ry for forced oscillations
systems, Preprint, ETH

ordism Theorem, Math.

connection matrices, Er-
1988).

le l'article de M. Gromov
ic manifolds', preprint,

dex and Floer homology,

jecture in higher dimen-

és Lagrangiennes fonc-
Hamiltoniens, Preprint,

edding Lagrangian tori,

Theory, J. Diff. Equa-

s compactness of pseu-
University, 1986.

xed points of symplec-
niltonian systems, Pro-
Mathematicians, Berke-

Determinants of Laplacians; Heights and Finiteness

Peter Sarnak

Stanford University
Palo Alto, California

To J. Moser on the occasion of his 60^{th} birthday.

Section 0. Introduction.

Most of the results described in this paper have been obtained jointly with R. Phillips and B. Osgood in the series of papers [OPS, 1,2,3]. The problem with which we will be concerned is, as it was coined by M. Kac [KA], "hearing the shape of a drum". Put another way what can be said about an isospectral set of planar regions. Here and more generally an isospectral set of compact Riemannian manifolds M (possibly with boundary) is a set of such manifolds which all have the same spectrum for the Laplace-Beltrami operator (with Dirichlet boundary conditions if M has boundary). For the basic problem of planar "drums" it is possible, as far as is known to date, that such isospectral sets consist of one element only, i.e. the drum is determined from its spectrum. However, in the more general setting of Riemannian manifolds such uniqueness fails. For example in dimensions greater than 4 isospectral sets may even contain 1-parameter families of nonisometric manifolds, see Gordon and Wilson [G-W]. In dimension 2, Vigneras [V] and more recently Sunada [SU] has given examples of isospectral sets of arbitrary large cardinality. Our main

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result is the compactness of isospectral sets of Riemannian manifolds in dimension 2, precisely.

Theorem 0.1 [OPS 1,2,3]:
(A) An isospectral set of closed (i.e. no boundary) Riemannian two manifolds is compact in the C^∞ topology.
(B) An isospectral set of planar drums is compact in C^∞.

We explain the C^∞ topology on metrics used above. Firstly, in (A) and (B) above it is known [KA] that the spectrum determines the topology of the underlying manifold so it suffices to topologize the space of all Riemannian structures on a given manifold M. This we do for general compact M. Fix on M a smooth Riemannian metric σ_o which is used as a background metric. The space $\mathcal{G} = \mathcal{G}^\infty(M)$ is the space of all C^∞ metric tensors on M with the usual C^∞ topology. Let $D = D^\infty(M)$ be the group of smooth diffeomorphisms of M (which take $\partial M \to \partial M$ smoothly if $\partial M \neq \phi$). D acts on $\mathcal{G}$ by pullback of metric. The quotient space (with its quotient topology)

$$\mathcal{R}(M) = \mathcal{G}^\infty(M)/D^\infty(M) \tag{0.1}$$

consists of the space of Riemannian metrics (up to isometry) on M. It is Hausdorff and gives the natural C^∞ topology on the space of metrics. A class in $\mathcal{R}(M)$ will be denoted by [g]. Convergence in $\mathcal{R}(M)$ may be described as follows

$$[g]_n \to [g]$$

iff there are $g_n \in [g]_n$, $g \in [g]$ such that

$$g_n \to g \quad \text{in} \quad \mathcal{G}^\infty(M).$$

Using Theorem 0.1 and the rigidity techniques of Guillemin-Kashdan [G-K] we will show in Section 2 finiteness of isospectral sets under certain assumptions. It seems plausible that in two dimensions isospectral sets are finite. We discuss this and related questions in Section 2. We turn now to the basic tool in our analysis which is the height function on $\mathcal{R}(M)$.

Section 1. Heights.

As above let (M, σ) be a compact manifold with smooth Riemannian metric σ. Let Δ denote the Laplace Beltrami operator acting on functions on M. Let $0 \leq \lambda_0 < \lambda_1 \leq \lambda_2 \ldots$ denote the eigenvalues of Δ. The notion of the determinant of the Laplacian, $\det' \Delta$, was introduced in Ray and Singer [R-S] in the context of analytic torsion. More recently it appears in Polyakov's string model [PO] as part of the integrand of a Feynman integral. Formally

$$\det'\Delta = \prod_{\lambda_j \neq 0} \lambda_j .$$

To give meaning to this product we use the standard zeta regularization. Let

$$Z(s) = \sum_{\lambda_j \neq 0} \lambda_j^{-s}. \tag{1.2}$$

This converges absolutely for $\text{Re}(s)$ large. $Z(s)$ has the integral representation

$$Z(s) = \frac{1}{\Gamma(s)} \int_0^\infty TR'(e^{\Delta t}) t^s \frac{dt}{t} \tag{1.3}$$

which follows from term by term integration. From the well known small time asymptotics for $TR'(e^{\Delta t})$ [GI 1], one easily deduces the meromorphic continuation of $Z(s)$ to the complex plane. The zero of $\Gamma(s)^{-1}$ at $s = 0$ ensures that $Z(s)$ is regular at $s = 0$. Formally from (1.1) and (1.2) we have $\log \det' \Delta = -Z'(0)$ and so we define $\det' \Delta$ by

$$\det'\Delta = e^{-Z'(0)}. \tag{1.4}$$

The height of (M, σ) denoted $h(\sigma)$ is defined to be

$$h(\sigma) = -\log \det'\Delta = Z'(0). \tag{1.5}$$

From its definition the height is clearly an isospectral invariant. The reason for the name height should become clear from the following analysis as well as the fact that it is related to certain height functions in the theory of arithmetical geometry, see the Appendix.

