Review of

Harish-Chandra Collected Papers

Edited by **V. S. Varadarajan**

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Group-representation theory is a broad topic which impinges on many domains of mathematics. The characters of abelian groups are of course central to Fourier analysis and entered at an early stage the theory of numbers, one of several spurs to the study of representations of finite nonabelian groups. Finite-dimensional representations of continuous groups, the groups with which the work of Harish-Chandra is largely concerned, arose in invariant theory. Interest in representation theory was stirred in the twenties and thirties by its utility in quantum mechanics, which probably encouraged the emphasis on unitary representations and, the Lorentz group having no interesting finite-dimensional unitary representations, led to the investigation of infinite-dimensional representations, whose mature theory has in turn profoundly influenced our thinking about zeta-functions.

Harish-Chandra's principal contributions were to the theory of infinite-dimensional representations of continuous groups, and it is a subject — a branch of analysis with algebraic prerequisites and geometric consequences — which still bears his imprint more than anyone else's. He was trained in India and England as a physicist and his papers on physics come to more than two hundred pages, almost half of the first of the four volumes of his *Collected Papers*. Nonetheless the last of them was written when he was only twenty-five and already turning to mathematics in Princeton, where he was exposed to the mathematical traditions of representation theory in which, following Cartan and Weyl, who had woven together Lie theory and invariant theory, the groups to be represented were arbitrary semisimple groups.

Specific semisimple groups or algebras are familiar, indeed everyday, objects, but not every mathematician is able to assimilate the general theory. Harish-Chandra, however, had great strength as a formal algebraist, and considerable experience, and from the very beginning worked not with concrete but with abstract semisimple algebras and groups. This is justified on all grounds but that of accessibility. As mathematics grows and sprawls, less and less of it can be regarded as common knowledge, and if it is not to break apart into several distinct sciences, it is best not to erect unnecessary obstacles to communication,. There is little in the papers of Harish-Chandra which is not as important for the symplectic group as in the general case, and it is to be regretted that those who are persuaded of the significance for number theory of the existence of the discrete series or who could appreciate the spectral theory of

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his later papers, but who are chary of roots and weights, will not be able to refer easily to the master.

Nonetheless, Harish-Chandra was one of the leading mathematicians of recent decades, with a difficult style but unique gifts, and some notion of the contents of his eighty-odd published papers is probably indispensable to an understanding of what mathematics has achieved in our time.

It is best when describing these contents to leave aside at first the papers in physics, the early papers on Lie algebras and the later papers on p-adic groups, and to consider only the work on real semisimple (later reductive) groups which began about 1950, reached a high point with the proof of the existence of the discrete series in the early sixties, and was completed toward 1970 with the proof of the explicit Plancherel theorem.

We can take the reductive group to be $G = \mathrm{GL}(n, \mathbf{R})$. It has a Lie algebra $\mathfrak{g}$, the algebra of $n \times n$ real matrices with bracket product $[X, Y] = XY - YX$, and the maximal compact subgroup K of orthogonal matrices. The first thing to notice, or rather to prove, is that the essential features of an irreducible representation π of G on a Banach space are captured by a representation, again denoted π, of the pair $(\mathfrak{g}, K)$ on a sense subspace V_0 of V, which consists of those $v \in V$ such that $\{\pi(k)v | k \in K\}$ spans a finite-dimensional space. Thus V_0 is a sum over the classes δ of irreducible representations of K of isotypical subspaces V_δ, and it is basic that $dim\ V_\delta \leq dim^2\, \delta$, for from this follows easily that for any smooth function f of compact support on G the operator $\pi(f) = \int_G f(g)\pi(g)dg$, defined by a Bochner integral, is of trace class and that $T_\pi\colon f \mapsto \operatorname{trace} \pi(f)$ is a distribution on the manifold G. It characterizes π (up to equivalence).

The universal enveloping algebra $\mathfrak{A}$ of $\mathfrak{g}$ is the associative algebra generated by $\mathfrak{g}$ and subject to the relations $X \cdot Y - Y \cdot X = [X, Y]$, $X, Y \in \mathfrak{g}$, the multiplication on the left being that in $\mathfrak{A}$ and not matrix multiplication. It is impossible to study representations of $\mathfrak{g}$ without considerable information on $\mathfrak{A}$, of which the most important piece is perhaps that its centre $\mathfrak{Z}$ is isomorphic to the algebra of symmetric functions in n variables.

The algebra $\mathfrak{g}$ is also an algebra of vector fields on G and the elements of $\mathfrak{A}$ are differential operators, and if π is irreducible Schur's lemma implies easily that T_π is a simultaneous eigenfunction of all $z \in \mathfrak{Z}$, so that $zT_\pi = \lambda(z)T_\pi$, $\lambda(z) \in \mathbf{C}$.

