(1)

UM

Density theorem.

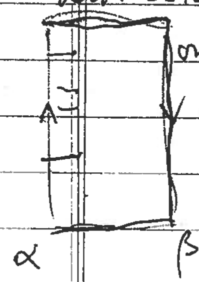
Methods.

Counting method.

F. Carlson Auxilliary function. $M_\xi(s) = \sum_{n=\xi} \frac{\mu(n)}{n^s}$

Jensens formula for circle, then

Littlewoods formula counting with weights depending on one side the zeros in a rectangle.



$$\sum_{\substack{\text{extended}\\ \text{over zeros.}}} (\sigma_i - \alpha) = \int \log|f(\alpha+it)|\,dt$$

$$\zeta(s) = \sum_{n=1} n^{-s} + \cdots + O(\cdots)$$

$$\zeta(s)\, M_\xi(s) = 1 + \sum_{\xi}^{X} \frac{a_n}{n^s} + \cdots + R.$$

essentially either by geometric arithmetic mean inequality or by $\log|1+u| \leq \cancel{\ldots}\ R\,u + \frac{|u|^2}{2}$

get integral.

$$\int_T^{2T} \left| \zeta(s)\, M_\xi(s) - 1 \right|^2 dt.$$

or other power, | ~~using~~ approx

Carlson $N(\sigma, T) = O(T^{4\sigma(1-\sigma)} \log^6 T)$

Titchmarsh approx functional equation

$\lambda(\sigma) \leq 4\sigma$, Carlson

$\leq \frac{4}{3-2\sigma}$, Titchmarsh.

Ti

	$1+2\sigma$	$2+4c$
1940	$\frac{3}{2-\sigma}$	if $\zeta(\frac{1}{2}+it) = O(|t|^c)$

Ingham. convexity theorems.

2 $\frac{2}{5}$ bombs 1969.

2 We have summing over a well-spaced set of zeros $\gamma - \gamma' \geq \frac{1}{q_1 T}$

$$M \leq 2 \sum_j \sum_n \frac{a_n}{n^{s_j}} \quad = \quad \text{if replacing.}$$

$\sum |\phi(s_j)|^2$ by integral over circle.

$$\frac{q_1^2 T}{2\pi} \iint_{|s - s_j| < \frac{1}{q_1 T}} |\phi(s)|^2 \, ds$$

Could also write

$$\phi(s) = \phi_1(s) + \cdots + \phi_R(s)$$

where $|\phi_1(s_j)| + \cdots + \phi_R(s_j) \geq \frac{1}{2}$

$\frac{1}{2R}$ at least.

$$M \leq \sum_{\nu=1} \sum_{j,\ \nu=1/R} (2R)^{k_\nu} |\phi_\nu(s_j)|^{k_\nu}$$

$$M \leq \sum_{\nu=1}^{R} (2R)^{k_\nu} \sum_j |\phi_\nu(s_j)|^{k_\nu}$$

get back advantage of higher powers ~~easily~~ with constants much simpler to determine

$M(i)$ no. of points s_j for which

$$|\phi_i(s_j)| \geq \frac{1}{2R}$$

$$M \leq \sum_i M(i)$$

$$M(i) \leq (2R)^{k_i} \sum_j (2R)^{k_i} |\phi_i(s_j)|^{k_i}$$

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$$\int_{\sigma=\beta} |f(s)|^p \qquad \int_{\sigma=\alpha} |H(s)|^p$$

and later also double convexity convex theorems with $\int |f(s)|^{p_1}$ $\int |H(s)|^{p_2}$

$0 < \sigma < 1$

$$\int_T^{2T} \left| \sum_{x_1}^{x_2} a_n n^{-\sigma - it} \right|^2 dt$$

$\leq T \sum_{x_1}^{x_2} \frac{|a_n|^2}{n^{2\sigma}} + O(x_2 \log x_2 \, x_2^{1-\sigma} \sum_{x_1}^{x_2} |a_n|^2 \ldots)$ if $|a_n| \leq 1$

