Dr. Selberg

The Theory of Discontinuous Groups

I think I shall start by making some remarks about the title of this lecture which will seem one that covers a lot of ground. The real title of this lecture was given over the phone from Maine to Dr. Rauch and he, I believe, ~~ought~~ was to leave the next day for somewhere else so he transmitted it again, presumably by phone, to someone else, and then I met Professor Gelbart in Stockholm; he told me that they hadn't been quite sure how the title really should go so they had it copied down which was probably the sensible thing to do under the circumstances. The title really should have been something like ''On Certain Recent Developments and Open Problems ~~of the Problems~~ in the Theory of Discontinuous Groups'' which I think is somewhat a more modest title.

The term, ''discontinous group'' I shall not define in general, in the case that ~~which~~ we shall be considering here, although, of course, the term has a more general meaning; it shall simply mean a discreet subgroup of a Lie-group, actually one of the semi-simple Lie groups and we shall be dealing only with the discontinuous groups that operate on symmetric spaces.

~~The~~ Historically, of course, the theory, one may say, really took the beginning or started with the discovery first of the modular group and then later as the theory of uniformization came, ~~over~~ large classes of discontinuous groups operating in the hyperbolic plane or in the ~~unities~~ unit disc or the upper half plane of the complex plane, to mention two of the ~~marvels~~ models, and already there one notices that the, in a sense the sensible class of groups to restrict oneself to, ~~is~~ if one wants to do function theory being without ~~ortho~~ automorphic ~~xxxxxxx~~ functions or forms or other things, is to limit oneself to the class of groups with finite volume or in this case, since it's two-dimensional, area of the fundamental domain. Well, the fundamental domain, if we have a discontinuous group acting on a space, Fundamental domain is simply a set usually chosen of course to be a somewhat sensible set of points, which represent one from each equivalence class, if we consider them ~~a~~ equivalent under the transformations of the discontinuous group. Well, if we consider the ~~xxxxx~~ class of groups in the hyperbolic plane with finite area of the fundamental domain, or the narrower class, the one with compact fundamental domain, we know, of course, from the uniformization theory that there are very many such groups. Actually the uniformization theory makes it possible to construct families of groups of this kind which ~~xxxxx~~ depend in a continuous, even in an analytic way, on as many parameters as we would like. For instance, if you uniformize a Riemann surface in a standard way, the parameters can be a set of ~~moduli~~ moduli. The Riemann surface genus ~~being~~ greater than 1.

TAPE CHANGE

Now there are two other ways to obtain groups which have been used in the hyperbolic plane, groups that belong in this class with finite volume of fundamental domain. One is by certain geometric construction using the principle of reflection and the third way ~~is~~ uses groups that are in a certain way arithmetically defined . Now, when one started to consider the same questions, for the higher dimensional hyperbolic spaces, (and I shall at once say that when I speak of higher dimensional hyperbolic spaces, now, I shall limit myself to such spaces that contain no ~~contact~~ compact or euclidean factor space, because it simplifies the way in which I can make statements, ~~that~~ they will be more complicated if I had admitted of this possibility.) Now, that in the higher dimensional symmetric spaces, there exists as far as I know in the literature ~~such a~~ one attempt at constructing a group geometrically, namely in the very simplest case, the three dimensional hyperbolic space, there is something known as the Löbel ? example, of which I can tell you very little because I never studied it, but I think it's the only example of a geometrically constructed group. And all the others known in higher dimensions are either those that we could construct trivially from the hyperbolic plane, I mean where they would have some factors, I mean let us say that we have one or several hyperbolic planes as factors and that we could of course construct a group of the full space by considering groups of various factors but apart from that, the only way has been the arithmetically defined, groups, that were arithmetically defined and [I think that I shall rather try to use this thing there] and of course, ~~in their~~ there, very large classes of groups has been considered, in ~~xx~~ especially very early years/very significant contributions were made by Siegel; well there also have been earlier things by Hilbert and ~~Hecker~~ Hecke ? and Poincaré et al but principally by Siegel, I should say, and in the very last one or two years, the concept of an arithmetic group may have ~~been~~ in a sense, ~~finally, may have~~ reached its final formulation by work by ~~XXXX~~ Harishamba Harish-Chandra and Borrer. Borel ? So this has ~~left~~ led then to perhaps as large a class of groups as ~~XXXXXXX~~ we are at present able to define in these higher dimensional spaces and which satisfy the condition of the finiteness ~~nurser~~ of the volume of the ~~xx~~ fundamental domain.

