Chaos, Quantum Mechanics and Number Theory

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Mahler Lectures 2011

Hamiltonian Mechanics
(x, ξ) generalized coordinates: x space coordinate, ξ phase coordinate.
$H(x, \xi)$, Hamiltonian
Hamilton's equations:

$$\dot{x} = -\frac{\partial H}{\partial \xi}$$
$$\dot{\xi} = \frac{\partial H}{\partial x}$$

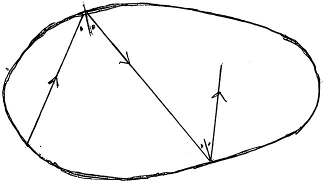
First order o.d.e.
H is a constant of the motion, and on this level set, the motion can be very complex.

We restrict to 2-degrees of freedom, as they carry all the features of interest.

$$(x, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2$$
$$H(x, \xi) = \frac{|\xi|^2}{2}.$$

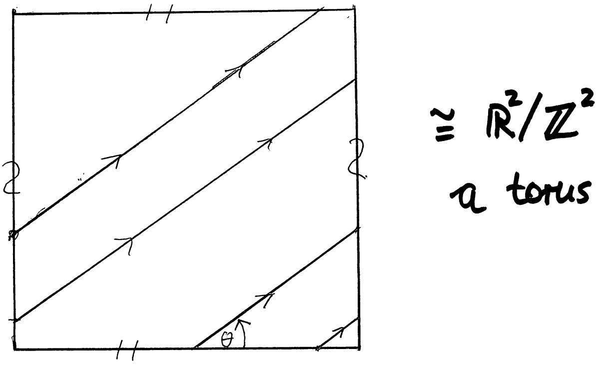
Motion is linear in the direction ξ.

Billiards: Ω a region in $\mathbb{R}^2$.



Motion is linear at unit speed and the angle of incidence equals the angle of reflection at the boundary.

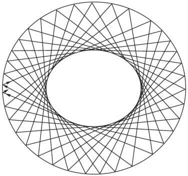
Simplest example is a square — it is convenient to identify sides.



θ is preserved; integral of the motion.

Trajectory is periodic iff θ is rational. If $\theta \notin \mathbb{Q}$, then the orbit is dense in the torus and equidistributed (Kronecker–Weyl).

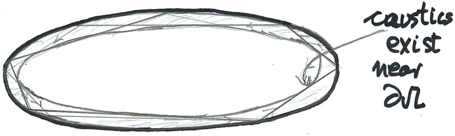
Integral example: Ellipse.



Once tangent to a confocal ellipse, always so. This invariant is the second integral. (Boscovich 1757)

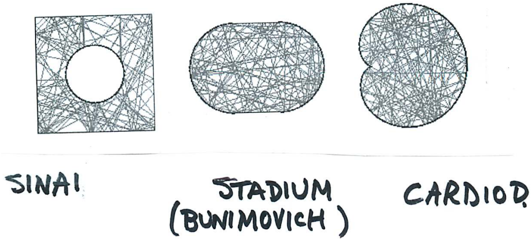
- In appropriate coordinates, this is again straight line motion on a torus.

More general smooth strictly convex domain is neither integrable nor chaotic.



"KAM theory" Lazutkin.

Examples of Chaotic Billiards

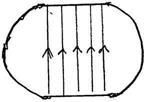


- Ergodic with respect to $\dfrac{dx\,dy\,d\theta}{2\pi\,\mathrm{Area}(\Omega)} = d\mu$

meaning almost all trajectories are equidistributed in phase space with respect to $d\mu$.

- Postive entropy (exponential divergence of orbits)
- K system, ...

NB:

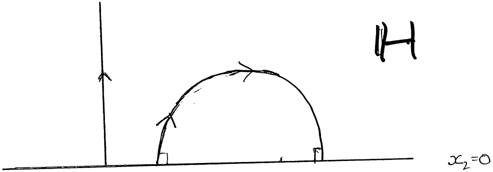


The stadium has bouncing ball trajectories — not "uniformly chaotic".