We now investigate the properties of the height function $h : \mathcal{R}(M) \to \mathbf{R}$. Unlike the individual eigenvalues $\lambda_j : \mathcal{R}(M) \to \mathbf{R}$, it is a smooth function. Its most important property is that of being a proper function (in certain cases). In general we know very little about h, so for the rest of this section M is taken to be a two-dimensional manifold. We use $\Sigma_{p,n}$ instead of M to stand for a fixed topological surface which is a compact surface of genus p with n distinct open disks removed. Thus $\Sigma_{p,o}$ is a closed surface of genus p, while for $n \geq 1$, $\partial\Sigma_{p,n}$ consists of n circles. When dealing with surfaces with boundary, our aim has been to capture the classical case of "planar drums". These all have flat metrics (i.e. zero Gauss curvature K in the interior of Σ) on $\Sigma_{o,n}$. So when considering more generally $\Sigma_{p,n}$, $n \geq 1$ we restrict ourselves to flat metrics in $\mathcal{R}(\Sigma_{p,n})$. We denote this subspace of flat metrics by

$$\mathcal{R}_F(\Sigma_{p,n}) \tag{1.6}$$

(when $n = 0$ the symbol F and flatness is to be ignored). The set of planar drums of connectivity n is a closed subspace of $\mathcal{R}_F(\Sigma_{o,n})$. We need to further normalize the metrics. If a given metric σ on Σ is scaled by γ^2 i.e. $\sigma^1 = \gamma^2\sigma$, $\gamma \in \mathbb{R}$, then the eigenvalues are scaled down and one can easily check from (1.2) and the definition of h that

$$h(\gamma^2\sigma) = \begin{cases} \frac{\chi(\Sigma)}{3}\log\gamma + h(\sigma) & \text{if } \partial\Sigma \neq \phi \\ 2(\frac{\chi(\Sigma)}{6} - 1)\log\gamma + h(\sigma) & \text{if } \partial\Sigma = \phi. \end{cases} \tag{1.7}$$

Here χ is the Euler number. So unless the coefficient of $\log\gamma$ is zero we need to normalize the metrics. We do so as follows:

$$\begin{cases} \text{Area}_\sigma(\Sigma) = 1 & \text{if } \Sigma \text{ is dosed} \\ \text{length}_\sigma(\partial\Sigma) = 1 & \text{if } \partial\Sigma \neq \phi. \end{cases} \tag{1.8}$$

The resulting space of normalized Riemannian metrics on Σ will be denoted by

$${}_o\mathcal{R}_F(\Sigma). \tag{1.9}$$

A key ingredient in the study of the height on ${}_o\mathcal{R}_F(\Sigma)$ is the Polyakov-Alvarez [AL, PO] variation formula. For $\phi \in C^\infty(\Sigma)$ and $\sigma \in \mathcal{G}(\Sigma)$ the variation formula gives the variation $\delta_\phi(h(e^{2\phi}\sigma))$. When integrated this gives rise to the following relation between the heights of conformal metrics [OPS 1]: Under the normalization 1.8

$$h(e^{2\phi}\sigma) = \frac{1}{6\pi}\left\{\frac{1}{2}\int_{\Sigma}|\nabla\phi|^2 dA + \int_{\Sigma} K\phi dA + \int_{\partial\Sigma} k\phi ds\right\} - \frac{1}{4\pi}\int_{\partial\Sigma}\partial_n\phi ds + h(\sigma) \tag{1.10}$$

where ∇, dA ... etc. are all in the σ metric. Since in case $\partial\Sigma \neq \phi$ we are assuming the metrics to be flat i.e. σ and $e^{2\phi}\sigma$ are flat, the above formula simplifies since then $k \equiv 0$, ϕ is σ harmonic and $\int_{\partial\Sigma}\partial_n\phi ds$ is independent of ϕ.

(1.10) allows us to study the height in conformal classes of metrics and to identify the extremal metrics for the height.

Definition 1.1.

(A) For Σ closed we say a metric u on Σ is uniform if it is of constant curvature.

(B) For $\partial\Sigma \neq \phi$ a metric u on Σ is uniform if it is flat on Σ and if $\partial\Sigma$ has constant geodesic curvature in the u metric.

Examples of uniform metrics are S^2 with its round metric, flat tori on $\Sigma_{1,0}$ and hyperbolic metrics on $\Sigma_{p,o}$, $p \geq 2$. These are all of them for closed surfaces. The unit disk $K = \{z : |z| \leq 1\}$ with its usual metric is uniform, in fact the only one up to scale on $\Sigma_{0,1}$. Flat cylinders give all uniform metrics on $\Sigma_{0,2}$. On $\Sigma_{1,n}$, a flat torus with n disjoint disks of radius equal $1/(2\pi n)$ removed gives a uniform metric.

Theorem 1.1 [OPS 1]. In a conformal class of metrics in ${}_0\mathcal{R}_F(\Sigma)$ there is a unique uniform metric (of type A if $\partial\Sigma = \phi$ and type B otherwise) and it is the unique global minimum of the height function in the conformal class.

The theorem is proven by extremizing $h(\phi)$ in (1.10) subject to the constraint $1 = \int_{\Sigma} e^{2\phi} dA_{\sigma}$ or $1 = \int_{\partial\Sigma} e^{\phi} ds_{\sigma}$ depending on whether $\partial\Sigma = \phi$ or not. Existence of the minimum is the trickier part especially if Σ is $\Sigma_{0,0} = S^2$ or $\Sigma_{0,1} = K$. In fact for S^2 Theorem 1.1 is equivalent to the following sharp form of an inequality of Moser [MO]:

For $\phi \in C^\infty(S^2)$

$$\log \int_{S^2} e^{\phi} \frac{dA_0}{4\pi} \le \frac{1}{4} \int_{S^2} |\nabla_0 \phi|^2 \frac{dA_0}{4\pi} + \int_{S^2} \phi \frac{dA_0}{4\pi} \tag{1.11}$$

where dA_0, ∇_0 correspond to the $K \equiv 1$ metric on S^2. Moreover equality holds in (1.11) iff $\phi = 2\log|V'| + \beta$ where $V : S^2 \to S^2$ is a Möbius transformation and $\beta \in \mathbf{R}$. (1.11) was proved by Onofri [ON] using work of Aubin [AU] but it can be derived directly from Moser's inequality [OPS 1].