From the other direction one ~~may~~ might want to investigate what properties a group would have if you ~~donxk~~ just make the assumption about the finiteness ~~x nurser~~ of the volume or of the fundamental domain, or stronger, about the compactness of the fundamental domain, what this would imply about the group and especially one might ask the question if there ~~xx~~ existed the families depending on continuous parameters in ~~an~~ a non trivial way (I will define shortly what I mean by this.) Or, you might even ask if ~~there~~ it, apart from certain obvious exceptions, ~~whether~~ there even would exist ~~essentially~~, groups that were essentially different from those which we get from arithmetical construction. ¶ Now, when I say essentially different, I shall in the following consider two groups as being equivalent if, one of them is obtained from the other by subjecting the continuous group to which it belongs or of which it is a discreet subgroup to an inner automorphism and secondly, I would consider two groups as essentially equivalent (and the term we will use for that is "commensurate,") If by an inner automorphism of the full group I can bring them on such a form that their intersection is of finite index in both ~~symbols~~. Two such groups I shall also consider essentially equivalent. And further there is one additional thing in order to formulate simply the exceptions, ~~also I will speak about~~, we need to speak about the groups that operate on products of symmetric spaces, I mean a symmetric space that can be split into factors. There, two possibilities can occur, namely, either the discontinous group is commensurate to a direct product of a group, discontinuous group, that operates on one factor, and one that operates on the other factor space. I mean there is a split, there may be such a splitting and also, correspondingly, a splitting of a group that is commensurate. If the group has this property, we shall call it reducible and if it does not have this property, then we shall call it irreducible, and it is simplest to just restrict oneself to the irreducible cases because there are no exceptions in that way. ¶ Well, in recent years a number of results have been obtained, most of them under assumption that the fundamental domain is compact. These results ~~do not go by any means,~~ still far from ~~they~~ are very ~~hard on~~ indicating that the arithmetically defined groups might be the only ones. (That means up to commensurability.) But they give some, I think, indications in this direction but again I think I will say that none of the methods that have been used to prove the present results are likely to give anything much further towards proving that theorem, if it be true. But at least one has been able to show the rigidity of the groups. If we assume compact fundamental domain, and that the space that I refer to is not the hyperbolic plane and that the group is an irreducible one, then one has been able to prove the rigidity. I mean that the groups can only be deformed in a trivial way. The only continuous deformation possible is what corresponds to an inner automorphism of the continuous group. Now something could be indicated about the methods that ~~they~~ were used and who obtains these results. The names to mention are Calabi, and Vesentini. Actually Calabi obtained this result for the n-dimensional hyperbolic space if n is greater than 3 and Calabi - Vesentini for the bounded complex

