First some remarks on groups Γ acting on ~~product~~ product space S ~~$S_1 \times S_2$~~ call reducible if Γ commensurable with (has common subgroup of finite index with a direct product $\Gamma_1 \times \Gamma_2$ where Γ_1 acts on S_1; Γ_2 on S_2 and $S = S_1 \times S_2$ otherwise "irreducible". The irreducible groups on S are in a sense the only groups the properly belong to that space. Consider only those Irreducible; (1) projection of Γ on any factor G_1 of G is everywhere dense in G_1 (2) if γ in Γ projects in identity on any factor G_1 of G then $\gamma = e$.

I shall for simplicity first consider product of two copies of $Sl(2,R)$ (or Space product of two hyperbolic planes) not much diff if one factor is $Sl(2,C)$ or hyp-3-space). Assume for simplicity group Γ acting on H^2 has no element ~~with~~ of finite order, (otherwise could go to subgroup of finite index). Volume D finite and for time being assume D compact. It is easy to show that in this case $\frac{V(D)}{(4\pi)^2} = \text{integer}$ later =

Consider for integer $l \geq 1$ functions in $z = (z_1, z_2)$ which under γ transform in way

$$\gamma z = \left(\frac{a_1 z + b_1}{c_1 z + d_1}, \frac{a_2 z + b_2}{c_2 z + d_2}\right)$$

$$F(\gamma z) = \left(\frac{c_1 z_1 + d_1}{c_1 \bar{z}_1 + d_1}\right)^l F(z), \text{ and}$$

which furthermore are such that

$$\frac{\partial\, y_1^{-l} F(z)}{\partial \bar{z}_1} = 0 \;\Big|\; y_1^{-l} F(z) \text{ analytic in } z_1 \text{ holomorphic}$$

Form the kernel

$$\left(\frac{y_1\eta_1}{\left(\frac{z_1-\bar{\xi}_1}{2i}\right)^2}\right)^{\ell} k\left(\frac{|z_2-\xi_2|^2}{y_2\eta_2}\right).$$

where k is a function tending reasonably to zero say as $t\to\infty$, $k(t)=O\left(\frac{1}{(t)^{1+\varepsilon}}\right)$

This for $\ell>1$ is an admissible kernel and we can at once write down the resulting trace formula for Γ, since the various integrals that occur split into direct products of two integral of the kind that occur for a single copy of H.

Denoting by r_j the values for which

$$y_2^2\,\Delta_2\,F(z)+\left(\tfrac{1}{4}+r_j^2\right)F(z)=0$$

has a (over D) square integrable solution (r_j depend on ℓ)

let $h(r)$ and $g(u)$ be the functions which derive from k in the way stated in my lectures here in 1956, we can now write down T. formula

(1) only elements $\neq e$ which contribute are those where first component of γ is elliptic, second hyperbolic.

Denote the primitive equivalence classes by $\{\gamma\}_p^*$ call them primitive if not positive power with exp >1 of other element in group

prim. class described by two parameters
$\varepsilon = e^{i\varphi}$ rotation angle φ; $\left(\frac{\varphi}{2\pi}\text{ rat.}\right)$ and
the minimal distance ρ that the
element moves a point in second copy of H.
get

$$\frac{4\pi}{2l-1}\sum_j h(r_j) = \frac{V(D)}{2\pi}\int_{-\infty}^{\infty} h(r)\, r\, \frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}\,dr$$

$$+\frac{4\pi}{2l-1}\sum_{\{\gamma\}_p^x}\sum_{m=1}^{\infty}\frac{\varepsilon^{lm}}{1-\varepsilon^m}\frac{\rho}{e^{\frac{m\rho}{2}}-e^{-\frac{m\rho}{2}}}\,g(m\rho)$$

or
$$\sum h(r_j) = (2l-1)\frac{V(D)}{8\pi^2}\int_{-\infty}^{\infty} h(r)\, r \tgh \pi r\, dr$$

$$+\sum_{\{\gamma\}_p^*}\sum_{m=1}^{\infty}\frac{\varepsilon^{lm}}{1-\varepsilon^m}\frac{\rho}{e^{\frac{m\rho}{2}}-e^{-\frac{m\rho}{2}}}\,g(m\rho)$$

