Carlson $T^{4\sigma(1-\sigma)} \lg^6 T$

Titchmarsh $T^{\frac{4(1-\sigma)}{3-2\sigma}} \lg^6 T$

Ingham $\frac{3}{2-\sigma}(1-\sigma) \lg^6 T$

② $x^{6-6\sigma} + T x^{8-4\sigma}$

$x = T^{\frac{1}{4-2\sigma}}$

$\frac{3\ldots}{3-\sigma}$

$x^{4-4\sigma} + T x$

$x^{3-2\sigma} = T$

Also Ingham on Lindelöf ~~hypot~~ hypothesis.

G. Halasz. H. Montgomery, Huxley

problem. ζ-function. (or other with Euler product. $f(s) = \sum \frac{a_n}{n^s}$ a_n "small"

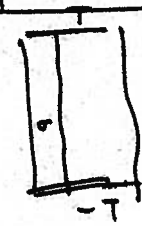
$$\sum_{x}^{2x} |a_n|^2 d_k(n) \ll x \log^K x$$

$$\frac{1}{f(s)} = \sum \frac{b_n}{n^s}$$

b_n "small"

est. no of zeros of. $f(s)$ in

$N(\alpha, T)$.



$\rho = \beta + i\gamma$

$\beta > \alpha$

$|\gamma| < T$

$$\psi_z(s) = 1 + \sum_{n \le z} \frac{b_n}{n^s}$$

$$f(s)\psi_z(s) = 1 + \underbrace{\sum_{n \ge z} \frac{c_n}{n^s}}_{\varphi_z(s)}$$

formally.

F. Carlson.

$q_z(s)$ on average "small". (mean value)

either Jensens inequality or Littlewoods.

$$\int_{\sigma=\beta}^{\alpha} N(\sigma, T)\, d\sigma = \frac{1}{2\pi}\int_{-T}^{T} \log|F(\alpha+it)|\, dt - \frac{1}{2\pi}\int_{-T}^{T} \log|F(\beta+it)|\, dt - \frac{1}{2\pi}\int_{\beta}^{\alpha} \arg F(\sigma+iT)\, d\sigma + \frac{1}{2\pi}\int_{\beta}^{\alpha} \arg F(\sigma-iT)\, d\sigma.$$

$\beta > 1$. (large const.).

if $|F(s)| \ll T^{A}$ for $\sigma > 0$; $|t| \le T$.

then $\arg F(s) = O(\log |T|)$ (Jensens inequality on $F(z+it) + \overline{F(\bar{z}+it)}$ circle radius a center $z = a$ suff large.)

$$N(\alpha, T) \le \log T \int_{\beta}^{\alpha - \frac{1}{\log T}} N(\sigma, T)\, d\sigma .$$

Other approach. (not useful for σ very close to $\frac{1}{2}$ or very close to 1, if one wants very precise results reason).

If $f(s_j) = 0$ then $|g_z(s_j)| = 1$

similarly if

$$g_z(s_j) = \sum_{r=1}^{R} v_r(s_j) \quad \text{then if}$$

$f(s_j) = 0$ we have $|v_r(s_j)| \ge \frac{1}{R}$ for at least one r. We may thus obtain an upper bound for the number of zeros in the region if we can determine an upper bound for the number of a set of points s_j for which at least one $|v_r(s_j)| \ge \frac{1}{R}$. Clearly can't be done unless requires that s_j be well separated say $|s_j - s_j| \ge \delta$ with $\cancel{\delta \ge \frac{1}{\log T}}$.

"Drawback" since there may be several zeros in a circle of radius 1; (but as Jensens formula shows at most $O(\log T)$, we would get bound by multiplying with $A \log T$, thus approach not good where best power of $\log T$ is most important.

