^r$ is <u>consistent automorphy factor</u> for weight r. If f holomorphic in B, we say f is a regular form of weight r. Question of determining for a given Γ for which weights r there exist consistent automorphy factors unsolved in general. Clear that if r is an integer we can always take $\varepsilon_\gamma = 1$ and we have a consistent a.f., and the same of r is such that $(j_g(z))^r$ is singlevalued on the group G. Two consistent a.-f. for same r differ only by a factor $\chi(\gamma)$ where χ is a (one dimensional) groupcharacter of Γ. If we for simplicity assume r integral and $\varepsilon_\gamma = 1$ in (4) and that f is regular, we get from (3), writing $B = \sum_{\gamma\in\Gamma} \gamma^{-1}\mathcal{D}$,

$$f(z) = c(r) \sum_{\gamma\in\Gamma} \int_{\mathcal{D}} \left(\frac{k(z, \gamma^{-1}\bar{\zeta})}{k(\gamma^{-1}\zeta, \gamma^{-1}\bar{\zeta})} \right)^r f(\gamma^{-1}\zeta)\, d\omega_\zeta$$

$$= c(r) \sum_{\gamma\in\Gamma} \int_{\mathcal{D}} \left(\frac{k(\gamma z, \bar{\zeta})}{k(\zeta, \bar{\zeta})} \right)^r (j_\gamma(z))^r f(\zeta)\, d\omega_\zeta$$

$$= c_r \int_{\mathcal{D}} \frac{K_r(z, \bar{\zeta})}{(k(\zeta, \bar{\zeta}))^r} f(\zeta)\, d\omega_\zeta,$$

where we have put

$$K_r(z, \bar{\zeta}) = \sum_{\gamma\in\Gamma} (j_\gamma(z))^r (k(\gamma z, \bar{\zeta}))^r.$$

From this we get that the number of linearly independent regular forms of weight r (belonging to our a.-f.) is

$$N^{(r)} = c(r) \int_{\mathcal{D}} \frac{K_r(z, \bar{z})}{(k(z, \bar{z}))^r}\, d\omega_z,$$

this can be computed exactly by combining the terms in K_r where the γ are conjugate within Γ. First approximation is

(5) $N^{(r)} = c(r) V(\mathcal{D}) + O(r^{n-1})$.

We now specialize to the one dimensional case where B is the unit circle, which as is well known with the invariant metric is a model of the hyperbolic plane. Prefer to map unit circle into upper half plane H, $z = x+iy$; $y>0$, then the group G consists of $z \to \frac{az+b}{cz+d}$, with a, b, c, d real and $ad - bc = 1$. ($\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ real unimodular matrix).

invariant metric

$$ds^2 = \frac{dx^2+dy^2}{y^2}, \quad dw_z = \frac{dx\,dy}{y^2}$$

and invariant Laplacian $y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = y^2\Delta$. Geodesics circles orthogonal to Real line.

Fundamental domain $\mathcal{D}$ is now a polygon with even no. of sides corresponding in pair under Γ. Area of a polygon with n sides is $(n-2)\pi - \sum_1^n \alpha_i$ where α_i are the interior angles. Apart from identity elements g of Γ fall in 3 classes; hyperbolic if $|a+d|>2$, elliptic of $|a+d|<2$ and parabolic if $|a+d| = 2$. For the time being we still consider only case $\mathcal{D}$ compact, then no parabolic elements are present in Γ. If there are k inequivalent elliptic elements in Γ (there can only be a finite number < number of vertices in polygon) of order $m_1, \ldots, m_k$ respectively and g is the genus of the closed surface we get by identifying corresponding sides then

(6) $A(\mathcal{D}) = 2\pi\left(2g-2 + \sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right)\right)$

If $gz = \frac{az+b}{cz+d}$ then $j_g(z) = (cz+d)^{-2}$.

Let $N^{(r)}$ be the number of regular forms of weight r (r positive integer) then (5) takes the form

$$(7) \quad N^{(r)} = \frac{2r-1}{4\pi} A(\mathcal{D}) + O(1),$$

(if no elliptic elements the term $O(1)$ is actually zero, otherwise it is periodic in r).

For a regular form of weight r we can determine the number of inequivalent zeros (zeros in a fundamental domain), say N_r. We see that

$$y^r |f(z)|$$

is invariant under Γ. Let

$u = \log y^r |f(z)|$, $v = 1$, we have

$$r\, A(\mathcal{D}) = \iint_{\mathcal{D}} (u\, y^2 \Delta v - v\, y^2 \Delta u) \frac{dx\,dy}{y^2}.$$

If we remove a small geodesic circle with radius ρ around each zero of $f(z)$ (for those on the boundary we only remove the part inside $\mathcal{D}$), and call the resulting domain $\mathcal{D}^*$, we get using Green's formula that

$$r A(\mathcal{D}^*) = \iint_{\mathcal{D}^*} (u\, y^2 \Delta v - v\, y^2 \Delta u) \frac{dx\,dy}{y^2}$$

$$= \int_{\partial \mathcal{D}^*} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds ,$$

In last integral the contribution of $\partial \mathcal{D}$ cancels out (corresponding sides give opposite and equal contributions), so we are left with the contribution from the integrals over our small circles; this contribution is seen to be $2\pi l + O(\rho)$ for a zero of order l not at a fixpoint and for a zero at a fixpoint of order l it is $\frac{2\pi l}{m} + O(\rho)$ where m is the order of the subgroup of Γ leaving the point fixed. Letting $\rho \to 0$ we get

$$r A(\mathcal{D}) = 2\pi N_r$$

where N_r is the number of zeros counted so that at a fixpoint of order m we count the multiplicity of the zero with the weight $\frac{1}{m}$.