Hyperbolic Geometry:

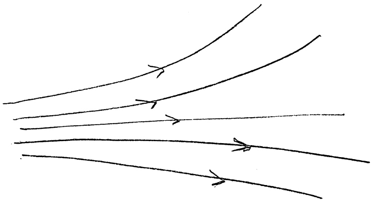
$$\mathbb{H} = \{x_1 + ix_2 : x_2 > 0\}$$

$$ds^2 = \frac{dx_1^2 + dx_2^2}{x_2^2}$$

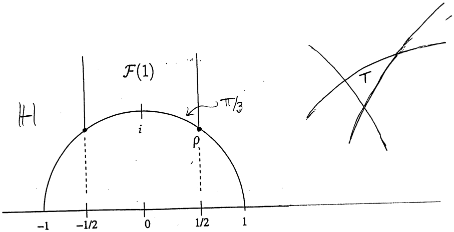


Geodesics (infinite) are semicircles subtended on $x_2 = 0$ and vertical lines.

- Gauss curvature $\kappa \equiv -1$;



$\kappa \equiv -1 \implies$ geodesics diverge exponentially fast in t; this is the key source for entropy and chaos in billiards in hyperbolic polygons:

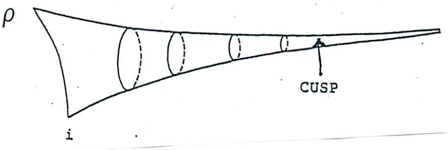


$\mathcal{F}(1)$ is a hyperbolic triangle of finite area. T is a compact triangle.

In $\mathcal{F}(1)$, glue $x_1 = -1/2$ with $x_1 = 1/2$ using the isometry of $\mathbb{H}$, $z \mapsto z + 1$ $(z = x_1 + ix_2)$ and glue the segment $-\rho$ to i with i to ρ using the isometry $z \mapsto -1/z$. These isometries generate

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

the modular group.
The resulting hyperbolic surface $X = \Gamma \setminus \mathbb{H}$ is the modular surface.



- The geodesic (Hamilton) flow on negatively curved surfaces are the best understood fully chaotic flows (Hopf, Morse, Sinai, ...)

Quantizer H:

Schrödinger : $$i\hbar\frac{\partial\psi}{\partial t} = \widehat{H}\psi, \qquad \widehat{H} \text{ “quantization of } H\text{”}$$

time independent equation

$$\widehat{H}\phi + \lambda\phi = 0$$

For our billiards,

$$\widehat{H} = \Delta \text{ the Laplacian}$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \qquad \text{in } \Omega \subset \mathbb{R}^2$$

$$= x_2^2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) \qquad \text{for } \mathbb{H}$$

$$\Delta\phi + \lambda\phi = 0 \qquad \text{on } \Omega$$

$\int_\Omega |\phi|^2\, dA = 1$, $\phi|_{\partial\Omega} = 0$, if there is a boundary.
ϕ is a “mode” or “eigenstate” or ...

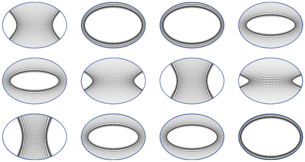
$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots$$

$$\lambda_n \sim \frac{4\pi n}{\mathrm{Area}(\Omega)} \quad \text{as } n \to \infty \text{ (Weyl)}.$$

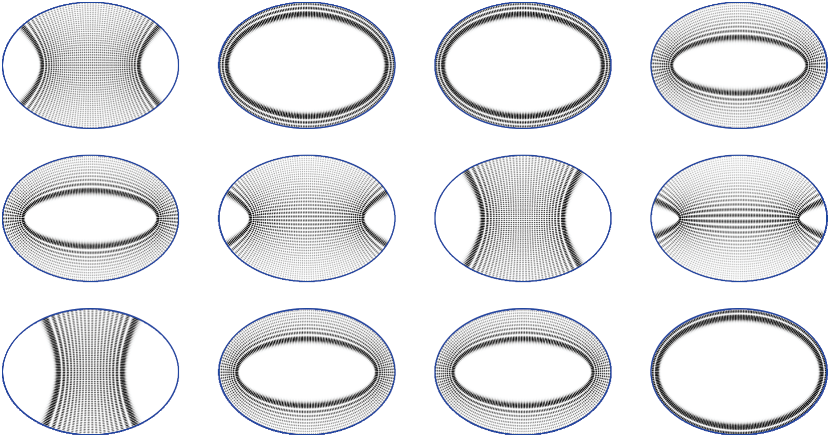
ϕ_n's corresponding modes.