In the case of the unit disk K, Theorem 1.1, via (1.10) above, is equivalent to the inequality:

For $\phi \in C^\infty(\overline{K})$

$$\log \int_{\partial K} e^{\phi} \frac{d\theta}{2\pi} \le \frac{1}{4} \int_{K} |\nabla \phi|^2 \frac{dx\,dy}{\pi} + \int_{\partial K} \phi \frac{d\theta}{2\pi} \tag{1.12}$$

with equality iff $\phi = \log|\tau'| + \beta$ where $\tau : K \to K$ is a Möbius transformation and $\beta \in \mathbf{R}$. (1.12) is the so-called first Milin-Lebedev inequality [D] (which is usually stated in terms of power series). Theorem 1.1 gives it geometric meaning i.e. of all plane simply connected domains of fixed boundary length the circle has maximum determinant for Laplacians. A geometric proof of (1.12) as well as sharpenings of it to cases when ϕ satisfies the extra hypothesis of $\int_0^{2\pi} e^{\phi} e^{i\theta} d\theta = 0$ (which are needed for the proof of Theorem 0.1) is given in [OPS 1]. Yet another proof of (1.12) and its sharpenings under $\int_0^{2\pi} e^{\phi} e^{im\theta} d\theta = 0$ for $m < n$ has been given by Widom [WI] using Szegö's theory [SZ].

Let $\mathcal{M}_u(\Sigma)$ denote the subspace of ${}_0\mathcal{R}_F(\Sigma)$ consisting of uniform metrics. It is closed and in view of Theorem 1.1 it is homeomorphic to the moduli space of conformal structures on Σ. As is well known this space is finite dimensional, of dimension $6p - 6 + 3n$ if $(p,n) \ne (0,0), (0,1)$ and $(0,2)$. The dimensions in these three cases are $0, 0$ and 1 respectively.

For $A \subset {}_0\mathcal{R}(\Sigma_{p,o})$ we say A is weakly precompact in W^α if the following holds: For any sequence $[g]_n \in A$ there is a subsequence $[g]_{n_j}$ satisfying

(1) there are $g_{n_j} \in [g]_{n_j}$ of the form $g_{n_j} = u_{n_j} e^{2\phi_{n_j}}$ where $[u]_{n_j} \in \mathcal{M}_u(\Sigma)$ and $\phi_{n_j} \in C^\infty(\Sigma)$.
(2) $u_{n_j} \to u$ in $\mathcal{G}^\infty(\Sigma)$ and ϕ_{n_j} is bounded in $W^\alpha(\Sigma)$; the Sobolev α space on Σ relative to our background metric.

Similarly for $A \subset {}_0\mathcal{R}_F(\Sigma_{p,n})$, $n \geq 1$, being weakly precompact in W^α. In this case the ϕ_{n_j}'s are u_{n_j} harmonic so we could, and will, use the Sobolev norms on $\partial\Sigma$ for the functions ϕ.

By the usual embedding arguments weakly precompact in W^α implies usual precompactness in a smaller $W^{\alpha'}$ and a set which is W^α weakly precompact for all α is precompact in ${}_0\mathcal{R}_F(\Sigma)$. We can now state the basic property of the heights in the cases $pn = 0$.

Theorem 1.2 [OPS 2,3].
(A) A subset of ${}_0\mathcal{R}(\Sigma_{p,o})$ of bounded height is W^1 weakly precompact.
(B) A subset of ${}_0\mathcal{R}_F(\Sigma_{o,n})$, $n \geq 1$, of bounded height is $W^{\frac{1}{2}}$ weakly precompact.

Theorem 1.2 is deduced from the following weaker but still central fact which shows that h is a proper function on the finite dimensional space $\mathcal{M}_u(\Sigma)$.

Theorem 1.3 [OPS 2,3]. For $pn = 0$

$$h(u) \to \infty \quad \text{as} \quad u \to \partial\mathcal{M}_u(\Sigma_{p,n}).$$

We outline a derivation of Theorem 1.2 from Theorem 1.3. Let $A \subset {}_0\mathcal{R}(\Sigma)$ be of bounded height. Let $[g]_n$ be a sequence in A. If $[u]_n$ corresponds to $[g]_n$ as in Theorem 1.1 then $h([u]_n)$ is bounded above. Hence by Theorem 1.3 $\{[u]_n\}$ lies in a compact. We may therefore extract a subsequence $u_{n_j} \in [u]_{n_j}$ such that $u_{n_j} \to u$ in $\mathcal{G}^\infty$. We then have the representation for suitable $g_{n_j} \in [g]_{n_j}$ and $\phi_{n_j} \in C^\infty(\Sigma)$

$$g_{n_j} = u_{n_j} e^{2\phi_{n_j}}.$$

A careful analysis (especially in the $\Sigma_{0,0}$ and $\Sigma_{0,1}$ cases where ϕ_{n_j} needs further adjusting by the action of Möbius (Σ)) using (1.10) and the fact that the heights are bounded, allows one to bound the ϕ_{n_j} in the Sobolev space claimed.

The proof of Theorem 1.3 is rather complicated especially for the case $n \geq 1$. For $\Sigma_{0,0}$ and $\Sigma_{0,1}$ there is of course nothing to prove. For $\Sigma_{1,0}$ and $\Sigma_{0,2}$ the moduli space and heights are easily described and computed explicitly, see the Appendix. From the formulas the result is verified directly. For $\Sigma_{p,o}$, $p \geq 2$, the result was derived by Gava-Ieng-Jagaraman-Ramachandran [G-I-J-R] and Wolpert [WO] using the Selberg zeta function. These authors were examining the boundary behavior of the determinant in connection with the Tachyon divergence of the Bosonic string [WO]. In these cases the degeneration of a hyperbolic surface is well known [BER]. The way it can degenerate is rather simple and basically comes from the development of long collars with short closed geodesics. We will give a direct method of analyzing the height in such cases. For the general $\Sigma_{o,n}$ case the boundary of $\mathcal{M}_u(\Sigma_{o,n})$ is apparently much more complicated. Here are some techniques developed in [OPS 3] to deal with these cases.