Thus

$$(8) \qquad N_r = \frac{r A(\mathcal{D})}{2\pi} ,$$

and combining this with (7)

$$(9) \qquad N^{(r)} - N_r = O(1) \text{ as } r \to \infty .$$

Clearly if we have $N^{(r)}$ linearly independent forms of weight r, we can at a given point z_0 produce a linear combination, not vanishing identically which has a zero of at least multiplicity $N^{(r)}-1$ at z_0. If z_0 is a fixpoint of order m we see that we can produce a form with a zero of at least multiplicity $(N^{(r)}-1)m$.

Let us denote such a form by $f_r(z, z_0)$ and norm it so that

$$\max y^r |f_r(z, z_0)| = 1.$$

We wish to show that <u>as $r \to \infty$ $\frac{1}{N^{(r)}} \log y^r |f_r(z, z_0)|$ tends to a limit function $g(z, z_0)$</u>, such that $g(z, z_0)$ is invariant under Γ, $y^2 \Delta g(z, z_0) = -\frac{2\pi}{A(\mathcal{D})}$; $g(z, z_0) - \frac{2\pi}{A(\mathcal{D})} \log y$ is harmonic and regular for all $z \neq \gamma z_0$ with $\gamma \in \Gamma$, $g(z, z_0) - m \log |z - z_0|$ is regular at $z = z_0$ if z_0 has ~~fix~~ a point of order m.

If we renormalize $g(z, z_0)$ by adding a suitable constant so that for the new function $g^*(z,z_0)$ we have

$$\iint_{\mathcal{D}} g^*(z,z_0)\,\frac{dx\,dy}{y^2} = 0,$$

then a simple application of Green's theorem shows that we have the reciprocity relation for $z_1 \neq \gamma z_0$

$$g^*(z_1, z_0) = g^*(z_0, z_1).$$

Constructing the analytic function which has

$\frac{1}{m}\left(g(z,z_0) - \frac{2\pi}{A(\mathcal{D})}\log y\right)$ as real part

(if z_0 is a point of order m)

and exponentiating we get a regular form of weight $\frac{2\pi}{m\,A(\mathcal{D})}$ which has a simple zero at z_0 and the equivalent points γz_0, but no other zeros. We call this the prime form $p(z,z_0)$, it is uniquely determined up to a factor of absolute value 1. ~~It is the fundamental~~ Any automorphic function or form can be expressed as

$$c\;\frac{\prod p(z,\alpha)}{\prod p(z,\beta)}$$

where α runs over the zeros and β over the poles of the form and c is a constant.

Since $p(z, z_0)$ is a form of weight $\frac{2\pi}{m\,A(\mathfrak{D})}$ if z_0 is a point of order m, we see that we can produce forms of all weights that are integral multiples of

$$(10) \quad \frac{2\pi}{[m_1, \dots, m_k]\, A(\mathfrak{D})} = \frac{1}{[m_1, \dots, m_k]\left(2g-2+\sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right)\right)},$$

where $[m_1, \dots, m_k]$ denotes the least common multiple of $m_1, \dots, m_k$. Also for a sufficiently large positive integral multiple of this expression there always exist regular forms of that weight.

If $\mathfrak{D}$ is not compact, but has finite area, the fundamental domain will have a finite number of vertices on the boundary of H (on the real line or at ∞) these are called cusps and are connected with the presence of parabolic elements in Γ which leave ~~the~~ a cusps fixed. We can still use our arguments with a little modification to prove the existence of a primeform even if we put z_0 at a cusp, only this primeform has no zeros in the interior of H so we can form arbitrary real powers of it, and so produce regular forms for all positive real r. In this case there are forms and consistent a.f. for all real weights.

For compact $\mathcal{D}$, the set of all consistent a.f for a given weight that is an integral multiple of (10) can be seen to depend on $2g$ continuous real parameters and a discrete parameter that can take

$$\frac{m_1 \cdots m_k}{[m_1, \ldots, m_k]}$$

values. For noncompact $\mathcal{D}$ with $A(\mathcal{D})<\infty$ if we have $\varkappa > 0$ inequivalent cusps it depends on $2g+\varkappa-1$ continuous real parameters and a discrete parameter that can take $m_1 \cdots m_k$ values.

For compact $\mathcal{D}$ one sees that there are no forms of half odd weight if

$$[m_1, \ldots, m_k]\left(2g-2+\sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right)\right)$$

is an odd number. This is the case iff some m_i are even and the m_i of highest parity occur in an odd number. In this case, and only in this case, is the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ contained in the commutator subgroup of the matrix group Γ.