The semiclassical limit $\hbar \to 0 \Longleftrightarrow \lambda_n \to \infty$ for these systems

When $\lambda_n \to \infty$, how does the classical mechanics impact the behaviour of ϕ_n? Correspondence principle.
Gallery of Modes (λ large):



Ellipse (A. Barnett)



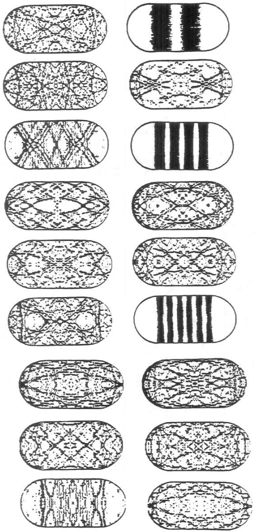
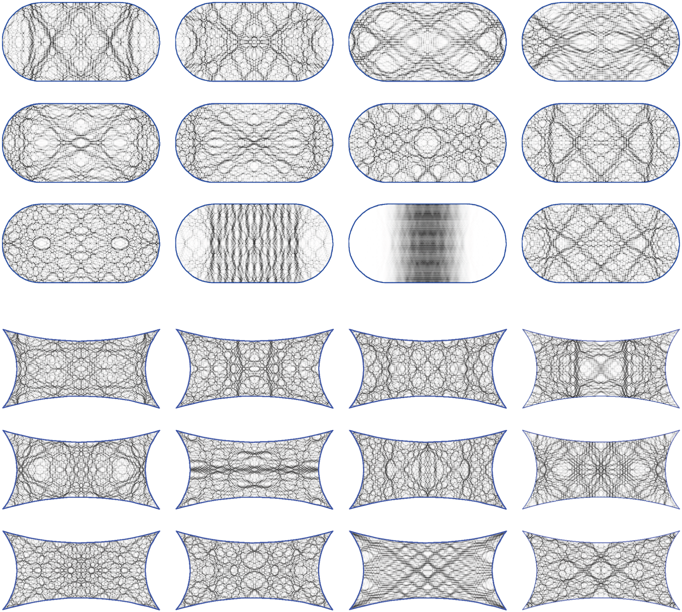
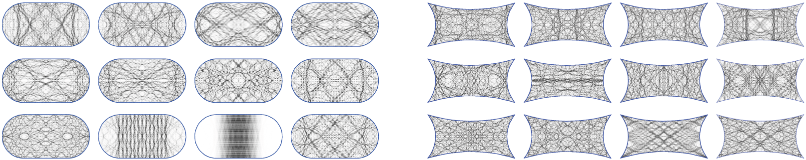


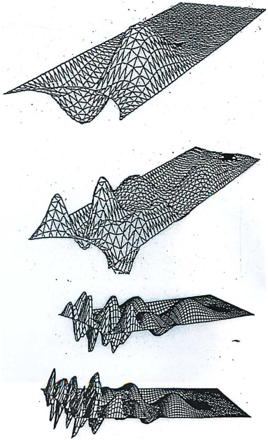
Figure 2.3. Density plot of $|\Phi(x)|^2$ for eigenstates of the stadium (Black signifies high density) for eigenvalues $\sqrt{\lambda} = k$, where going from top to bottom, $k = 110.389$, 119.413, 119.417, 119.451, 119.499, 119.512, 119.512, 119.525, 119.547, 119.587, 119.637, 119.672, 119.691, 119.701, 119.740, 119.802, 119.809, 119.839.