Division of $q_z(s_j)$

$a > 1$

$$1 + q_z(s) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^{s} \Gamma$$

$$1 + \sum_1^\infty \frac{c_n}{n^s} e^{-\frac{n}{x}} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^{z} \Gamma(z) f(z+s) \psi_z(z+s)\, dz$$

for $-\frac{1}{4} < \beta < 0$

$$= f(s)\psi_z(s) + \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} x^{w} \Gamma(w) f(s+w) \psi_z(s+w)\, dw$$

$f(s)\psi_z(s)$.

$$q_z(s) = \sum_1^\infty \frac{c_n}{n^s} e^{-\frac{n}{x}} + \int_{-\infty}^{\infty} x^\beta\, f\, \psi_z(\sigma+\beta+i(t+v)) \Gamma(\beta+iv)\, dv$$

if $f(s)$ has pole of order 1 at $s=1$ term $x^{1-s}\Gamma(1-s)\psi_z(1)$

$$|\Gamma(\beta+iv)| \le C_\beta\, e^{-|v|}$$

$a > 0$ (4)

$$\cdots \frac{1}{\pi i}\int_{a-i\infty}^{a+i\infty} x^{w}\,\Gamma(w)\,dw = e^{-\frac{1}{x}}$$

A) For how many "well dispersed" s_j in rectangle Σ can we have

$$\left|\sum_{y}^{2y} \frac{c_m}{m^{s_j}}\right| \geq \delta \quad \text{where } \delta \text{ is a small constant.}$$

call ~~M~~

1)st approach. surround each s_j with circle $\mathcal{C}_j$ rad $\frac{1}{\lg T}$; then circles do not overlap and

$$\left|\sum_{y}^{2y} \frac{c_m}{m^{s_j}}\right|^2 \leq \frac{\lg^2 T}{\pi}\int_{\mathcal{C}_j}\left|\sum \frac{c_m}{m^s}\right|^2 d\sigma\,dt$$

thus

$$M\,\delta^2 \leq \frac{\lg^2 T}{\pi}\int_{1}^{\sigma - \frac{1}{\lg T}}\left|\sum_{y}^{2y}\frac{c_m}{m^{\sigma+it}}\right|^2 d\sigma\,dt$$

$y \leq T^2$

$$\leq c\,\frac{\lg^2 T}{\lg y}\,y^{\frac{2}{\lg T}}\, T \sum_{y}^{2y}\frac{|c_m|^2}{m^{2\sigma}} + c\sum_{m\neq n}\sum \frac{|c_n\, c_m|}{(mn)^{\sigma}\left|\lg\frac{m}{n}\right|}$$

$$\ll \lg^{k} T\left(T y^{1-2\sigma} + y^{2-2\sigma}\right)$$

take power of sum. so that y^k "near" T.

2) second approach. Hilbert space

$F, \phi_i \ldots$

$$\sum |(F, \phi_i)|^2 \le \|F\|^2 \max_i \sum_j |(\phi_i, \phi_j)|$$

proof.

Choose $F(n)$ defined on pos integers as

for $y \le n \le 2y$; $\quad \rho_n = e^{-\frac{n}{2y}} - e^{-\frac{n}{y}} \ge c$ for $y \le n \le 2y$

$$F(n) = \frac{c_n}{\sqrt{\rho_n}\sqrt{n}} \; ; \quad \phi_i(n) = \frac{\sqrt{\rho_n}}{n^{s_i - \frac{1}{2}}}$$

otherwise 0.

$$\sum_i \left| \sum_y^{2y} \frac{c_n}{n^{s_i}} \right|^2 \ll \sum_y^{2y} \frac{c_n^2}{n} \cdot \max_i \sum_j \left| \sum_n \frac{e^{-\frac{n}{2y}} - e^{-\frac{n}{y}}}{n^{s_i + \bar{s}_j - 1}} \right|$$

$\downarrow \; \log^k y$

$$\sum_n \frac{e^{-\frac{n}{2y}} - e^{-\frac{n}{y}}}{n^s} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left((2y)^{w} - y^{w}\right) \Gamma(w) \, \zeta(s+w) \, dt$$

$t^{\frac{1}{2}} \log$

$\sigma + w = -\sigma$.

$$= \left((2y)^{1-s} - y^{1-s}\right) \Gamma(1-s) + O\left(y^{-\sigma} \int_{-\infty}^{\infty} e^{-|v|} |\zeta(i(t+v))| \, dv \right)$$

$s = s_i + \bar{s}_j - 1$

$\sigma = \sigma_i + \sigma_j - 1$

$t = t_i - t_j$

$$O\left(y^{-\sigma} (1+|t|)^{\frac{1}{2}} \log (2+|t|) \right)$$

$$= O\left(y^{1-\sigma_i-\sigma_j} H^{\frac{1}{2}} \log H \right)$$

for $2 \le H \le |t_i - t_j| < H$.