$= O(T x_1^{1-2\sigma}) + O(\log x_2 \cdot x_2^{2-2\sigma})$

$\frac{1}{\log T}$

Ingham . used integration on $\frac{1}{2}$ and on $1+\delta$ with small δ. and either mean value theorems on $\zeta(s)$ on $\sigma = \frac{1}{2}$ or Max order. $\zeta(s) = O(t^c)$. Results . I could no doubt have obtained more by (1) using lines other than $\sigma = \frac{1}{2}$ (2) using the results on est of exp. sums directly

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in 1945 to early 1946

Own work. using inequalities

$$\log |1 + u_1 + \cdots u_r| \leq R \sum_{\frac{i_1}{k_1} + \cdots \frac{i_r}{k_r} < 1} a_{i_1 \ldots i_r} u_1^{i_1} \cdots u_r^{i_r}$$

$$+ A \sum_{i=1}^{r} |u_i|^{k_i}$$

Advantages, can break up $\zeta(s) M_\xi(s) - 1$ into parts and choose each k_i (as an even integer) such that it gives the best bound for the relevant part. Can use a much smaller ξ in relation to T. ($\xi = T^\delta$ with some arbitrary pos fixed δ.). Also easier to utilize results about exp. sums. (throw away parts for which $\sum_{n=x}^{(x)} n^{-s} = O(|T|^{-\delta})$; choose $\xi = T^{(\delta)}$ then $|M_\xi(s)| = O(\xi^{\frac{1}{2}}) = O(|T|^{-\frac{\delta}{2}})$; take $k \geq \frac{2}{\delta}$

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obtained results somewhat sharper than Inghams using more primitive machinery of exp ~~sums~~, also could show that $\lambda(\sigma)$ for σ sufficiently close to ~~one~~ is $\lambda(\sigma) < 2+\delta$; All of Inghams results followed although (not with as good powers of $\log T$), and the conjecture $\sum_{N}^{2N} n^{-it}$ for

$$\int_T^{2T} \left| \sum_N^{2N} n^{it} \right|^{2k} = O(T^{1+\varepsilon})$$

if $T^{\delta} \le N \cdot T^{\frac{1}{k}} \le T^{\delta(\varepsilon)}$; ~~$\delta$ suff small.~~

(holds for k an integer. would imply.

$\lambda(\sigma) \le 2+\varepsilon$. (for arbitrary $k \ge 1$ . . .

Turán was able to show also .. his instrument for detecting zeros was to look at high derivatives of $\log \zeta(s)$ on the line $2+it$

if $\beta+i\gamma$ is a zero. then $\left(\frac{\zeta'}{\zeta}^{(N+i)}(2+i\gamma)\right)$ would get large.

Anything based on Littlewoods formula could not give anything better than

$N(\sigma,T) = O(T^{2(1-\sigma)} \log^c T)$.

namely if $\zeta(s_j) = 0$ we have

$\zeta(s_j) M_\xi(s_j) - 1 = -1$ which means

$\sum_{\xi}^{X} \frac{a_n}{n^{s_j}} + R$ | if remainder $\le \frac{1}{2}$

$$\mathrm{Re} \sum_{\xi}^{X} \frac{a_n}{n^{4s_j}} \le -\frac{1}{2} \left| \left| \sum_{\xi}^{X} \frac{a_n}{n^{s_j}} \right| \ge \frac{1}{2} \right.$$

this can be used as zero detecting device.

?

$\log|f|$

sin

$$\int \log|f| \frac{\partial v}{\partial n} \qquad -\frac{f'}{f}$$

$v = 0$ on boundary.