The following insertion Lemma often allows one to decompose a degenerating family of metrics into simpler parts and to examine the height on each. Let Σ be a general surface and $\Sigma(\sigma)$ denote this surface with metric σ. Suppose Σ is decomposed into Ω_1 and Ω_2 by some curve Γ lying in Σ. In this way Ω_1, Ω_2 become spaces with the induced metric. By the minimax principle it follows that if $\lambda_1 < \lambda_2 \leq \lambda_3 \dots$ are the eigenvalues for $\Sigma(\sigma)$ and if those for Ω_1 together with Ω_2 are $\mu_1 \leq \mu_2 \leq \mu_3 \dots$ then

$$\lambda_j \leq \mu_j.$$

Hence at least formally

$$\Pi \frac{\mu_j}{\lambda_j} \geq 1 \quad \text{or} \quad h(\Sigma(\sigma)) \geq h(\Omega_1) + h(\Omega_2).$$

Unfortunately this inequality is not true in general but it is approximately true under certain conditions. We say a curve Γ in $\Sigma(\sigma)$ is a $C^\ell - K$ quasi-collar if Γ has a neighborhood N in Σ which is isometric with $g_{ij}dx^i dx^j$ on $\{z \mid \frac{1}{2} < |z| < 2\}$ where $x^{(1)} = x$, $x^{(2)} = y$ and

(1) $K^{-1} \leq g_{ij}(x) \leq K$,

(2) $\|g_{ij}\|_{C^\ell} \leq K$,

(3) under the isometry Γ goes over to $|z| = 1$.

Insertion Lemma 1.4 [OPS 3]:
If Γ decomposes $\Sigma(\sigma)$ into Ω_1 and Ω_2 where Γ has a $C^\ell - K$ quasi-collar with $\ell \geq 11$ then

$$h(\Sigma(\sigma)) \geq h(\Omega_1) + h(\Omega_2) + 0(1)$$

where the $0(1)$ term depends only on K.

To illustrate this technique consider again Theorem 1.3 for the cases $\Sigma_{p,o}$, $p \geq 2$. A typical degeneration of the hyperbolic surface is as follows:

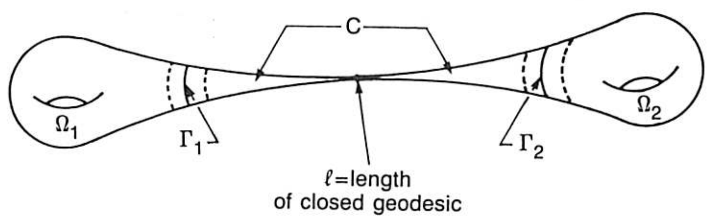

FIGURE 1.1

Here the degeneration is coming from $\ell \to 0$, Ω_1 and Ω_2 are converging. The curves Γ_1 and Γ_2 inserted have $C^{11} - K$ quasi-collars with K bounded and so by the lemma

$$h(\Sigma(\sigma)) \geq h(C) + h(\Omega_1) + h(\Omega_2) + 0(1).$$

Since $h(\Omega_1)$ and $h(\Omega_2)$ are bounded we need only check that $h(C) \to \infty$. For C we can compute the height directly as in Appendix 1 and one obtains the precise asymptotics of $h(C)$ as $\ell \to 0$. The above analysis can be pushed further to give directly the asymptotics of $h(u)$ as $u \to \partial\mathcal{M}_u(\Sigma_{p,o})$ without use of the Selberg Zeta function, see the thesis of Lundelius [LU].

Returning to the case of $\Sigma_{o,n}$, $n \geq 3$ the proof proceeds in three steps [OPS 3].

(1) An explicit description of the space $\mathcal{M}_u(\Sigma_{o,n})$ is developed. From this one determines the form of degeneration of these uniform metrics.

(2) Given a degenerating sequence of such metrics one tries to insert a family of $C^{11}-K$ quasi-collars (with K fixed) which decompose Σ into domains of smaller connectivity. If this is possible then one can proceed inductively.

(3) If the insertion cannot be done then through a notion of "valuation" [OPS 3] we are able to get an explicit enough handle on the height to verify that it $\to \infty$ directly.

Since this analysis becomes rather delicate, we do not pursue it further here.

The reader will have noticed that in Theorems 1.2 and 1.3 the cases $pn \neq 0$ are excluded. In fact these results are false in these cases! Khuri [KH] in her thesis has shown that the function $h : \mathcal{M}_u(\Sigma_{p,n}) \to \mathbf{R}$ is not a proper map if $np \neq 0$. She shows that there are curves $\gamma(t)$ with $\gamma(t) \to \partial\mathcal{M}_u(\Sigma)$ on which $h(\gamma(t))$ remains bounded from above (and even $h(\gamma(t)) \to -\infty$ if $(p,n) \neq (1,1)$). An example of such a curve in $\Sigma_{1,1}$ is the family u_t given by the following: Let $L_t = t\mathbf{Z} + \mathrm{i}t\mathbf{Z}$ be a lattice in $\mathbf{R}^2$. From the flat torus $\mathbf{R}^2/L_t$ remove the disk $\{z \mid |z| < 1\}$. The resulting metric u_t is uniform and normalized on $\Sigma_{1,1}$. For this curve $h(\gamma(t))$ is bounded above as $t \to \infty$. The insertion Lemma above and related techniques are used heavily in this analysis.

Section 2. The Isospectral Problem.

