H hyperbolic plane, upper half plane model $z = x + iy$, $y > 0$. G group of motions, elements g, $gz = \frac{az+b}{cz+d}$, matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ real, $\det = 1$.

Γ discrete subgroup of G, elements γ and such that quotient $\Gamma \backslash G$ has finite measure. For simplicity assume quotient compact (compact fundamental domain D_Γ in H), and no elliptic elements. Then consideration of eigenvalue problem

$$(1) \quad y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f + \lambda f = 0,$$

where f is a function of x and y or of z such that

$$f(\gamma z) = f(z)$$

for all γ in Γ leads to the trace formula

$$(2) \quad \sum_j h(r_j) = \frac{A(D)}{4\pi} \int_{-\infty}^{\infty} r \frac{e^{\pi r} - e^{-\pi r}}{e^{\pi r} + e^{-\pi r}} h(r)\, dr$$

$$+ \sum_{\{\gamma\}_\Gamma^*} \sum_{\nu=1}^{\infty} \frac{\log \rho_\gamma}{\rho_\gamma^{\frac{\nu}{2}} - \rho_\gamma^{-\frac{\nu}{2}}} g(\nu \log \rho),$$

here the r_j are derived from the eigenvalues λ_j of (1) by $\frac{1}{4}+r_j^2=\lambda_j$. The summation $\{\gamma\}_\Gamma^*$ extends over one representative of each "primitive" hyperbolic conjugacy class in Γ, ρ_γ denotes the square of the largest eigenvalue of the matrix γ. $A(\mathcal{D})$ is the (hyperbolic or invariant) area of $\mathcal{D}_\Gamma$, $h(r)$ is an even function holomorphic for $|\Im(r)|<\frac{1}{2}+\varepsilon$ for some $\varepsilon>0$ and such that

$$h(r)=O\left(\frac{1}{|r|^{2+\varepsilon}}\right), \text{ while}$$

$$g(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iur}\, h(r)\,dr.$$

(2) by its similarity with the well known explicit formulas connecting a series extended over the zeros of the Riemann zeta-function with another series over the primes, suggests defining

$$(3)\quad Z(s,\Gamma)=\prod_{\{\gamma\}_\Gamma^*}\ \prod_{\nu=0}^{\infty}\left(1-\rho_\gamma^{-s-\nu}\right),$$

Convergent for $\Re s>1$

Using (1), can prove Z is integral function of order 2. Functional equation

$$(4)\quad Z(s,\Gamma)\exp\left(-A(\mathcal{D})\int_0^{s-\frac{1}{2}} v \, tg\,\pi v \, dv\right) = Z(1-s,\Gamma)$$

Z has "trivial" zeros at at neg. integers (implied by 4), non-trivial at . points $\frac{1}{2} \pm i r_j$.

Form of weight l, l integer

$$f(z) = \varepsilon_\gamma^{l}(z)\, f(\gamma z) \text{ for } \gamma \in \Gamma$$

where $\varepsilon_\gamma(z) = \dfrac{cz+d}{c\bar{z}+d}$; $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Operator

$$(5)\quad \Delta_l = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2il\, y \frac{\partial}{\partial x}$$

carries form of weight l into form of weight l.

Operators

$$2i\, y^{1-l} \frac{d}{dz} y^{l} \text{ and } -2i\, y^{1+l} \frac{d}{d\bar{z}} y^{-l}$$

carries form of weight l into form of weight $l+1$ and $l-1$ respectively

N.B! (though a finite number of forms

may be annihilated in the process).

Integral operator

$$(6)\quad \int_H \frac{(4\eta)^{\ell}}{\left(\frac{z-\bar{\xi}}{2i}\right)^{2\ell}}\, k\left(\frac{|z-\xi|^2}{4\eta}\right) f(\xi)\, d\omega_\xi$$

for $\ell \geq 0$ carries form of weight ℓ into form of weight ℓ. For $\ell < 0$ we replace it by

$$(6')\quad \int_H \frac{(4\eta)^{-\ell}}{\left(\frac{\xi-\bar{z}}{2i}\right)^{-2\ell}}\, k\left(\frac{|z-\xi|^2}{4\eta}\right) f(\xi)\, d\omega_\xi .$$

A traceformula arises if we consider the restriction of (6) or (6') to the space of forms of weight ℓ, in the same way as the case $\ell=0$ led to (2), k giving rise to the h and g. Here Δ_ℓ plays the role of the $\Delta_0 = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ for $\ell = 0$.