$e^{x} - e^{-x} < 2xe^{x}$

$\frac{1-e^{-2x}}{2x} < 1$

$1-2x < e^{-2x}$

σ

$v = \underline{\sin \alpha t}$

$$\cos \frac{\pi}{4T} t \cdot e^{\frac{\pi}{2T}(\sigma-\frac{\alpha}{2})} - e^{-\frac{\pi}{2T}(\sigma-\frac{\alpha}{2})} \qquad \frac{4T}{\pi} \cdot 2\pi$$

$$\frac{\pi}{4T}\left(e^{\frac{\pi}{4T}(\sigma-\alpha)} + e^{-\frac{\pi}{2T}(\sigma-\frac{1}{2})}\right) \cos \frac{\pi t}{4T}$$

$\frac{\pi}{T} \qquad 2\pi \sum v(\rho)$

$$\sin \frac{\pi t}{2T} \quad e^{\frac{\pi}{2T}}$$

$$e^{\frac{\pi}{2T}(\sigma-\frac{1}{2})} - e^{-\frac{\pi}{2T}(\sigma-\frac{1}{2})} \qquad e^{\frac{\pi}{4}}$$

$$\int_{-2T}^{2T} \log|f(\alpha+it)| \cdot \cos \frac{\pi t}{4T}\, dt + \int_{\alpha}^{\infty} \left(\log|f(\sigma+2iT)| + \log|f(\sigma-2iT)|\right)\left(e^{\frac{\pi}{4T}(\sigma-\alpha)} - e^{-\frac{\pi}{4T}(\sigma-\alpha)}\right) d\sigma$$

$$= \sum_{|\gamma|<2T} 8T \cos \frac{\pi\gamma}{4T} \cdot \left(e^{\frac{\pi}{4T}(\beta-\alpha)} - e^{-\frac{\pi}{4T}(\beta-\alpha)}\right)$$

$$> 2\sqrt{2} \sum_{|\gamma|\le T} (\beta-\alpha)$$

$$\sum_{|\gamma|\le T} (\beta-\alpha) < \frac{1}{2\sqrt{2}} \int_{-2T}^{2T} \log|f(\alpha+it)|\, dt + \frac{1}{T\sqrt{2}} \int_{\alpha}^{\infty} (\sigma-\alpha) e^{\frac{\pi}{4T}(\sigma-\alpha)} \log|f(\sigma+2iT)| + \log|f(\sigma-2iT)|\, d\sigma$$

1

Density theorem.

Combing methods. | approx or repn. of ζ function

Carlson. 'Jensens'

Littlewoods lemma

$$N(\sigma,T) = O(T^{\lambda(\sigma)(1-\sigma)} \log^6 T)$$

~~Carlson~~

$\lambda(\sigma) = 4\sigma$

$\frac{4}{3-2\sigma}$ Titchmarsh

$1+2\sigma$ | $2+4c$ Ingham

$\frac{3}{2-\sigma}$ | if $\zeta(s) = O(|t|^c)$ $s = \frac{1}{2}+it$.

$\frac{2}{\sigma}$ (Montg. 1969) Turan-Halasz 1968

$$\zeta(s) = \sum_{n \le x} n^{-s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma})$$

$x > |t|$

$xy = \frac{|t|}{2\pi}$

$$\zeta(s) = \sum_{n \le x} n^{-s} + \chi(s) \sum_{n \le y} n^{s-1} + O(x^{-\sigma} + |\chi(s)| y^{\sigma-1})$$

$y^{\frac{1}{2}-\sigma}$

$$\zeta(s) = \chi(s)\zeta(1-s)$$

$$\zeta(s) = \sum_{n \le x} n^{-s} - \frac{x^{1-s}}{1-s} + \int^{\sigma} \ldots + O\left(\frac{|t|^c (1+\log|t|)}{x^{\sigma-\frac{1}{2}}}\right)$$

For $|t| > 1$

$x \to \infty$.

$$M_\xi(s) = \sum_1^{\xi} \frac{\mu(n)}{n^s}$$

zeros s_ξ.

$$\zeta(s) M_\xi(s) = 1 + \sum_{\xi \le n} \frac{a_n}{n^s} + R(s)$$

summability

abel sum.