Stadium (E. Heller)

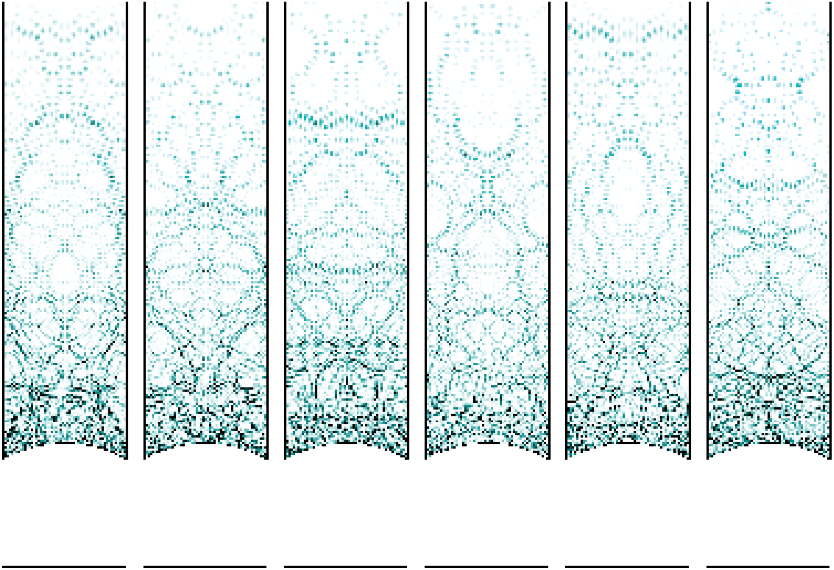




Barnett



Low modes modular surface



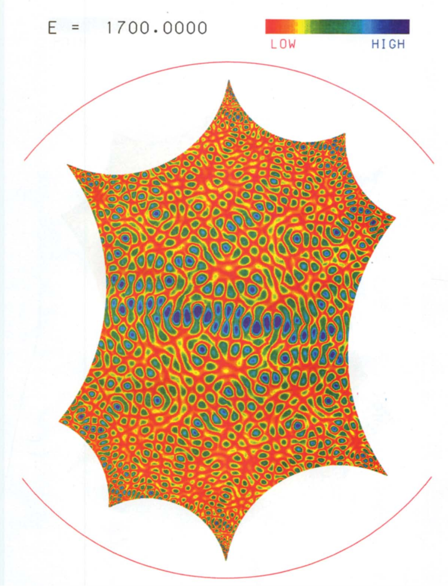
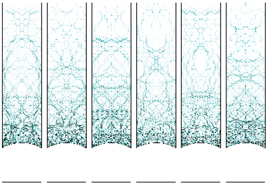
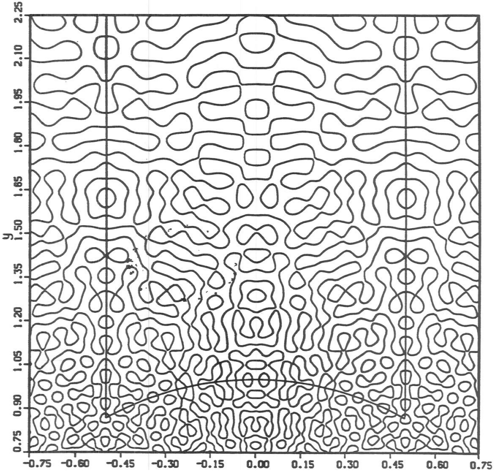


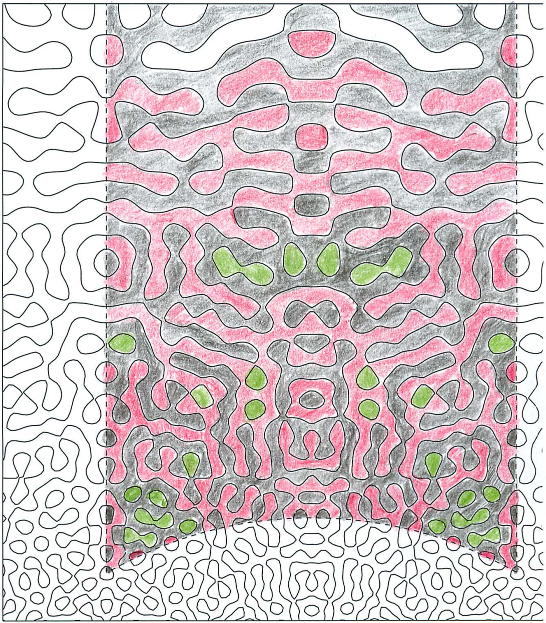
Fig. 14. The intensity $|\Psi(z)|^2$ of the random wavefunction having the largest overlap with the Gaussian wavepacket at energy $E = 1700$.