For different ℓ, we get essentially the same formula since with a finite number of exceptions we have the same eigenvalues.

For $|\ell| > 1$ we may use (5) or (5')

with $k \equiv 1$, then the trace formula collapses to a finite formula which counts only the multiplicity of the eigenvalue $|l|(1-|l|)$ for $l > 1$ the forms for which $y^{-l} f(z)$ is an holomorphic function of z, and for $l < -1$, those for which $y^{l} f(z)$ is an holomorphic function of $\bar{z}$. For $l > 1$ this is equivalent to counting the number of holomorphic analytic automorphic forms of weight l for Γ.

II Groups on products of hyperbolic planes.

For simplicity we shall first consider the case of product space of two hyperbolic planes, $H_1 \times H_2$, our G is now $G_1 \times G_2$, elements $g = (g_1, g_2)$ and points in space represented by $z = (z_1, z_2)$. We now assume we have a discrete subgroup of G, Γ such that the measure of the quotient $\Gamma \backslash G$ is finite; for simplicity again we assume the quotient is compact (or fundamental domain D_Γ compact). Elements of Γ denoted by $\gamma = (\gamma_1, \gamma_2)$.

We assume that Γ is irreducible, that is its projection on G_1 or G_2 is not discrete (it does not contain any subgroups of finite index which is a direct product $\Gamma_1 \times \Gamma_2$ where Γ_1 and Γ_2 are discrete groups acting on H_1 and H_2 respectively). The projection of Γ on G_1 (or G_2) is then everywhere dense, also if one component of γ is the identity (e), the other component must also be.

For $l = (l_1, l_2)$ where l_1 and l_2 are integers we may consider forms of weight l, that is: functions that transform by the rule

(7) $f(z) = \varepsilon_{\gamma_1}^{l_1}(z_1)\, \varepsilon_{\gamma_2}^{l_2}(z_2)\, f(\gamma z)$,

and which are eigenfunctions of the operators:

Δ_{ℓ_1, z_1} and Δ_{ℓ_2, z_2}, defined by (5), where we have added subscripts z_1 and z_2 to indicate which variable they act on.

We can then develop a general trace formula, but here we consider instead the case $\ell = (\ell, 0)$ where we have dropped the subscript on the first component. We consider first the integral operator

$$(8) \quad \int_{H_1} \int_{H_2} \frac{(y_1 \eta_1)^\ell}{\left(\frac{z_1 - \bar{\xi}_1}{2i}\right)^{2\ell}} k\left(\frac{|z_2 - \xi_2|^2}{y_2 \eta_2}\right) f(\xi) \, d\omega_\xi$$

if ℓ is positive, for ℓ negative we replace

$$\frac{(y_1 \eta_1)^\ell}{\left(\frac{z_1 - \bar{\xi}_1}{2i}\right)^{2\ell}} \quad \text{by} \quad \frac{(y_1 \eta_1)^{-\ell}}{\left(\frac{\xi_1 - \bar{z}_1}{2i}\right)^{-2\ell}} .$$

The restriction of this integral operator to forms of weight $(\ell, 0)$ for Γ will have only eigenfunctions corresponding to the eigenvalue $|\ell|(1-|\ell|)$ with respect to Δ_{ℓ, z_1}, and with eigenvalues $\frac{1}{4}+r_j^2$ with respect to Δ_{0, z_2} (these r_j will depend on ℓ of course). If we for simplicity assume that Γ has no element of finite order (or both components elliptic) and denote the volume of $\mathcal{D}_\Gamma$ by $V(\mathcal{D})$ (in this case $\frac{V(\mathcal{D})}{(4\pi)^2}$ is an integer), we get when we compute the trace of (8) in two ways that

$$(9)\quad \frac{4\pi}{2|\ell|-1}\sum h(r_j) = \frac{V(\mathcal{D})}{4\pi}\int_{-\infty}^{\infty} r\,\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}\,h(r)\,dr$$