~~domains, symmetric~~

domains, symmetric complex domains in the sense of Cartan; and, then there was actually previous, somewhat previous to this, certain results that dealt with certain other symmetric spaces and which were obtained by rather different methods, namely Calabi-Vesentini worked with differential geometric methods and I, myself, have obtained results about rigidity using I shall say rather elementary methods ~~operating~~ working directly ~~on~~ with some representation of the groups. Now finally the rigidity ~~theory~~ theorem was proved for all the groups with compact fundamental domain, with the obvious exception of hyperbolic plane, of course, all the irreducible groups, again by differential-geometric methods and by André Weil ??? . There are also certain results in this direction which give, I shall say, more insight into the structure of the groups, by Matsushima. These results ~~were~~ however, well, the exclusions ~~of~~ do not coincide with the exclusion of the other theorems. The exclusion of ~~the spaces, I mean~~, the spaces that correspond to the real or the complex unit ball . Namely what Matsushima has proved in these cases, also on the assumption of compact fundamental domain, is that the first Betti number of the manifold, if you have a group without fixed points necessarily zero and that the commutator group of the group, necessarily has finite index . This of course is not true in the hyperbolic plane. It's quite possible that it might be true in hyperbolic space of dimension at least 3 ~~planes in space~~ for n greater than 3. I don't know if anyone has ~~xxx~~ looked at the Löbel example from this point of view, enough to check if that would contradict ~~whetherxthatx~~ such a theorem or not.

Now I wanted today to speak not of the compact case, but of the case of ~~long~~ non-compact fundamental domain but still with finite volume. There, results are much more incomplete, and this is not surprising because very little is known, even now, about what the finite volume actually implies, what this condition implies, I mean what type of noncompactness of the fundamental domain is compatible with the finite volume condition. Well, I will mention something about open problems now in this connection. One does not know if such a group is necessarily finitely generated. In the case of compact fundamental domain, that is an obvious and trivial result that it is finitely generated, but one does not know in the case of the finite volume, ~~rather~~ whether the group is necessarily finitely generated, and still less, if we construct a fundamental domain ~~by~~ in what I would call a canonical way by taking a base point, and counting as belonging to a fundamental domain those points in a given equivalence class under the group, that was closest to the base point. It's ~~now~~ not known ~~whether~~ whether, if we construct a fumdamental domain in this way, whether it's necessarily bounded by a finite number of ~~phases~~ faces. This of course will be stronger than the question of whether it's finitely generated. And a third thing that is not known, if we have a matrix representation of ~~the~~ this, I mean if the semi-simple Lie group is given by a matrix representation, in general does the condition of non-compactness but finiteness of volume of the fund. domain, does that necessarily always imply that there are what we would call parabolic transformations in the group or what is the same thing~~, I mean~~, unipotent matrices, I mean matrices with all their eigenvalues one, that cannot be diagonalized, that are not identity matrices. This is not known. Alright. Now certain results have been obtained, however, on various assumptions ~~on~~ for particular types of symmetric spaces, and I think I should begin by indicating one that is not, well, it is stated in the literature but the proof has ~~not~~ never been published anyway and it is actually the oldest ~~written~~ rigidity theorem of them all. Namely, if we consider a product of hyperbolic planes and we have a discontinuous group with finite volume of the fundamental domain, then one can prove rigidity of the group without any additional condition and this is the only case ~~left~~ where the rigidity is proved without any additional condition. The proof of this which I could indicate some points, uses very heavily the particular properties of two by two real unimodular group, or the properties of a hyperbolic plane, namely it uses first of all one property that is common for all the irreducible groups, ~~but~~ that if we have an irreducible group acting on a product of symmetric spaces, then one can show that, if we ~~xxxxd~~ consider the component on any factor, ~~x~~ (~~the~~ each transformation could be split into a number of components, I mean one acting on each component of the space) Now, if the group is irreducible and the fundamental domain has finite volume, then the, if we consider the components ~~xf~~ that act on one factor then ~~khix~~ they form a group whose closure actually contains the full connect~~ive~~ed component of the group of motions of this component space, ~~of~~ this factor space, ~~of~~ this symmetric space. So this will in particular in the present case imply that if I had a group that acts on a product of a hyperbolic planes and I consider the component of a transformation which acts on one particular of these ~~xhingx~~ planes, for simplicity we may consider that we have two hyperbolic planes and I consider the part of the transformations that act on the first. So this is ~~done firxk~~ then dense in the full group of motions ~~in~~ of the hyperbolic plane into itself. It follows therefore, it is in particular dense in the set of validity rigidity?