Structure quite similar to formula for
a single hyp. plane only factor $\frac{\varepsilon^{lm}}{1-\varepsilon^m}$
seems inconvenient. Combine γ with γ^{-1}
same ρ but conjugate ε's.

and
$$\frac{\varepsilon^{lm}}{1-\varepsilon^m}+\frac{\varepsilon^{-lm}}{1-\varepsilon^{-m}} = -\frac{\varepsilon^{lm}-\varepsilon^{-(l-1)m}}{\varepsilon^m-1}$$

$$= -\sum_{|i|<l}\varepsilon^{mi}$$

Form
$$Z_l(s,P) = \prod_{\substack{\nu\geq 0\\ |i|<l}}\left(1-\varepsilon^i e^{-(s+\nu)\rho}\right)^{-1}$$

has analytic cont. is integral function
with functional equ.

$$Z_\ell(1-s) = Z_\ell(s)\exp\left[-(2\ell-1)\frac{V(D)}{4\pi}\int_0^{s-\frac{1}{2}} v \operatorname{tg}\pi v\,dv\right]$$

zeros in critical strip at $\frac{1}{2}+ir_j$ and trivial zeros at negative integers $-n$ no poles.

For $\ell=1$; kernel is not admissible not in L_1. Formally if we put $\ell=1$ above we get

$$Z_1(s) = \prod_{\nu\geq 0}\left(1-e^{-(s+\nu)\rho}\right)^{-1}$$

which clearly must have a singularity somewhere on the real axis, so something will have to change.

For $\ell=1$ we modify our kernel by introducing a convergence factor

$$\left(\frac{y_1\eta_1}{\left|\frac{z_1-\bar{\xi}_1}{2i}\right|^2}\right)^\alpha \quad \text{with } \alpha>0$$

our eigenfunctions are now eigenfunctions of two operators

$$y_1^2\Delta_1 - 2iy_1\frac{\partial}{\partial x_1} \quad \text{and } y_2^2\Delta_2$$

In the trace formula we get for this we could try to let $\alpha\to 0$; this would be extremely messy. Instead we also write down the T formula from kernel

$$\frac{\alpha}{1+\alpha}\left(\frac{y_1\eta_1}{\left|\frac{z_1-\bar{\xi}_1}{2i}\right|^2}\right)^{1+\alpha} k\left(\frac{|z_2-\xi_2|^2}{y_2\eta_2}\right)\Bigg|_{y_2^2\Delta_2}^{y_2^2\Delta_2}$$

The spectra in the two cases are mostly identical, except that the functions with $y_1^{-1}F(z)$ holomorphic in z_1 do not occur in second formula, and the point corresponding to the constant eigenfunction does not occur in the first. On other side also most terms drop out when taking the difference. After dividing with 4π the resulting formula for $l=1$ is

$$\sum_j h(r_j) - h(\tfrac{i}{2}) - h(-\tfrac{i}{2}) = \frac{V(D)}{8\pi^2}\int_{-\infty}^{\infty} h(r)\, r \operatorname{tgh} \pi r\, dr$$

$$+ \sum_{\{\gamma\}_p^*} \frac{\varepsilon^m}{1-\varepsilon^m}\, \frac{\rho}{e^{\frac{m\rho}{2}} - e^{-\frac{m\rho}{2}}}\, g(m\rho)$$

We see that statements about $Z_\rho(s)$ remain true about $Z_1(s)$ except that $Z_1(s)$ has a pole of order 1 at $s=1$.

product of n hyp planes use special kernel for first $(n-1)$ factors

$$Z_{l_1\cdots l_{n-1}}(s,P) = \prod_{\substack{\{\gamma\}_p^* \\ |i_1|<l_1 \cdots |i_{n-1}|<l_{n-1} \\ \nu>0}} \left(1-\varepsilon_1^{i_1}\cdots\varepsilon_{n-1}^{i_{n-1}} e^{-\rho(s+\nu)}\right)^{(-1)^{n-1}}$$

for $l_1=\cdots=l_{n-1}=1$; have pole or zero at $s=1$ depending on whether n is even or odd.

angles φ_i equidistributed with respect to measure $\frac{\sin^2 \frac{\varphi}{2}}{\pi} = \frac{1-\cos\varphi}{2\pi}$

and statistically independent. ~~surprised me at first.~~

other cases of hybrid.

$|z_1|^2 \cdots + |z_n|^2 \leq 1 \quad n > 1$

and also