2-f. ans with groups & hybrid trace formulas.

1.

Hybrid T-formulas. . . .

While T-formula in general is transcendental in nature asserting the equality of ~~two sides~~ when both sides are infinite series over ~~functions~~ summed over, ~~there~~ it is well known that there are also cases where it reduces to a finite identity counting the multiplicity of a single point in the spectrum, For $SL(2,\mathbb{R})$, say or of regular analytic forms of a given dimension, or for general complex bounded symmetric domains, by using powers of $\left(\frac{k(z,\xi)}{\sqrt{k(z,z)\,k(\xi,\xi)}}\right)^{l}$ with l positive and sufficiently large to insure admissibility.

by suitable choice of point pair function as eigen function:

The object of this lecture is to illustrate some instances, where we consider groups acting on products of symmetric spaces, and use a point pair representation which is ~~an~~ eigen function for some factors and a general point pair invariant for others, we get then what I will call a hybrid trace formula.

First some remarks on groups Γ acting on $S_1 \times S_2$; call reducible if Γ is commensurable (that is has a common subgroup of finite index with) a direct product $\Gamma_1 \times \Gamma_2$ where Γ_1 acts on S_1, Γ_2 on S_2 otherwise "irreducible". The irreducible groups ~~deny~~ any factorisation of S are in a sense the only that properly belong to that space, and we shall restrict ourselves to those. an irreducible group Γ has following 2 properties (1) its projection onto any factor ~~group~~ G_i of G acting on a factor of S is everywhere dense in G_i (2) if an element γ of Γ projects on the identity element of any factor of G, γ is the identity element.

I shall ~~begin by~~ consider the case of a product of two copies of hyperbolic plane or one copy of hyperbolic plane, one of hyperbolic 3-space. In most cases no irreducible group (with finite volume of D) for $S_1 \times S_2$ (compatibility) of spaces or groups.

Assume for simplicity that Γ acting on $H_2 \times H_2$ has no element with fixpoints (except identity element) could otherwise go to a subgroup of finite index which has that property and also that $\mathcal{D}$ is compact $V(\Gamma \backslash H_2 \times H_2^{\#}) < \infty$ in that case $\frac{V(\mathcal{D})}{(4\pi)^2}$ is an integer.

in $H_2 \times H_3$ we have no similar statement about volume, but as we shall see it is not needed there

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We consider for l integral ≥ 1, functions in $z = (z_1, z_2)$ which under γ transform in the way that if $\gamma z = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2}\right)$

$$F(\gamma z) = \left(\frac{c_1 z_1 + d_1}{c_1 \bar{z}_1 + d_1}\right)^l F(z)$$

and furthermore are of the form that

$\frac{\partial y_1^{-l} F(z)}{\partial \bar{z}_1} = 0$ or $F(z) = y_1^l f(z)$ where

$f(z)$ ~~analytic~~ holomorphic in z_1

Forming the point pair function z, ξ

$$\left(\frac{y_1 \eta_1}{\left(\frac{z_1 - \bar{\xi}_1}{2i}\right)^2}\right)^{l} k\left(\frac{|z_2 - \xi_2|^2}{y_2 \eta_2}\right)$$

if we assume $l > 1$; k a function tending reasonably to zero, say $k(t) = O\left(\frac{1}{t^{1+\varepsilon}}\right)$ as $t \to \infty$, this is admissible and we can at once write down the resulting trace formula for Γ, since the various integrals that occur split into direct products of the kind that occur for a single hyperbolic plane.

Denoting by r_j the eigenvalues for which

$$y_2^2 \Delta_2 F(z) + \left(\frac{1}{4} + r_j^2\right) F(z) = 0 \quad \left(\begin{matrix} r_j \\ \text{depend} \\ \text{on } l \end{matrix}\right)$$

and let $h(r)$ and $g(u)$ be the functions that derive from k in the way given in my paper in the Indian journal, we shall proceed to write down the T-formula

(1) only elements $\neq e$ whose first component is elliptic give a contribution. Let us denote by $\{\gamma\}_\Gamma^*$ equivalence classes in Γ for which 1st component is elliptic, then second is hyperbolic

call them primitive if not positive power with exp > 1 of other element in group; ~~γ is then primitive class~~ ~~described by rotation angle φ~~, and minimal ~~distance ρ that the components move~~ the two copies of H. put $\varepsilon = e^{i\varphi}$ ($\frac{\varphi}{2\pi}$ irrational)

$$\frac{4\pi}{2l-1} \sum_j h(r_j) = \frac{V(\mathcal{D})}{2\pi} \int_{-\infty}^{\infty} r\,h(r)\, \frac{e^{\pi r} - e^{-\pi r}}{e^{\pi r} + e^{-\pi r}}\, dr$$

$$+ \frac{4\pi}{2l-1} \sum_{\{\gamma\}_p^*} \sum_{m=1}^{\infty} \frac{\varepsilon^{lm}}{1-\varepsilon^m} \frac{\rho}{e^{\frac{m\rho}{2}} - e^{-\frac{m\rho}{2}}}\, g(m\rho)$$

or

$$\sum_j h(r_j) = \frac{V(\mathcal{D})}{8\pi^2}(2l-1) \int_{-\infty}^{\infty} r\,h(r)\, \frac{e^{\pi r} - e^{-\pi r}}{e^{\pi r} + e^{-\pi r}}\, dr$$

$$+ \sum_{\substack{\{\gamma\}_p^* \\ m \geq 1}} \frac{\varepsilon^{lm}}{1-\varepsilon^m} \frac{\rho}{e^{\frac{m\rho}{2}} - e^{-\frac{m\rho}{2}}}\, g(m\rho)$$