$$+\ \frac{4\pi}{2|\ell|-1}\sum_{\{\gamma\}_\Gamma^*}\sum_{\nu=1}^{\infty}\frac{\varepsilon_\gamma^{\ell\nu}}{1-\varepsilon_\gamma^{\nu\,\mathrm{sgn}\,\ell}}\,\frac{\log\rho_\gamma}{\rho_\gamma^{\frac{\nu}{2}}-\rho_\gamma^{-\frac{\nu}{2}}}\,g(\nu\log\rho_\gamma)$$

Here h and g are functions related as in (2) and satisfying the same conditions. $\{\gamma\}_\Gamma^*$ here means that summation is over one representative of each primitive conjugacy class in Γ where the first component is elliptic and the second hyperbolic. $\varepsilon_\gamma = e^{i\varphi}$ where φ is the rotation angle implied by γ_1, and $\rho_\gamma = \rho_{\gamma_2}$. (9) can be rewritten

$$(9')\quad \sum h(r_j) = \frac{(2|l|-1)V(\mathfrak{D})}{(4\pi)^2}\int_{-\infty}^{\infty} r\,\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}\,h(r)\,dr$$

$$+\sum_{\{\gamma\}_p^*}\sum_{\nu=1}^{\infty}\frac{\varepsilon_\gamma^{l\nu}}{1-\varepsilon^{\nu\,\mathrm{sgn}\,l}}\,\frac{\log\rho_\gamma}{\rho_\gamma^{\frac{\nu}{2}}-\rho_\gamma^{-\frac{\nu}{2}}}\,g(\nu\log\rho_\gamma).$$

The structure of (9') is very similar to (2) only the factor $\dfrac{\varepsilon_\gamma^{l\nu}}{1-\varepsilon_\gamma^{\nu\,\mathrm{sgn}\,l}}$

seems inconvenient. However, if we for $l>1$ combine (9') for l and $-l$ and observe

$$\frac{\varepsilon^{l\nu}}{1-\varepsilon^{\nu}}+\frac{\varepsilon^{-l\nu}}{1-\varepsilon^{-\nu}} = -\,\frac{\varepsilon^{l\nu}-\varepsilon^{-(l-1)\nu}}{\varepsilon^{\nu}-1} =$$

$$= -\sum_{|i|<l}\varepsilon^{\nu i},$$

we see that we are led to consider the expression

$$(10)\quad Z_\ell(s,\Gamma)=\prod_{\{\gamma_p\}^*}\ \prod_{\substack{\nu=0\\ |i|<\ell}}^{\infty}\left(1-\varepsilon_\gamma^{i}\,\rho_\gamma^{-s-\nu}\right)^{-1},$$

which can be shown to be convergent for $\mathcal{R}\, s>1$. From (9′) we can deduce that it is an integral function of order 2 with the functional equation

$$Z_\ell(1-s,\Gamma)=Z_\ell(s,\Gamma)\exp\left(-(2\ell-1)\frac{V(\mathcal{D})}{2\pi}\int^{s-\frac{1}{2}} v\,\mathrm{tg}\,\pi v\,dv\right),$$

again with trivial zeros at the negative integers and non-trivial at the points $s=\frac{1}{2}\pm i r_j$.

If we instead had a group Γ acting on a product of n hyperbolic planes $H_1\times H_2\times\cdots\times H_n$, and Γ is irreducible and again with compact fundamental domain $\mathcal{D}_\Gamma$, and no elements of finite order, we can in a similar manner consider forms of weight

$\ell = (\ell_1, \ell_2, \ldots, \ell_{n-1}, 0)$, where

$|\ell_i| > 1$ for $1 \le i < n$, and so that

for $z = (z_1, z_2, \ldots, z_n)$ our

functions for $1 \le i < n$ are eigen-

functions of the operator

Δ_{ℓ_i, z_i} with eigenvalue $|\ell_i|(1-|\ell_i|)$.