We now indicate how the properties of the height function, especially Theorem 1.2, when combined with the 'heat invariants' lead to Theorem 0.1. First let's recall what these local heat invariants are. Quite generally for a compact manifold M with metric σ let $k(t,x,y)$ be the fundamental solution of the heat equation

$$u_t + \Delta u = 0 \tag{2.1}$$

on $[0,\infty] \times M$. Then

$$\mathrm{TR}(e^{\Delta t}) = \sum_{n=0}^{\infty} e^{-\lambda_n t} = \int_M K(t,x,x)dx. \tag{2.2}$$

The asymptotics of $k(t,x,x)$ as $t \downarrow 0$ are well known [M-S] and involve certain universal (depending on dimension) polynomials in the curvatures and its covariant derivatives at x [GI 1]. Integrating these local invariants over M gives the small time asymptotics for $\mathrm{TR}(e^{\Delta t})$. For example if (M, σ) is of dimension 2 and closed then

$$\mathrm{TR}(e^{\Delta t}) \sim \frac{1}{t}\sum_{j=0}^{\infty} a_j(\sigma)t^j \quad \text{as} \quad t \downarrow 0. \tag{2.3}$$

Here $a_j = \int_M U_j dA$ with U_j a universal polynomial of degree $2j$ in K and Δ.

$$a_0 = \frac{A}{4\pi}, \quad a_1 = \frac{\chi(M)}{6}, \quad a_2 = \frac{\pi}{60}\int_M K^2 dA \quad \text{etc.} \tag{2.4}$$

For a flat surface with boundary ∂M we have

$$\mathrm{TR}(e^{\Delta t}) \sim \frac{1}{t}\sum_{j=0}^{\infty} a_j(\sigma)t^{j/2} \quad \text{as} \quad t \downarrow 0 \tag{2.5}$$

with $a_j(\sigma)$ integrals of polynomials in $k(s)$ and $k'(s)$ over ∂M, s being arc length. The first few a_j's are [M-S]

$$a_0 = \frac{A}{4\pi}, \quad a_1 = -\frac{\ell(\partial M)}{8\sqrt{\pi}}, \quad a_2 = \frac{\chi(M)}{6}$$

$$a_3 = \frac{1}{2^8\sqrt{\pi}}\int_{\partial M} k^2(s)ds, \quad a_4 = \frac{1}{315\pi}\int_{\partial M} k^3(s)ds$$

$$a_5 = \frac{37}{2^{15}\sqrt{\pi}}\int_{\partial M} k^4(s)ds - \frac{1}{2^{12}\sqrt{\pi}}\int_{\partial M}(k'(s))^2 ds, \quad \text{etc.}$$

The functions $a_j(\sigma)$ are of course isospectral invariants and are well suited for bounding curvature. Thus for example Melrose [ME] shows that one can prove the compactness of the curvature functions $k(s)$ in $C^\infty(\partial\Sigma)$ ($\partial\Sigma$ being parametrized by arc length s) for isospectral sets of planar domains using only the heat invariants. From a_2 and a_5 above one can easily get a bound on $k(s)$ in the Sobolev 1 norm.

What Melrose shows is that the $2j+1$ heat invariant in (2.5) is of the form

$$\alpha_{2j+1}\int_{\partial\Sigma}|k^j(s)|^2 ds + \int_{\partial\Sigma} (\text{lower order derivatives of } k)\, ds \quad (2.6)$$

where the constant $\alpha_{2j+1} \neq 0$ (this being the crucial point). With (2.6) it is straightforward to bound k in the higher Sobolev norms inductively. Note that in (2.6) above it seems rather difficult to evaluate the lower order terms explicitly. Luckily only the nonvanishing of the leading terms is needed.

We need a similar evaluation of the leading terms of $a_j(\sigma)$ in (2.3). In [OPS 2] we showed that the highest derivative terms of a_j (which recall is an integral of a polynomial in Δ and K) is

$$C_j \int_M \left(\Delta^{\frac{j-2}{2}} K\right)^2 dA \quad (2.7)$$

where

$$C_j = \frac{(-1)^j}{8\pi}\frac{j!}{(4j^2-1)(2j-3)!}, \quad j \geq 2.$$

Again the important point is that $C_j \neq 0$.

Subsequently Gilkey [GI 2] has evaluated the leading terms of the j^{th} heat invariant for closed manifolds of arbitrary dimensions ≥ 3. This will be used later.

It is important to observe that the local heat invariants alone do not suffice to ensure the compactness in $\mathcal{R}(\Sigma)$ of isospectral sets. For example for planar domains the family shown below in Figure 2.1 clearly has all its local heat invariants uniformly bounded (i.e. k bounded in $C^\infty(\partial\Sigma)$) yet the family degenerates.

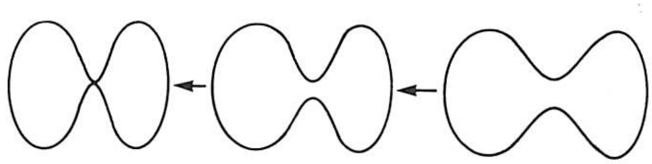

FIGURE 2.1

Similarly the family in Figure 2.2 (or 1.1) has all heat invariants bounded and it degenerates.

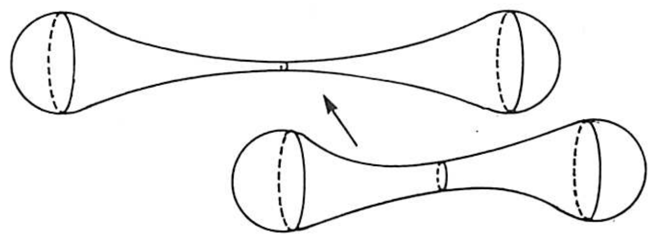

FIGURE 2.2

What is remarkable is that the height (which is not local i.e. is not an integral over Σ of local quantities) supplies the missing isospectral invariant in proving compactness. To see this consider say the closed surface case. The first heat invariant (2.4) fixes the area $A(\sigma)$ while the third fixes

$$\int_\Sigma K^2 dA. \tag{2.8}$$

The height is fixed and so by Theorem 1.2 the isospectral family is bounded in $W^1(\Sigma)$. That is we can write the metrics in the isospectral set as

$$e^{2\phi}u$$

with ϕ bounded in $W^1(\Sigma)$ and u in a compact set in $\mathcal{G}(\Sigma)$, also $[u] \in \mathcal{M}_u(\Sigma)$. We therefore have the bounds

$$\int_\Sigma (\Delta\phi)^2 e^{-2\phi} dA(u) << 1$$

$$\int_\Sigma e^{2\phi} dA(u) << 1$$

and

$$\int_{\Sigma} |\nabla \phi|^2 dA(u) + \int_{\Sigma} |\phi|^2 dA(u) << 1.$$

It is then easy to see that ϕ is uniformly bounded and hence that ϕ is bounded in $W^2(\Sigma)$. Proceeding with the heat invariants and (2.7) inductively one obtains the C^∞ compactness of the isospectral sets for closed Σ. The compactness in ${}_0\mathcal{R}_F(\Sigma_{o,n})$ of isospectral sets, is deduced in a similar way from Theorem 1.2 (B) and Melrose's result 2.6. The result for ${}_0\mathcal{R}_F(\Sigma_{o,n})$ implies the result for planar domains viz Theorem (0.1)B.