transformations and now if I consider a transformation of the group that has one, I think I can even give this ~~group~~ proof, the idea of it, without writing anything, if I may not have the ? theory. If we consider the transformation, one of whose components is, say the first component is elliptic, then if we had a continuous deformation of this group, depending on some parameter, retaining the structure of the group, this would necessarily have to preserve the rotation angle of this elliptic transformation. It would not only have to remain elliptic, but the, an elliptic transformation of course represents a rotation of a certain angle and this angle would have to remain fixed. Namely, if it changed it would change from rational to irrational or vice versa. And of course whenever a component is the angle, its ~~is~~ the rational multiple of π. That means a certain multiple of that transformation is the identity. And for ~~any~~ an irreducible group one can easily see that one component is the identity only if the other component of the transformation is also the identity. So, the rotation angles of the elliptic transformations contained in the one component of the group would have to remain fixed. Now, if I ~~went to~~ then took two elliptic transformations acting on the first hyperbolic plane, and that ? [illegible] components of transformations in the discontinu~~ed~~ous group, it is easily seen that I could always form certain power products

Dr. Selberg

of these that would again be elliptic transformations. That's a very simple computation, to show that this is true ~~That is true~~. You have only to show that you can make the trace ? -2 and +2 . Now, the rotation angle of that transformation would have to remain constant too. But, It ~~this~~ is very easily seen that ~~and~~ this implies that the distance between ~~of~~ the centers of rotations of these have to remain constant. And from there on it is quite easy to see that the whole change in what ~~happends~~ happens with the, if I look at the first ~~third~~ component of the transformation, they can just undergo an inner automorphism of the continuous groups I mean it's a motion of the first hyperbolic plane into itself. And of course the same argument goes for the second component. So, the deformation should have to be a ~~trivaxx~~ trivial one. Now, this result does not tell you anything, of course, beyond the fact that the group is rigid. It does not tell you whether ~~that~~ it is finitely generated or it has any of the other properties or whether it necessarily has parabolic transformations, if the fundamental domain is not compact. ~~If we make however, and~~ I should say the same result in the same form is valid ~~valued~~ for product of n hyperbolic planes.

¶ If we assume, however, that the group contains one transformation which is unipotent, there is one transformation both of whose components are matrices ~~xx~~ just for the eigenvalues 1 and this should be a transformation which is not the identity, First of all, because the group is assumed irreducible we can very easily conclude that both of the components have ~~thx~~ to be parabolic. And so we may assume then that we have a parabolic transformatinn. I'd use the upper half plane model of the two hyperbolic planes, so that z_1 goes into $z_1 +$? descriminant And the other one is something like this. Well, these two numbers then cannot be zero. It was first actually shown by Pyatetskij Karpetski Shapiro ~~x~~ that from this fact, if one assumes the existence of one single such parabolic transformation in the group, this in combination with the assumption ~~ofxthe~~ about ~~thexassumptionxofx~~ the finite volume and also the group of ~~the~~ ir-reducibles, made it possible to draw certain conclusions about a larger part of the group, namely he showed that the group necessarily, and I think for this I shall go over to write, rather, these indices on top inaudible. He showed that if we have ~~certain~~ such a parabolic transformation, then we have of necessity a full group, a sub~~b~~group of a full group ~~which consists~~, which is of the following ~~formxformx~~ form, (or can be brought on the following form by means of an inner automorphism of the group), namely that z^i, these are the components, that would be 1, 2 ~~of the~~ in the ~~dx~~ case we have two hyperbolic planes, and this is a group of triangular, I mean these are transformations represented by triangular matrices. Actually if we $z^i \to \eta^i z^i + \omega^i$