We are then led to form the function

$$(10') \quad Z_{\ell_1, \ldots, \ell_{n-1}}(s, \Gamma) =$$

$$= \prod_{\{\gamma\}^*_\Gamma} \prod_{\substack{\nu \ge 0 \\ |i_j| < |\ell_j|}} \left(1 - \varepsilon_{\gamma_1}^{i_1} \cdots \varepsilon_{\gamma_{n-1}}^{i_{n-1}} \rho_\gamma^{-s-\nu}\right)^{(-1)^{n-1}},$$

which again has similar properties.

A problem arises when we try to consider the case where some of the $|\ell_j| = 1$. For $n = 2$ this would formally lead to

$$(11) \quad Z_1(s, \Gamma) = \prod_{\{\gamma\}^*_\Gamma} \prod_{\nu \ge 0} \left(1 - \rho_\gamma^{-s-\nu}\right)^{-1},$$

which must have a singularity on the real line for some s, since we quite easily can establish that the product converges $Rs > 1$, the singularity would be in the interval $0 \le s \le 1$. Since for the weight $\ell = (\pm 1, 0)$, the integral operator (8) does not converge if we take absolute values of the kernel, and so our earlier manipulations could not be justified in this case, we have to use other more complicated means.

We introduce a factor, with $\alpha > 0$,

$$\frac{(y_1 \eta_1)^\alpha}{\left|\frac{z_1 - \bar{\xi}_1}{2i}\right|^{2\alpha}}$$

in order to obtain convergence, so for the case $(1, 0)$ we consider the operator (instead of (8)),

$$(12) \quad \int_{H_1}\int_{H_2} \frac{y_1 \eta_1}{\left(\frac{z_1 - \bar{\xi}_1}{2i}\right)^2} \frac{(y_1 \eta_1)^\alpha}{\left|\frac{z_1 - \bar{\xi}_1}{2i}\right|^{2\alpha}} k\left(\frac{|z_2 - \xi_2|^2}{y_2 \eta_2}\right) f(\xi)\, d\omega_\xi$$

on forms of weight $(1, 0)$ for Γ, and develop the trace formula for this operator (which of course will contain the full two parameter spectrum for weight $(1, 0)$), as the case $\alpha = 0$ would formally correspond to (8) for $l=1$, one could try to make $\alpha \to 0$ and see what happens, it is possible to proceed in this way but very messy (and even more so for $n>2$ if more than one $l_i = \pm 1$). Instead we consider also the weight $(0,0)$ and the operator with the kernel

$$(12') \quad \frac{\alpha}{1+\alpha} \frac{(y_1 \eta_1)^{1+\alpha}}{\left|\frac{z_1-\xi_1}{2i}\right|^{2+2\alpha}} k\left(\frac{|z_2-\xi_2|^2}{y_2\eta_2}\right),$$

and develop the resulting trace-formula for this kernel and forms of weight $(0,0)$ for Γ. The two spectra for weights $(1,0)$ and $(0,0)$ are largely identical, since we can pass from an eigenform for $(1,0)$ to one for $(0,0)$ by the operator

$$-2i\, y_1^2 \frac{d}{d\bar{z}_1} y_1^{-1}$$

and from an eigenform for $(0,0)$ to one for $(1,0)$ by the operator $2iy\frac{d}{dz}$. In the first case we see that we annihilate just those forms for which $\Delta_{1,z_1} f(z) = 0$ and no others. In the second case we only annihilate the constant eigenfunction, since this is the only eigenfunction which is holomorphic in $\overline{Z}$, (as easily follows using the maximum principle for Z, and the properties of an irreducible group Γ). So when we subtract the trace formula for the kernel (12′) from that for the kernel (12), and divide out a common factor depending only on α which is found in all remaining terms, we are on the left-hand side left with:

$$\sum_j h(r_j) - h\left(\frac{i}{2}\right)$$

and on the right-hand side

$$\frac{V(\mathcal{D})}{(4\pi)^2}\int_{-\infty}^{\infty} r\,\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}\,h(r)\,dr\;+$$

$$+\sum_{\{\gamma\}^*_\Gamma}\sum_{\nu=1}^{\infty}\frac{\varepsilon_\gamma^{\nu}}{1-\varepsilon_\gamma^{\nu}}\;\frac{\log\rho_\gamma}{\rho_\gamma^{\frac{\nu}{2}}-\rho_\gamma^{-\frac{\nu}{2}}}\;g(\nu\log\rho_\gamma).$$