Cesaro.

$$\sum_{n \le x} \left(1 - \frac{n}{x}\right)^k$$

$R(\omega) = (\frac{1}{2}-\sigma)$ $e^{-\frac{n}{x}}$

abel sum -

Mellin transf has $s=0$ $w=0$

$$\int_{\mathcal{X}}^{W} f(s+w)\phi(w)\, dw$$

$$\frac{1}{2\pi i}\int \Gamma(w) \left(\frac{x}{n}\right)^{w} dw$$

$e^{-\frac{n}{x}}$

$$\frac{1}{T}\int_T^{2T} \left|\sum_{\xi}^{x} \frac{a_n}{n^s}\right|^2 dt \le T \sum_{\xi}^{x} \frac{a_n^2}{n^{2\sigma}} + \log x \sum_{\xi}^{x} \frac{a_n^2}{n^\sigma} \cdot \sum_{\xi}^{x} \frac{1}{n^\sigma}$$

$$T \xi^{1-2\sigma} \log x + x^{2(1-\sigma)} \log^a x.$$

7 Ingham convexity

$\log|1+u| \leq R\, u + \frac{|u|^2}{2}$

$\log|1+u_1+\cdots+u_r| \leq R \sum_{\frac{i_1}{k_1}+\frac{i_2}{k_2}+\cdots+\frac{i_r}{k_r}\leq 1} a_{i_1\ldots i_r} u_1^{i_1}\cdots u_r^{i_r}$

$+ A \sum_{i=1}^{r} |u_i|^{k_i}$

Break $\phi(s) = \zeta(s) M_X(s) - 1$ up.

chose k_i most suitable for estimation of

$\int_T^{2T} |u_i|^{k_i} \, \underline{dt}$

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cannot give better than

$N(\sigma, T) = O\left(T^{2(1-\sigma)}\right)$

Turans instrument. if $s_j = \beta_j + i\gamma_j$ zero

uses $\left(\frac{\zeta'}{\zeta}\right)^{(\nu)}(2+i\gamma_j)$ for large ν

$\frac{(-1)^{\nu}\,\nu!}{(s-\rho_j)^{\nu+1}} \quad (2-\beta_j)$. establishes that one $\frac{\zeta'}{\zeta}^{(\nu)}(2+i\gamma_j)$ has

to be large in the range $\alpha \delta T \leq \nu \leq \beta \delta T$

~~Summation rather than integration~~

$\phi(s_j) = 1 - \zeta(s_j) M_\xi(s_j, \xi)$

$\phi(s_j) = 1$ could majorize ϕ by integral over disk radius $\frac{1}{\delta T}$

$M \leq \sum \phi(s_j)$

$\phi(s_j) = u_1(s) + u_i(s) + \quad u_k(s)$

at s_j at least one $|u_i(s_j)| \geq \frac{1}{r}$

$M \leq \sum_j \sum_i A^{k_i} |u_i(s_j)|^{k_i}$

3 estimate other than by integral.

$$\sum |\mu_i(s_j)|$$

$$\sum \left| \sum_{\xi_1}^{\xi_2} \frac{a_n}{n^s} \right|$$

$$\leq \sum \eta_j \sum_{\xi}^{\xi_2} \frac{a_n}{n^{s_j}}$$

$3 \times 2 = 12$

$$= \sum_{\xi_1}^{\xi_2} a_n \sum_{j}^{q} \eta_j n^{-s_j}$$

$Q^2 T$

3

$$\leq \sqrt{\sum_{\xi_1}^{\xi_2} \frac{a_n^2}{n} 100} \sqrt{\sum_{\xi_1}^{\xi_2} n \left| \sum_j \eta_j n^{-s_j} \right|^2}$$

$(Q^2 T)$ $\quad Q^2 T$

f is conductor of χ.

$$\leq q^{*} \xi_2 \sqrt{\sum_{j_1 j_2'} \left| \sum_{\xi_1}^{\xi_2} n^{1-s_j-\bar{s}_{j'}} \right|}$$