write them in the full matrix form as $\frac{\sqrt{\eta_1}}{\sqrt{\eta_n}}$ I believe something like this for the various components, if we want it to have determinant one ~~form~~. Well he showed that it could always be brought on such a form by an inner automorphism where the numbers were actually integers in a certain algebraic field, a totally real field of degree n ~~and~~ if it were a product of n hyperbolic planes; the η's that could occur, they would form an abelian group of course, would form a group that had finite index in the group of the totally positive units of the field, and the group of pure parabolic transformations, (~~so~~ that means where the η would be equal to one) on the η's, that would be a group of translations with compact fundamental domain in n dimensional euclidean space. ~~So, and~~ This actually then would be the group that describes a particular cusp or non-compact part of the fundamental domain. It represents, I mean we can define a ~~fundx~~ fundamental domain then which has something like a needle or so sticking out towards infinity. Now, however, much more can be said just from the assumption about the single parabolic transformation in the group, namely not only this is true, but one can show that the full group, the whole group, actually is such, that it is represented by ~~matrices~~ means of matrices with algebraic entries, ~~so that the~~ components, whose entries are algebraic numbers from the totally real field of degree n that I spoke of. I think that should be delta. And as long as we ~~dom~~ do not insist on the determinant being necessarily one, but just some totally positive number from the field, and this is not so very hard to show. Either actually it can be shown simply by assuming that you have some transformation in the group given by its n components one acting on each ~~component space,~~ component of the space, and this could be, ~~a~~ well I could write it in the form that I have it there, that I have normalized so that the determinant is one. Then it is very easy to see that if ~~h~~ this matrix was not of the form that ~~I~~ I stated earlier, I mean that it actually ~~would~~ could be eventually by multiplying it by a suitable number brought, on the form where it has as entries algebraic numbers from the totally real ~~fixed~~ field and with determinant~~s~~ of course totally positive. If it was not of this form, then adjoining it to what I had earlier would lead to something that was no longer a discreet group. END OF TAPE

That seems to be a general phenomenon, by the way, that ~~the~~ if you have arithmetically defined~~x~~ groups, (well this ~~I~~ of course has not been shown to be necessarily an arithmetically defined group.) If you have an arithmetically defined group with compact fundamental domain, and of course you will always try, ~~to~~ by an inner automorphism to bring it on such a form, that the numbers occurring as elements in the matrices, ~~or~~ in whatever matrix representation you are considering, that they are of minimal degree. These algebraic numbers belong ~~all the~~ to ulta- fields of the minimal degrees. But for compact fundamental ~~s~~ domain, there seems in general to be no bound on this degree. You can construct examples of arbitrary high degree: for instance if you have a product of two hyperbolic planes, you can construct groups with compact fundamental domain and irreducible groups, I shall ~~xhx~~ say, where the minimal degree of the field is arbitrarily high, and this can be done from certain quaternion algebras. But here you see, if you ~~should~~ assume the presence of the parabolic transformation, which, it is very likely, ~~in~~ is necessarily present under the assumption of the finiteness but the non-compactness of the fundamental domain, If you assume ~~the~~ this presence of the parabolic transformation, then the degree of the field is determined. In this case it has to ~~be~~ of degree n where n is the number of hyperbolic planes that you have, (Of course I exclude here the case that n = 1, ~~Of course~~ the case of one hyperbolic plane) Now, of course, it is well-known that there ~~is a~~ exist groups that are like this that I have described -- the so-called Hilbert ~~mxx~~ modular group of a totally real field of degree n, which simply consists of all matrices formed by integers in the field, with determinants equal to a totally positive unit. There are very strong indications that the presence of the one unipotent element in this case even probably means that the group is necessarily commensurate to one of the Hilbert modular groups of degree n, ~~And~~ the one that corresponds to that particular field that you have here. However, it has not been quite possible to prove that. You can prove a number of things that would seem very strong indications in that direction.