The compactness theorem gives us global information about an isospectral set. What remains is the local analysis, especially the question of the existence of local isospectral deformations. Guillemin and Kashdan [G-K] have shown that a negatively curved closed surface is isospectrally rigid, i.e. any 1-parameter family $g_t \in \mathcal{G}(\Sigma_{p,o})$ which is isospectral is an isometric family i.e. the projection of g_t in $\mathcal{R}(\Sigma_{p,o})$ consists of a single point. Using their techniques and Theorem 0.1 we can show

Theorem 2.1.

Let $I \subset \mathcal{R}(\Sigma_{p,o})$ be an isospectral set of metrics of curvatures $K \leq -\epsilon_0 < 0$ (variable curvatures) then $I \cap F$ is finite for any closed finite dimensional subspace F of $\mathcal{R}(\Sigma_{p,o})$.

This strongly suggests that I is in fact finite but so far this has defied proof. A special case of Theorem 2.1 is that an isospectral set in $\mathcal{M}_u(\Sigma_{p,o})$ is finite (by the result of Vigneras mentioned at the outset such a set may be arbitrarily large). This fact was proven by McKean [MC] using entirely different methods viz the trace identities in $SL(2,\mathbb{R})$ of Fricke-Klein [F-K].

The following is an outline of a proof of Theorem 2.1. Assume for simplicity that $I \cap F$ is contained in a fixed conformal class of metrics (the general case can be dealt with using the techniques in [G-K]). If $|I \cap F| = \infty$ then it follows from the compactness and above assumptions that there are functions $\rho_n \in C^\infty(\Sigma)$, $\rho_n \not\equiv 1$ with $\rho_n \to 1$ in $C^\infty(\Sigma)$ and a $\sigma_0 \in I \cap F$ such that

$$\rho_n^2 \sigma_o \in I \cap F.$$

Using the trace of the wave operator and especially the analysis of its singularities [G-K] it follows that the set of lengths of the closed geodesics of the metrics $\rho_n^2\sigma_o$ are all independent of n. Writing $\rho_n = 1 + \epsilon_n$ we have $\epsilon_n \not\equiv 0$. Let $v_n = \epsilon_n/\|\epsilon_n\|_{C^o}$. Since F is finite dimensional we can choose a subsequence of $\{v_n\}$ which converges to a nonzero limit v. By checking the variations of the lengths of closed geodesics one finds that, since $\rho_n^2\sigma_o$ all have identical lengths for their closed geodesics, we must have

$$\int_\gamma v ds_o = 0, \tag{2.9}$$

for every closed geodesic γ in the σ_o metric. However Guillemin and Kashdan have shown that 2.9 implies $v \equiv 0$. This gives a contradiction.

Returning to the general metrics on Σ and especially the planar domains we unfortunately know very little about local rigidity. The following is a basic problem.

Question 2.2: Are planar domains isospectrally rigid; i.e., is every 1-parameter family of isospectral planar domains necessarily an isometric family?

If as we expect the answer to 2.2 is yes then in view of the compactness we should expect isospectral sets of planar domains to be finite.

To end this section we describe some progress that has been in 3 dimensions. The main result is the following due to Chang and Yang.

Theorem 2.3 [C-Y]: Let M be a compact closed 3 manifold then in a fixed conformal class of metrics any isospectral set is compact in $\mathcal{R}(M)$.

Some cases of this theorem were established in Brooks-Perry-Yang [B-P-Y]. The proof of Theorem 2.3 does not make use of the height function. One of the difficulties with using the height in dimensions greater than 2 is that one does not have the simple Polyakov variational formula (1.10), see Parker-Rosenberg [P-R]. On the other hand in 3 dimensions the first heat invariant is $\int_M k dV$, (k being the scalar curvature) which, unlike in dimension 2 where it is independent of metric, turns out to be very useful. It allows one to bound the "conformal factor" in W^1. To get bounds in W^2 of the conformal factor the non-local invariant λ_1 is used in an ingenuous way together with the next heat invariants. To then obtain the bounds in higher Sobolev norms inductively they use the evaluation, due to Gilkey, of the highest derivative term of each heat invariant.

It is interesting that this Chang-Yang method allows one to give a proof of Theorem 0.1 for $\Sigma_{o,o}$ without using the height. However in all other cases, i.e. $\Sigma_{p,n}$, $pn = 0$, $(p,n) \neq (0,0)$, this approach cannot succeed in proving Theorem 0.1. The reason is that it is easy to construct degenerating families like those in Figures 1.1, 2.1 and 2.2 satisfying:

(1) Each of the local heat invariants is uniformly bounded above.
(2) For some fixed $\eta > 0$ and integer k

$$\eta \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \eta^{-1}.$$

The obvious question for these higher dimensional closed manifolds is:

Question 2.4: Are isospectral sets in $\mathcal{R}(M)$ always compact? Concerning this question it should be noted that the families constructed in Gordon-Wilson apparently all lie in a compact in $\mathcal{R}(M)$, [G-W].

Finally there is a recent excellent account of the inverse spectral problem in geometry, by Guillemin [GU]. The reader will find there a host of results and relations between the isospectral problem and problems in dynamical systems related to billiards and geodesic flows. The connection between these comes from the analysis of the wave equation and propagation of singularities [GU].

Appendix.

During the course of the proof of Theorem 1.3 we needed the asymptotics of h at the boundary of $\mathcal{M}_u(\Sigma_{p,n})$ for $(p,n) = (0,2)$ and $(1,0)$. In these and some few other cases the height can be computed explicitly.