The distributions T_π are clearly invariant under conjugation and one form of an explicit Plancherel formula would be a relation

$$f(1) = \int_{\prod} T_\pi(f) d\pi = \int_{\prod} \operatorname{trace} \pi(f) d\pi,$$

valid for smooth compactly supported f, where $\prod$ is a collection of irreducible unitary representations and $d\pi$ a measure on it. Thus our problem is to expand the invariant distribution

δ: $f \mapsto f(1)$ as an integral of eigendistributions of $\mathfrak{Z}$. So formulated it is a somewhat unusual problem in spectral theory, and it is to be stressed that the methods used by Harish-Chandra to solve it are on the whole elementary — curvilinear coordinates, Fourier transforms, variation of parameters — although they are heaped up in elaborate logical progressions.

The character is the simplest invariant attached to π but the matrix coefficients are also important. For simplicity take π to be unitary. If δ and ε are two classes of irreducible representations of K and if $\{x_i\}$, $\{y_i\}$ are bases of two subspaces of V, one transforming under K according to δ and one according to ε, the functions $f_{ij}(g) = (\pi(g)x_i, y_j)$ are matrix coefficients of π. Since every element of G may be written as a product $k_1 \exp H k_2$, where H is a real diagonal matrix with descending coefficients $h_1 \geq \ldots \geq h_n$, and since the matrix of spherical functions (f_{ij}) transforms in a prescribed way under K, it is determined by the restriction $(F_{ij}(H)) = (f_{ij}(\exp H))$. Each f_{ij} is an an eigenfunction of each z in $\mathfrak{Z}$ with eigenvalue $\lambda(z)$. Writing these equations in terms of $F = (F_{ij})$ we see that it satisfies a maximally overdetermined system of linear differential equations $\Delta_z F = \lambda(z) F$, for as we observed there are n independent operators in $\mathfrak{Z}$. Such equations have a finite-dimensional space of solutions and behave in many respects like ordinary differential equations.

The problem of obtaining an explicit Plancherel formula can also be formulated as the problem of obtaining (for each ε and δ) the spectral decomposition of the commuting family of operators Δ_z on the domain $h_1 \geq \ldots \geq h_n$, and those who are familiar with the spectral theory of ordinary differential equations will be pleased to see the theory reappear intact in higher dimensions and fascinated by the interplay between the two formulations. They may not be too surprised to discover that the spectral measure $d\pi$ is obtained from the asymptotic behavior of the functions F, although they may not easily follow the group theory that leads to the final explicit formula; but they will probably be startled to see how one passes to the invariant problem to determine the spectrum precisely.

Harish-Chandra discovered quite early the principles which allowed him to do this but he overcame the obstacles to their proof only slowly. The critical notions are those of a Cartan subgroup, of a parabolic subgroup, of an induced and of a square-integrable representation.

For $G = \mathrm{GL}(n, \mathbf{R})$ a typical parabolic subgroup P is obtained from a partition $n = n_1 + \ldots + n_r$ by embedding $M = \mathrm{GL}(n_1, \mathbf{R}) \times \ldots \times \mathrm{GL}(n_r, \mathbf{R})$ in G by diagonal blocks and then multiplying it by the group N of matrices with ones along the diagonal and zeros below the blocks to obtain $P = MN$. A typical Cartan subgroup T of G or of M is obtained by fixing 1×1 and 2×2 blocks along the diagonal and taking matrices which are zero outside the blocks, invertible in the blocks, and in a 2×2 block of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

In the context of reductive groups an induced representation of G is obtained from one of M by extending it to P, making it trivial on N, and then inducing to G in a manner familiar from the theory of finite groups, taking just a little care to ensure that unitary representations remain so upon induction.

The group M is the direct product of its connected centre A and the group M^0 of those matrices whose blocks have determinant ± 1. An irreducible unitary representation σ of M is the product of a character χ of A, which depends on r parameters, and a unitary representation σ^0 of M^0. The representation σ^0 is said to be square-integrable if its matrix coefficients are square-integrable functions on M^0. The square-integrable representations of M^0 clearly form a discrete family or, in the jargon of the subject, series. The first principle is that the representations of G induced from those σ for which σ^0 is square-integrable suffice for the Plancherel formula. The second is that M^0 has square-integrable representations if and only if there are Cartan subgroups T such that $M^0 \cap T$ is compact.

For $G = \mathrm{GL}(n)$ this forces the connected component of M^0 to be a product of several copies of $\mathrm{SL}(2, \mathbf{R})$, a group which illustrates much of the general theory but for which it is not necessary. It is, however, for the symplectic group, which also plays a role in algebraic geometry denied $\mathrm{GL}(n)$. The two facts are linked. We continue nonetheless with $\mathrm{GL}(n, \mathbf{R})$, overlooking the simplifications possible for it.