$|\tau(\bar\chi)|^2 = f$ $\quad$ r squarefree

$s_1 = \bar{s}_j'$

$$\sum \frac{\mu(r)}{\varphi(r)} = \sum' \mu(d)$$

$q = fr$ and $(r, f) = 1$ $\quad$ ~~q~~ $= fr$ $\quad r, X$

$$\frac{\mu(d)}{d} \sum_{Q \leq T^{1+\varepsilon}}$$

$$\sum_{\substack{r \leq Q/f \\ (r,f)=1}} \qquad \sum_{r \leq X} \frac{\mu^2(r)}{\varphi(r)} \leq \text{ yes, my dear.}$$

$$1 + \frac{p}{p-1}$$

$$\sum_{r \leq X} \frac{\mu^2(r)}{\varphi(r)} \ll \prod \left(1 - \frac{1}{\varphi(p)}\right)$$

$$\sum_{\substack{r \leq X \\ (r,f)=1}} \frac{\mu^2(r)}{\varphi(r)} \cdot \prod_{p/f} \left(1 + \frac{1}{\varphi(p)}\right) \geq \log X$$

$$\geq \frac{\varphi(f)}{f} \log X$$

$$\sum \log \frac{Q}{f}$$

4

$\sum$ why not do it. the $\binom{n}{m} e^{-\frac{u}{\sigma}}$

$$\left| \sum_{N}^{2N} a_n n^{-s} \right|^2$$

u

$\boxed{\text{the range.}}$

$\int |a_n|^2 \geq \sqrt{a_n}$

1 is represented which way every

$$\frac{a_n a_m}{(nm)^\sigma} \quad \left(\frac{m}{n}\right)^{it}$$

$\sum \frac{a_n}{n^2}$

$3\zeta(s) + 4\zeta(s-it)$ yeah.

$\int \underline{n} \frac{p}{q} \sum a_n$

$\sqrt{n}$

$\sum_q (1 + \ldots + \cos 2\theta$

$1 + 3 + 4\cos\theta$

M^s

$\left(\frac{m}{n}\right)$

$\sigma = \frac{1}{2} \quad \sum \frac{a_n a_m}{(n m)^\sigma}$

$\frac{m}{n} = s$

$$\frac{1}{m^\sigma n^\sigma}$$

$M H^2 \leq$

yeah

$p > q$

λp

$$M^2 H^4 \leq \sum_{N}^{2N} \frac{a_n a_m}{n^\sigma m^\sigma} \cdot \sum \frac{p}{q}$$

λq

$\lambda p \leq N_2$

$q \left(\frac{1}{T}q\right)\frac{u}{T}$

$a_n a_m \sum \quad \frac{1}{q} - \frac{1}{p}$

the distance N $\leq \lambda q$

$\leq \frac{N_2}{p}$

$\frac{q+1}{q} = \left(1+\frac{1}{p}\right)$

$\frac{N_1}{q} \leq \lambda$

$\left(1+\frac{1}{q}\right)$

$$\frac{p}{q} - \frac{p'}{q'} = \frac{1}{qq'}$$

$\frac{1}{Q^2} \quad \frac{1}{q}$

yeah

$\log \frac{p}{q} \left(1 + \frac{1}{qq'} \quad \frac{1}{pq'}\right)$

$$\frac{\log Q}{Q^2}$$

$\leq \frac{1}{pQ}$

How many terms are

$q < p$

$\log \frac{p}{q} \leq \frac{1}{pQ}$

$\sum \lambda^{2\sigma}$

$$\frac{N_1}{q} \leq \lambda \leq \frac{N_2}{p}$$

$$\frac{a}{\lambda^{2\sigma}(pq)^\sigma}$$

$$\frac{q^{2\sigma-1}}{N_1^{1-2\sigma}}$$

$p \leq q$.

$\sigma = \frac{1}{2}$

$$\frac{1}{(pq)^\sigma}$$

$p^\sigma q^{1-\sigma}$

$\frac{1}{N_1}$

$$\frac{1}{\sqrt{pq}} \log \frac{N_2}{N_1} \frac{q}{p}$$