Some of this could be generalized. Of course the ~~a~~ case of a product of hyperbolic planes is a particularly simple one in many ways. Some of this could be generalized to ~~in~~ certain other cases, if we for instance consider the Siegel ~~field~~ symplectic space, so ~~the~~ z being now a symmetric complex matrix of the form iy, y pos. definite, And we assume that n is not equal to one again, Then of course it is known that there are ~~is~~ again a continuous group of motions that can be represented in this form by the matrices a, b, c, d are $n \times n$ matrices that satisfy certain conditions that I shall not write. Now if we make a certain assumption about the presence of a certain type of element, we may be able to draw certain conclusions. First of all if I assume the presense of one such element of this type where H, well H will have to be a symmetric matrix necessarily since this should satisfy the conditions of the simplectic group. E denotes identity matrix If H is a definite matrix, then conclusions of a somewhat similar ~~type~~ kind as those made by Karpeski-Shapiro ? about the presence of a group that actually describes a certain ~~part~~ cusp of the fundamental domain. If the H is either non-definite, or it might even be singular, have determinant 0, then however, it seems more complicated and we can't ~~xxxxxxxxxx~~ quite draw the same conclusions. Then again if I ~~had~~ have reached the fact that there was a group of this form, a subgroup of this form that would describe a cusp where ~~x~~ actually the nilpotent elements, where A was the identity matrix, should form ~~the~~ an abelian group, if that has $\frac{n^2+n}{2}$ generators, if the transformations of this form that occur are enough to have finite fundamental domain in the group of all transformations of this kind, in the continuous group; And, if furthermore, the group of A, matrices A, that occur, are such, well you can show that they will have to have determinants one more of them. But if they have finite volume in the smaller space of the real $n \times n$ unimodular group, then again you can easily conclude ~~that~~ first of all that this can be brought on such a form that all the elements are algebraic, and furthermore, that actually the whole group is rigid, and that once you have brought it on this form the full group is in such a form that all the matrix elements in the matrices a,b,c, and d are algebraic numbers from a certain field. This again would be shown simply by showing that if we adjoin to this group any element that was not of that kind it would lead to a set of elements that was not discrete. So it seems, and again although it cannot be shown, it is a pretty, I would say a rather good bet that ... Actually if a group in the simplectic space has finite volume of the fundamental domain, but not compactness of it, i.e. by a suitable inner automorphism can be brought on such a form, that it does contain such a group which will describe one of the cusps. I mean that by and large it seems that ~~the cusps are very often, o~~r the groups that describe the cusps are strong enough to keep the group rigid in itself, and actually imposes a much stronger restriction on the full group than any part of a group with compact fundamental domain does. So that in one sense this would point in the direction that in a certain sense one could say there are probably fewer groups with finite volume of the fundamental domain, but not compact. There are groups with compact fundamental domain, I should say in a very loose sense. Of course there are infinitely many of both kinds, but there are more probably possibilities for variation of structure of the group in one with compact fundamental domain than one with the non-compact fundamental domain. (Again excluding the obvious exception, I mean in the hyperbolic plane, the presence of a parabolic transformation of a single cusp imposes no more restriction or so on the group than the presence of any other kind of transformation.

~~Now, I think I shall look a little~~ ... Similar things hold for other spaces than the simplectic space, not for all symmetric spaces, with the exclusion of the hyperbolic plane though. I am unable to make any similar kind of argument for the groups that operate on ~~from~~ the complex unit ball for $n > 1$. It is possible that the presence of cusps, I mean of non-compactness, might be in a sense less restrictive for a group in such a space, but it's also of course possible that this is a definiency that is due to the