The simplest case where this can be done is for the simplest Riemannian manifold - the unit circle S^1. The eigenvalues of its Laplacian are n^2 with $n = 0, \pm 1, \pm 2, \ldots$. Hence its Z function from 1.2 is

$$Z(s) = 2\zeta(2s) \qquad (A.1)$$

where $\zeta(s)$ is the Riemann zeta function. Using the definition (1.5) and a well known evaluation for $\zeta(s)$ we have

$$h(S^1) = 4\zeta'(0) = -2\log(2\pi). \tag{A.2}$$

Next consider the more interesting case of the height on $\mathcal{M}_u(\Sigma_{1,0})$ i.e. on the space of area 1 flat metrics on a torus. As is well known from the theory of elliptic curves this space is parametrized by

$$\mathsf{H}/\Gamma \quad \text{where} \quad \mathsf{H} = \{z \mid \mathrm{Im}(z) > 0\}$$

and $\Gamma = \mathrm{PSL}(2,\mathbf{Z})$. Each point $z \in \mathsf{H}/\Gamma$ gives rise to a flat torus R^2/L where L is the lattice $\mu\langle 1, z\rangle$, μ a scalar so chosen the R^2/L has area 1. With this identification $z \to \partial\mathcal{M}_u(\Sigma_{1,0})$ corresponds to $\mathrm{Im}(z) \to \infty$ with z in the usual fundamental domain for Γ. The eigenvalues of R^2/L are

$$(2\pi|\ell|)^{-2} \tag{A.3}$$

where $\ell \in L^*$, the lattice dual to L in R^2. Hence

$$Z(s) = {\sum_{\ell\in L^*}}' (2\pi|\ell|)^{-2s}. \tag{A.4}$$

This series is a so-called Eisenstein series. It is usual to set

$$E(z,s) = {\sum_{m,n}}' \frac{y^s}{|m+nz|^{2s}} \tag{A.5}$$

then

$$Z(s) = (2\pi)^{-2s} E(z,s). \tag{A.6}$$

The derivative of $E(z,s)$ w.r.t. s at $s=0$ was evaluated by Kronecker and is known as Kronecker's limit formula, see [WE]. He showed

$$E(z,0) = -1,$$
$$\left.\frac{\partial E}{\partial s}\right|_{s=0} = \log(\sqrt{2\pi}\ \ y^{\frac{1}{2}}|\eta(z)|^2) \tag{A.7}$$

where $\eta(z)$ is the Dedekind Eta function

$$\eta(z) = e^{\pi i z/12}\prod_{n=1}^{\infty}(1-e^{2\pi i n z}). \tag{A.8}$$

We conclude that

$$h(z) = -\log(y|\eta(z)|^4). \qquad (A.9)$$

From the rapidly convergent integral (A.8) we find

$$h(z) = \frac{\pi y}{3} - \log y + o(1) \quad \text{as} \quad y \to \infty. \qquad (A.10)$$

This proves Theorem 1.3 for $\Sigma_{1,0}$ and is a key ingredient in the proof of the other cases. For $\Sigma_{0,2}$, $\mathcal{M}_u(\Sigma_{0,2})$ consists of flat cylinders $[0,a]\times S^1$ i.e. $0 < x < a$, $0 \leq \theta < 2\pi$ (we use a slightly different normalization of the boundary length). The eigenvalues and eigenfunctions of Δ are computed by separating variables, one finds

$$Z(s) = \frac{1}{2}\sum_{m,n}{}' \frac{1}{|m+nz|^{2s}} - \frac{1}{2}\zeta(2s)$$

and hence from (A.5)

$$h(a) = -\log(\sqrt{2\pi}\ |\eta(\frac{\pi i}{a})|^2). \qquad (A.11)$$

From this and the behavior of $\eta(iy)$ as $y \to 0$ and ∞ (the behavior at 0 is obtained from $y^{\frac{1}{4}}|\eta(z)|$ being Γ invariant) we see $h \to \infty$ as $a \to 0$ or $a \to \infty$. (Moreover we get the precise asymptotics). These of course correspond to the degenerations of u in $\mathcal{M}_u(\Sigma_{0,2})$. Hence Theorem 1.3 for $\Sigma_{0,2}$ is also established by direct calculation.

From (A.11) we can deduce the behavior of h for the collars in Figure 1.1. Such a hyperbolic collar is after a change of coordinates isometric to

$$ds^2 = \sin^{-2}\phi(d\rho^2 + d\phi^2) \qquad (A.12)$$

where $0 \leq \rho \leq \ell$ and is a periodic variable and $\ell \leq \phi \leq \pi - \ell$. Thus this metric is conformal to the flat metric on a cylinder with explicit conformal factor $(\sin\phi)^{-2}$. Using (1.10) the height of the collar is easily evaluated.

We end by describing the relation of h to some other measures of complexity. In the arithmetic theory of heights, the height measures the arithmetic complexity of points defined over a number field [LA] and is a basic tool in proving finiteness results. Faltings [FA] developing Arakelov's theory of heights has introduced a discriminant function which measures in some sense the degeneration of the surface much like the height function h. For elliptic curves Faltings evaluates his δ-function explicitly and comparison of his results in Section 7 of his paper and (A.9) above shows that δ and h coincide on the moduli space of elliptic curves. For higher genus surfaces this is no longer true. In fact Bost, [BO] using variational methods developed by Quillen [Q] and Belavin-Knishnik [B-K], has shown that δ is related to determinants of Laplacians, but not for the uniform metric in a conformal class, but rather the so-called Arakelov metric [BO]. For algebraic purposes the Arakelov metric seems more appropriate while for our analytic purposes we are forced in view of Theorem 1.1 to use the uniform metric (in genus 1 the Arakelov and uniform metric coincide). In any event it is clear that the height is a natural measure of the complexity of a manifold. Further confirmation of this is the following finite analogue of the theory.