The central problem now becomes the existence of the discrete series of square-integrable representations. The proof of existence is elaborate, and is tied to uniqueness results which flow from basic but unforeseen properties of the distributions T_π. On the open subset G' of G formed by matrices with distinct eigenvalues we can use the eigenvalues as radial coordinates. The angular coordinates are irrelevant since T_π is invariant. So it is a distribution in the n radial coordinates. The n independent equations which it satisfies force it to be a function F_π, indeed (when the algebra is worked out) a simple elementary function, which becomes infinite as one approaches the singular set $G - G'$. It turns out, although difficult to prove, that F_π is nonetheless locally integrable on G, so that it defines a distribution. The difference $T_\pi - F_\pi$ of the two distributions is thus supported on the singular set. The astonishing fact, which takes a long series of papers to prove and on which the whole theory turns, is that it is zero. So $T_\pi - F_\pi$ is not only a distribution but a true function.

Simple experiments with the δ-function and its derivatives show that whether a differential equation admits singular distributions as solutions depends upon its delicate numerical properties. To deal with T_π and show it is a function one has to exploit not only the differential equation but also the invariance. It is not possible to designate any result in a theory with the elaborate architecture of Harish-Chandra's harmonic analysis as the fundamental one, but the theorem that the character is a function is as important as any, and in addition, so far as I know, an unprecedented result in partial differential equations.

Although it is the individual achievements which are the striking, perhaps even the lasting, part of the theory, it was the goal of an explicit Plancherel formula that shaped it. Harish-Chandra maintained a certain rhythm as he proceeded, which of course slowed as he grew older — a quick burst of announcements, followed by a long series of papers providing the details, and then another burst of announcements, and so on. The announcements are remarkably clear, and can be recommended to those who want to see the theory being formed but not necessarily to master it.

The first papers on physics, some written in collaboration with Bhabha, treat from a classical standpoint interactions between particles and fields, and are followed by a series of papers in which the form taken by the most general relativistic wave equation for a simple particle is discussed. Traces of these papers can be found in the physics literature, but most readers of the *Collected Papers* will look at them, and are urged to do so, to see the young algebraist developing his skills. There are two papers written under the influence of Dirac, one on the irreducible representations of the Lorentz group, whose part in preparing him for his later career is evident, and another, very short, little more than an exercise, a quantum-mechanical treatment of the motion of an electron in the field of a monopole, in which it is touching to see the classical techniques, spherical coordinates and the method of Frobenius, which are applied so relentlessly and carried so far in the later papers, being exploited by him for the first time.

In Princeton, Harish-Chandra took courses from Artin and Chevalley. The influence of Chevalley is manifest in his first mathematical papers, on finite-dimensional representations and on the structure of the universal enveloping algebra; that of Artin, because it did not correspond to a natural bent, was more subtle. Harish-Chandra's encounter with class-field theory awakened in him ambitions which he was first able to satisfy toward 1960 when, benefiting from his experience with representation theory, he proved a general finiteness theorem for automorphic forms and, in collaboration with A. Borel, extended the classical reduction theory to arbitrary groups. These are important papers, but the influence of Harish-Chandra on the theory of automorphic forms goes far beyond them. Automorphic forms can be viewed in several contexts, and one of them is now representation theory, which has blown away a lot of must.

The relation between representation theory and automorphic forms is indeed very close. One notion from automorphic forms of which Harish-Chandra was very fond was that of a cusp form. He expressed the results of his last papers on harmonic analysis on real groups in terms of it, and exploited it brilliantly in a brief but influential paper on representations of finite groups of Lie type.

It dominated his thinking on p-adic groups, a topic to which many papers in the final volume are devoted. There are sound mathematical grounds for studying the representations not merely of $\mathrm{GL}(n, \mathbf{R})$ or $\mathrm{GL}(n, \mathbf{C})$ but of $\mathrm{GL}(n, F)$ where F is any local field, or of the F-valued points on any reductive group. These representations are known to reflect the structure of the set of Galois extensions of F, and for that reason the theory over a nonarchimedean field is quite different from that over the real or complex field, and is still incomplete. What appears in these last papers is that from one point of view it is possible to go a long way in the harmonic analysis on p-adic groups — as far as a concrete, if no longer completely explicit, Plancherel formula — without any consideration whatsoever of the arithmetic of the field, in which Harish-Chandra was perhaps not much interested, or of the structure of the discrete series.

Often the publication of collected or selected works is no more than a tribute paid to the achievement and influence of a friend, teacher or colleague, springing from affection and respect but with no claim on time. However, occasionally they are to be a monument, to endure and remind coming generations that our age did not lack all greatness. So it should be with Harish-Chandra. The editor, V. S. Varadarajan, has understood this, has overcome difficulties caused by Harish-Chandra's final illness and untimely death, and has prefaced the papers with an account of his achievement, to which all readers will turn with profit, and a moving homage to the conviction that sustained him.

It is supplemented by comments by Nolan Wallach on individual papers and by an essay on Harish-Chandra's work on p-adic groups by Roger Howe which perhaps errs on the side of modesty. Some critical ideas are due to Howe himself.

The volumes also contain three important but previously unpublished papers and are accompanied by expressive, and revealing, photographs. It is a pity that those papers which appeared in camera-ready form were not set in type. They detract considerably from the appearance of the later volumes.