method in my type of reasoning. Now the thing that one would very much like to show, of course, would be, that ~~such a group is necessarily~~ a group fulfilling the condition of finiteness of the volume of the fundamental domain is necessarily commensurate ~~so that it is not essentially different from~~ . ~~It is commensurate~~ with one of the arithmetically defined groups. The only ~~All the~~ case where I would say that one at present would seem close to show anything like that would be for the case of n hyperbolic plane and, by the way, the non-compact case. Namely, there are two things missing. If we could show, on the one hand, that the non-compactness combined with the finite volume, implies the presence of the unipotent parabolic transformation, so that what I stated earlier could be dervied from that, would be valid, then we would just be left with the question of whether we could show that this group then necessarily was commensurate to one of the Hilbert modular groups or actually to the one that corresponded to the particular totally real field. And actually there's just one fact that would be lacking in order to prove such a thing. Namely, you would have to show, that in this representation which I gave of the matrices representing the transformations (the various components) as matrices with integral elements from this totally real field, but as I said where the determinant was not necessarily one, which means that if I divide to make the determinant one I introduce certain possible denominators, of course. And the only fact which would be necessary to show that such a group was commensurate to the Hilbert modular group, would be the fact that this denominator should be bounded, so that there should be, (if I write the elements in such a form, that they have determinant one) ~~there should~~ be some six bounded, say, integers d, but if I multiply all of these entries with this number d they all become algebraic

If we could prove that the denominations are bounded, then we could also show that the group would be commensurate with the Hilbert modular group. They would have an intersection that had finite index symbols. Now, both of these

inaudible

could not be entirely impossible that one could make in the near future. So in this case it might be possible to prove such a result , but this of course still would leave ~~the~~ completely unresolved the question of those with compact fundamental domain for which I have no corresponding thing to start with. And similarly in some of the other, for instance the n by n complex ~~XXXXXXX~~ space of symmetric matrices, I mean the simplectic space that I just defined. Again it might conceivably be possible to ~~pxxx~~ prove the presence of the necessary unipotent elements, that we could sort of get the whole group brought on this form which I, (well I did not really indicate that form ~~x~~ too precisely actually.) But there are ~~axx~~ a lot more possibilities, I mean I can't there just point to a clearly well-ordered set like the Hilbert ~~mxx~~ modular groups are ~~wxxxxxx~~ against which I could measure these. So, although you could do some parts of this. Possibly there it would be a very much harder task to show there are no groups ~~apark~~ with non-compact but fundamental domain but finite volume of it, which were not commensurate to some arithmetically defined group. And there again it would leave me entirely -- I would not know anything, really, about ~~a~~ the ones with compact domain; ~~but~~ of course I know that they are rigid, and it is known also that they can be brought on such a form that the matrices that ~~a~~ define the generators, they are finitely generated, have algebraic numbers as entries, algebraic numbers from some field, but I don not have any kind of well-defined and ordered set of aritmatically defined groups against which I could ~~like~~ try to measure any ~~extkhe~~ of these that operate on the higher complectic spaces. But I would say the only case where there seems any hope is, at present, the case with the product of hyperbolic planes and where you assume non-compactness. This is the only case where I think ~~any~~one ~~I think~~ is anything like close to proving the ~~gox~~ group is necessarily commensurate to an arithmetically defined group. Now, it is of course possible that this might not in general be true, that the groups were necessarily such. Just from the assumption of the finite volume of the fundamental domain. Again I would say I don't know if anyone has looked carefully at the Lubel example from this point of view, whether this is commensurate to any arithmetically defined group that operates on the 3-dimensional hyperbolic space. However, it would seem that if there~~is~~ is any exception from this, I mean if there groups that are not commensurate with the arithmetically defined groups and which belong in this class of finite volume, it would seem most ~~likely~~ unlikely that we should ever devise any way of getting hold of them. Because the only other way would seem to actually present the group by means of generators and relations and I think this, of course, in the higher dimensional cases one can say that this is a very complicated system. The set of generators is of course in general ~~very~~ large but the ~~relations~~ set of relations is even much larger, and in particular is of course what serves to hold the group ~~XXXX~~rigid. I mean this is one thing, it also imposes a lot of restraints on the structure of the group. These relations, and I would say, if the situation is not as I would say I would hope it is, that all the groups in the higher dimension are ~~is~~ commensurate ~~the the~~ to the arithmetically defined ones. I think that those that are not will probably remain undiscovered. I shouldn't say forever, but at least in any foreseeable future.