Notations: $f(s) = 1 + \sum_{n=2}^{\infty} \frac{a_n}{n^s}$; $\frac{1}{f(s)} = 1 + \sum_{n=2}^{\infty} \frac{b_n}{n^s}$

a_n, b_n "small" ; $\sum_{x}^{2x} |a_n|^2 d_k(x) \le A\, x \log^{c_k} x$

$f(s)$ regular for $\sigma \geq \frac{1}{2}$ with possibility of simple pole at $s = 1$.

$f(\frac{1}{2}) \le O(T^{\frac{1}{2}+}$ $\int_{-T}^{T} |f(\frac{1}{2}+it)|^4 dt = O(T \log^c T)$

and

$$M_z(s) = 1 + \sum_{n \le z} \frac{b_n}{n^s} \quad | \quad \psi_z(s) = \sum_{z \le n <} \frac{c_n}{n^s}$$

$$f(s) M_z(s) = 1 + \psi_z(s)$$

$$\psi_z(s) = \sum_{n \le \cancel{x \log x}} \frac{c_n}{n^s} e^{-\frac{n}{X}} + R$$

where $|R| \ll X^{-(\sigma - \frac{1}{2})} \int_{-2\log T}^{2\log T} |\Gamma(\frac{1}{2} + iu)\, \psi_z(\frac{1}{2} + i(t+u))| dt$

$$+ \frac{X^{1-\sigma} e^{-|t|}}{\cancel{|t<s|}} + \frac{1}{X^{(\sigma-\frac{1}{2})}\sqrt{T}^{0}} \quad \text{for } T_0(s)$$

$t_j \geq \boxed{\log T}$

$$= \sum_{n \le x \log x} \frac{c_n}{n^s} e^{-\frac{n}{X}} + x$$

$$R \ll X^{-(\sigma - \frac{1}{2})} \int_{-\log T}^{\log T} e^{-|u|} |\psi_z(\tfrac{1}{2} + i(t+u))| du$$

$\underline{\text{for } T \geq T_0}$

$+ \frac{1}{10}$

(1)

notation $f(s) = 1 + \sum_{n=2}^{\infty} \frac{a_n}{n^s}$; $\frac{1}{f(s)} = 1 + \sum_{n=2}^{\infty} \frac{b_n}{n^s}$

c_k

a_n, b_n "small" | $\sum_{n \le x} |a_n|^2 d_k(x) < A_k x \log x$

$f(s)$ regular for $\sigma > \frac{1}{2} - \varepsilon$ with possibility of simple pole at $s = 1$; $\int_{-T}^{T} |f(\tfrac{1}{2}+it)|^4 dt = O(T \log^c T)$

$$M_z(s) = 1 + \sum_{n \le z} \frac{b_n}{n^s}$$

$$f(s) M_z(s) = 1 + \psi_z(s) \; ; \quad \psi_z(s) \sim \sum_{n \ge z} \frac{c_n}{n^s}$$

$$\psi_z(s) = \sum_{n \le x \log x} \frac{c_n}{n^s} e^{-\frac{n}{x}} + R$$

$$|R| \ll x^{-(\sigma - \frac{1}{2})} \int_{-\log T}^{\log T} |\Gamma(\tfrac{1}{2} - \sigma + iu)\, \psi_z(\tfrac{1}{2} + i(t+u))|\, du$$

$$+ \underbrace{x^{1-\sigma} e^{-|t|} + \frac{1}{x} + \frac{1}{x^{(\sigma-\frac{1}{2})} T^c}}_{\le \frac{1}{10} \text{ for } T \ge T_0}$$

$$\underbrace{x^{-\sigma-\frac{1}{2}} \int_{-\log T}^{\log T} e^{-|u|} |\psi_z(\tfrac{1}{2} + i(t+u))|\, du + \frac{1}{10} \quad (T \ge T_0}$$

(2) $\sigma \geq \ldots$; $|t| < 2T$ w...

$$\left| \sum \frac{e^{-\frac{n}{2y}} - e^{-\frac{n}{y}}}{n^s} \right| = \frac{1}{2\pi i}\int \left((2y)^w - y^w \right) \Gamma(w)\, \zeta(s+w)\, dw$$

$$= \frac{2^{1-s}-1}{1-s}\, y^{1-s} + c\, y^{-\sigma} \int$$

$$\ll \frac{y^{1-\sigma} e^{-|t|}}{1+|t|} + y^{-\sigma} \int e^{-|u|} \zeta(i(t+u))\, du$$

For $y \leq n \leq 2y$

$$e^{-\frac{n}{2y}} - e^{-\frac{n}{y}} > \frac{1}{5} \qquad \ll \left(\frac{y^{1-\sigma}}{1+|t|} + y^{-\sigma} \sqrt{1+|t|}\, \log T \right)$$

(M) $\qquad f(n) = \frac{c_n}{\sqrt{n}} \qquad |t_i - t_j| \leq H.$

$$\sum_{j=1} \left| \sum_{y}^{2y} \frac{c_n}{n^{s_j}} \right|^2 \leq 5 \sum_{y}^{2y} \frac{|c_n|^2}{n}.$$

$$\max_i \sum_{i \neq j} \left| \sum_n \frac{e^{-\frac{n}{2y}} - e^{-\frac{n}{y}}}{n^{s_i + \bar{s}_j - 1}} \right|$$

$$\leq \log^c T \cdot \left(\sum_j \frac{y^{2-\sigma_i-\sigma_j}}{\cancel{1+|t_i-t_j|}} e^{-|t_i-t_j|} \right.$$

$$\left. + M \cdot y^{1-2\sigma} \sqrt{H} \log T \right)$$

$$H \leq \frac{y^{\ldots-2}}{(\log T)^{2c+\ldots}}$$

3

$| t_i - t.$

get. if $\left|\sum_{y}^{2y} \frac{c_n}{n^{s_j}}\right| > \delta$

$$M\delta^2 \ll \log^c T \left(y^{2-2\alpha} + \frac{M}{\log T} \right)$$

$\delta \geq \frac{c}{\log^2 T}$

M

$$M \ll \log^{c+4} T \; y^{2-2\alpha}$$

total. no in interval $2T$; $\left(1 + \frac{2T}{H}\right)$

$$\ll \log^c T \left(y^{2-\cancel{2}\alpha} + T y^{4-6\alpha} \right)$$

use k power instead.

$$\log^c T \left(y^{2k(1-\alpha)} + T y^{(4-6\alpha)k} \right)$$

Better than. former for $\alpha > \frac{3}{4}$ $3 - 4\alpha < 0$

$\sum |R(\rho_j)|$

y goes up to $x \log x$ $\qquad X =$