Let X be a finite graph with $|X| = n$ vertices. The Laplacian on functions on vertices of X is defined by

$$\Delta f(v) = d(v)f(v) - \sum_{w \sim v} f(w) \qquad (A.13)$$

where $d(v)$ is the degree of the vertex v i.e. the number of edges eminating from v and $w \sim v$ means w joined to v. Δ is a non-negative symmetric matrix and 0 is a simple eigenvalue of Δ iff X is connected which we now assume is the case. Let

$$\det{}'\Delta = \prod_{\lambda \neq 0} \lambda \qquad (A.14)$$

as in 1.1. This time the product is finite. The complexity of X denoted $K(X)$ [BO] is by definition the number of spanning trees in X. A spanning tree in X is a connected subgraph of X which is a tree and which contains all vertices of X. That $\det{}'\Delta$ measures the complexity of X is one of the oldest theorems in graph theory. It is due to Kirchhoff [BO] and is also known as the matrix tree theorem:

Theorem: $\det{}'\Delta = nK(X)$.

REFERENCES

[AL] Alvarez, O., "Theory of strings with boundary", Nucl. Phys. B. 216 (1983) 125-184.

[AV] Aubin, Th., "Meillerns constantes ...", J. Funct. Anal. 32 (1979) 148-174.

[BEL] Belavin, A. and V. Knishnik, Phys. Letters, B 168, (1986) 201.

[BER] Bers, L., "Spaces of degenerating Riemann surfaces", Ann. Math. Studies 79 (1974) 43-55.

[BOL] Bollobas, B., "Graph theory an introductory course", Springer Grad. Texts 1985.

[BOS] Bost, preprint 1987.

[BR] Brooks-Perry-Yang, "Isospectral sets of conformally equivalent metrics", preprint 1988.

[C-Y] Chang, A. and P. Yang, preprint 1988.

[D] Duren, P., "Univalent Functions", Springer Verlag 1983.

[FA] Faltings, G., "Calculus on arithmetic surfaces", Annals of Math. 1984, Vol. 119, 387-424.

[FR] Fricke, R. and F. Klein, "Vorlesungen über die Theorie der Automorphen Functionen", B. G. Teubner, 1965.

[GIJR] Gava, E., T. Iengo, T. Jayaraman and R. Ramachandran, "Multiloop divergences in the closed in the closed bosonic string", Phys. Letters 168 B (1986) 207-211.

[GI 1] Gilkey, P., "Invariance theory the heat equation and the Atiyah-Singer index theorem", Publish or Perish, Inc., Wilmington 1984.

[GI 2] Gilkey, P., "Leading terms in the asymptotics of the heat equation", preprint.

[GUL] Guillemin, V., "Inverse spectral problems in geometry", preprint 1988.

[G-W] Gordon, C. and Wilson, "Isospectral deformations of compact solvmanifolds", J. Diff. Geom. 19 (1984) 241-256.

[G-K] Guillemin, V. and D. Kashdan, "Some inverse spectral results for negatively curved manifolds", in Topology 1979.

[KA] Kac, M., "Can one hear the shape of a drum", Amer. Math. Monthly 73 (1966) 1-23.

ındary", Nucl. Phys. B.

. Funct. Anal. 32 (1979)

ters, B 168, (1986) 201.
ın surfaces", Ann. Math.

ıctory course", Springer

f conformally equivalent

ger Verlag 1983.
rfaces", Annals of Math.

über die Theorie der Au-
, 1965.
Ramachandran, "Multi-
ed bosonic string", Phys.

equation and the Atiyah-
ı, Inc., Wilmington 1984.
ptotics of the heat equa-

ıs in geometry", preprint

leformations of compact
:) 241-256.
ıverse spectral results for
gy 1979.
a drum", Amer. Math.

[KH] Khuri, H., Ph.D. thesis, Stanford in progress.

[LA] Lang, S., "Diophantine geometry", Springer, New York 1985.

[LU] Lundelius, R., Ph.D. thesis, Stanford in progress.

[ME] Melrose, R., "Isospectral drum heads are compact in C^∞", M.S.R.I. report, 1983.

[MO] Moser, J., "A sharp form of an inequality of Trudinger", Ind. J. (20) 1971, 1077-1092.

[M-S] McKean, H. and I. Singer, "Curvature and the eigenvalues of the Laplacian", J. Diff. Geom. 1 (1967) 43-69.

[Mc] McKean, H., "The Selberg trace formula ...", Comm. Pure and Appl. Math. 25 (1972) 225-246.

[ON] Onofri, E., "On the positivity of the effective action in a theory of random surfaces", Comm. Math. Phys. 86 (1982) 321-326.

[OPS 1] Osgood, B., R. Phillips and P. Sarnak, "Extremals of determinants of Laplacians", J. Funct. Analysis, 80 (1988) 148-211.

[OPS 2] Osgood, B., R. Phillips and P. Sarnak, "Compact isospectral sets of surfaces", J. Funct. Analysis, 80 (1988) 212-234.

[OPS 3] Osgood, B., R. Phillips and P. Sarnak, "Moduli space, heights and isospectral sets of plane domains", to appear Ann. Math.

[P-R] Parker, T. and Rosenberg, "Conformal Laplacians", J. Diff. Geometry 1988.

[PO] Polyakov, A., "Quantum geometry of bosonic strings", Phys. Letters 103 B (1981) 207-210.

[R-S] Ray, D. and I. Singer, "Analytic torsion for complex manifolds", Ann. Math. 98 (1973) 154.

[Q] Quillen, D., "Determinants of Cauchy Riemann operators", Funct. Anal. and Appl. 1984.

[SU] Sunada, T., "Riemannian coverings and isospectral manifolds", Ann. Math. 121 (1985) 169-186.

[SM] Smith, L., "The asymptotics of the heat equation for a boundary value problem", Invent. Math. 63 (1981) 467-493.

[SZ] Szego, G., Math. Ann. 76 (1915) 490-503.

[V] Vigneras, M. F., Ann. Math. 112 (1980) 21-32.

[WE] Weil, A., "Elliptic functions according to Eisenstein and Kronecker", Springer-Verlag, 1976.

[WI] Widom, H., "On an inequality of Osgood, Phillips and Sarnak", preprint 1987.

[WO] Wolpert, S., "Asymptotics of the Selberg zeta function for degenerating Riemann surfaces", Comm. Math. Phys. 112 (1987) 283-315.