Thus instead of (9') we get for $\ell = (1,0)$

$$(13) \quad \sum_j h(r_j) \quad - h\left(\tfrac{i}{2}\right) =$$

$$= \frac{v(\mathcal{D})}{(4\pi)^2} \int_{-\infty}^{\infty} r \frac{e^{\pi r} - e^{-\pi r}}{e^{\pi r} + e^{-\pi r}} h(r)\, dr +$$

$$+ \sum_{\{\gamma\}_p^*} \sum_{\nu=1}^{\infty} \frac{\varepsilon_\gamma^\nu}{1-\varepsilon_\gamma^\nu} \frac{\log \rho_\gamma}{\rho_\gamma^{\frac{\nu}{2}} - \rho_\gamma^{-\frac{\nu}{2}}}\, g(\nu \log \rho_\gamma).$$

Combining again this with the (identical) formula for $\ell = (-1,0)$. (or combining terms coming from γ and γ^{-1}), we can again simplify the terms in the last sum since

$$\frac{\varepsilon^\nu}{1-\varepsilon^\nu} + \frac{\varepsilon^{-\nu}}{1-\varepsilon^{-\nu}} = -1.$$

Thus the function $Z_1(s,\Gamma)$ defined in (11) can be continued analytically in the same way as the $Z_\ell(s,\Gamma)$ for $\ell > 1$, however the term $-h\left(\frac{i}{2}\right)$ creates a pole at $s=1$ in the same way as $h(r_j)$ produces a zero at $\frac{1}{2} \pm i r_j$. The functional equation is as that we found for $\ell > 1$, if we only replace l by 1.

For general n, we may use the same differencing argument for all the $l_i = \pm 1$, and only when all $|l_i| = 1$ for $1 \leq i < n$, does the new term $(-1)^{n-1} h(\frac{i}{2})$ enter on the left-hand side. For n even this produces a pole at $s=1$, for n odd it produces a zero. As to the points $\frac{1}{2} \pm ir_j$ it should be noted that for a weight $(l_1, \ldots, l_{n-1}, 0)$ the r_j that enter will depend on the l_i as well as on Γ, and that when we change the sign of some of the l_i in general the r_j will change, only if we change the sign of all do the r_j stay the same, since there are in all 2^{n-1} sign combinations, we get that in general zeros $\frac{1}{2} \pm ir_j$ in our Z come from 2^{n-2} different eigenvalue problems.

It is of interest to compare the formulas arising in this situation with those arising when we consider a group Γ acting on a single hyperbolic plane and where we have $(n-1)$ representations of Γ by unitary matrices (two by two) and let these representations each act on one of $(n-1)$ spheres and consider functions of a variable which consists of a point z_1 in the the hyperbolic plane and one point on each of the spheres and functions invariant under this action and eigenfunctions of the operator Δ_{0,z_1} and are spherical functions of level l_i on the sphere with index i. The formulas are quite analogous only the factor $(-1)^{n-1}$ that occurred earlier does not enter.

One may of course use these $Z_\ell(s,\Gamma)$ to investigate the distribution of the ρ_γ occurring, as well as the distribution of the rotation angles in the elliptic components (the arguments of the $\varepsilon_{\gamma i}$)

However, this can just as well be done directly from the trace formulas without any $Z_\ell(s,\Gamma)$.

If $\mathbb{Z}_n$ in the earlier development is replaced by a copy of the hyperbolic 3-space, one can repeat the analysis above, among the slight changes: the $Z_\ell(s,\Gamma)$ will now be functions of order 3, the functional equation is simpler, and there are no trivial zeros.

Finally, if we permit elements of finite order in Γ, the formulas get rather more complicated, but the main features remain the same. Permitting Γ with non-compact fundamental domain (but with finite volume) brings in more complication, particularly in the handling of the case when some or all $|\ell_i| = 1$, the functional equation gets further modified, but basically the nature of the $Z_\ell(s,\Gamma)$ remain the same.