Structurally quite similar to trace formula in case of single hyperbolic plane, only factor $\frac{\varepsilon^{lm}}{1-\varepsilon^m}$ ~~is somewhat inconvenient. We may~~ however combine term for class γ with class γ^{-1} that ~~they have same ρ, but φ changes sign~~ ~~so $\varepsilon \to \varepsilon^{-1}$~~

and we have

$$\frac{\varepsilon^{lm}}{1-\varepsilon^{m}} + \frac{\varepsilon^{-lm}}{1-\varepsilon^{-m}} = -\frac{\varepsilon^{lm} - \varepsilon^{-(l-1)m}}{\varepsilon^{m}-1}$$

$$= -\sum_{|i|<l} \varepsilon^{mi}$$

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form $Z_l(s, P) = \prod_{\substack{\nu \geq 0 \\ |i|<l}} \left(1-\varepsilon^{i} e^{-(s+\nu)\rho}\right)^{-1}$

This dirichet series has analytic continuation & functional equation

$$Z_l(1-s) = Z_l(s) \exp\left[-(2l-1)\frac{V(D)}{4\pi}\int_0^{s-\frac{1}{2}} v \operatorname{tg} \pi v \, dv\right]$$

zeros in critical strip at $s = \frac{1}{2} + ir_j$ and "trivial" zeros at ~~negative~~ integers $-n$ for $n \geq 0$ (or poles), so far we considered $l > 1$

For $l = 1$ our series Kernel does not converge absolutely; Formally ~~if~~ we put $l = 1$ in $Z_l(s, P)$ we get $Z_1(s, P) = \prod_{\nu \geq 0} \left(1-e^{-(s+\nu)\rho}\right)^{-1}$ which clearly must have singularity somewhere on real axis, so something has to change if we try to consider this case too.

We use a limit process

and we see that our statements about $Z_\rho(s)$ remain true in case $\ell = 1$ except that $Z_1(s)$ has a pole of order 1 at $s = 1$

From Z_1 can determine distribution of the ρ (prime geodesic theorem)

not so surprising in after thought

Perhaps more interesting and unexpected are the results of distribution of the φ for $\rho \le x$ find not equidistributed but has distribution function $\frac{1-\cos\varphi}{2\pi}$.

If second factor of our space was H_3 there is some simple modification.
factor in functional equation becomes of form $e^{a(s-1)^3 + b(s-1)}$; form of Zeta function modified in terms with $\nu > 0$

Noncompact case not much changed no continuous spectrum can enter here though parabolic terms do contribute to trace formula and form of functional equation.

NB: some factors act on lower half planes.

must add copies where factors are

Product of n hyperbolic planes

$$Z_{\ell_1 \cdots \ell_{n-1}}(s, \rho) = \prod_{\substack{\{\gamma\}^*_\rho \\ |i_1| < \ell_1, \cdots |i_{n-1}| < \ell_{n-1} \\ \nu \geq 0}} \left(1 - \varepsilon_1^{i_1} \cdots \varepsilon_{n-1}^{i_{n-1}} e^{-\rho(s+\nu)}\right)^{(-1)^n}$$

non integral ℓ_i; half integer quarter integer

$Z_{1,1,1}(s, \rho)$ has pole at $s = 1$ for n even, zero for n odd, Similar for case last factor is H_3.

Other cases, complex ~~spaces~~ symmetric domains, expect Z-function only in case of factors of rank 1, say products of unit balls $|z_1|^2+|z_2|^2+\cdots|z_m|^2<1$ for same m.

problem. $z_i \to \varepsilon_i z_i$

$$\frac{(\varepsilon_1\cdots\varepsilon_m)^{\ell}}{(1-\varepsilon_1)\cdots(1-\varepsilon_m)} + \frac{(\bar\varepsilon_1\cdots\bar\varepsilon_m)^{\ell}}{(1-\bar\varepsilon_1)\cdots(1-\bar\varepsilon_m)}$$ does

not simplify, ~~probably~~ perhaps this can be attained by bringing in representations of rotation group which are not one dimensional, or otherwise expressed consider differential forms of various order. I have not tried to look into this.

One can of course also look at hybrid trace-formulas where the last factor is of higher rank (or is a product of several factors of rank 1) in this case no connection with a zeta function would be expected, but the resulting formulas are still of interest and do throw additional light on the distribution of elements in such groups.