High Modes Modular Surface (Stromberg)



Hejhal–Rackner nodal lines for $\lambda = 1/4 + R^2$, $R = 125.313840$



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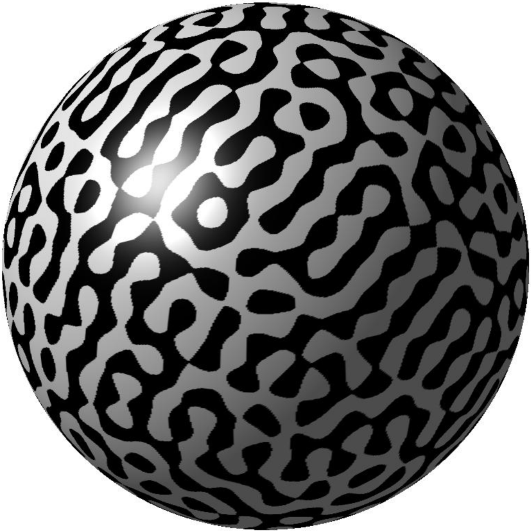

What conclusions can be drawn?

(A) In the integrable case, the ϕ_n's concentrate on invariant torii in a predictable fashion. Well understood analytically (W–K–B, quasi-modes)

(B) If H is strongly chaotic, much less clear. M. Berry suggests that at least the ϕ_n's behave like random waves of a fixed (growing) energy.
The study of the ϕ_n's in this case is known as Quantum Chaos.

Basic Conjectures for Strongly Chaotic H:

(all are specific to chaotic H's)

1) QUE (quantum unique ergodicity) Rudnick–S. '93:
   The probability measures on Ω

$$\mu_\phi = \phi_n^2 \, dA$$

   become equidistributed with respect to dA as $n \to \infty$ (plus microlocal version)

2) Size of ϕ_n (S. '94); For $\varepsilon > 0$,

$$\max_{x \in \Omega} |\phi_n(x)| \ll_\varepsilon n^\varepsilon.$$

3) Number of model domains $N(\phi)$ (Bogomolny–Schmit 2002)

$$N(\phi_n) \sim Cn, \qquad C = \frac{3\sqrt{3} - 5}{\pi}$$

   C comes from modeling the nodal set by a critical percolation model; exactly solvable!

Some results:

- The general Quantum Ergodicity Theorem (Shnirelman–Colin de Verdière–Zelditch):
  For any ergodic H, almost all the μ_{ϕ_n}'s (and their microlocal lifts) become equidistributed. Almost all in the sense of the density of the n's.
- (N. Anantharaman 2008):
  Any weak limit of the μ_{ϕ_n}'s (or better still, their microlocal lifts) has positive entropy (it is invariant under the Hamilton flow). In particular, the ϕ_n's cannot concentrate on an unstable periodic orbit!

- (A. Hassell 2009): Stadium
  Shows that for the stadium, QUE fails due to the existence of "bouncing ball modes" with arbitrary large n.

Number Theory (Arithmetic Quantum Chaos)

The main progress on the ϕ_n's in the chaotic cases comes from number theory. That is, when $X = \Gamma \setminus \mathbb{H}$ is an arithmetic surface such as the modular surface. In this case, the ϕ_n's are automorphic forms and the modern tools from this theory allow one to study individual ϕ_n's.

Tools include

- Automorphic L-functions, subconvexity, Riemann hypothesis, functoriality, Ramanujan conjectures, Hecke operators, …measure rigidity from p-adic flows.

Theorem 1 (E. Lindenstrauss 2006, K. Soundararajan 2009):
QUE holds for arithmetic X.

Theorem 2 (Iwaniec–S. '95):
X arithmetic,

$$\|\phi_n\|_\infty \ll n^{5/24}$$

(the local convexity bound is $n^{1/4}$).

Theorem 3 (A. Ghosh–A. Reznikov–S. 2011)
X the modular surface and assume the Riemann hypothesis for automorphic L-functions. Then locally the number of nodal domains goes to infinity with n, in fact

$$N_{\mathrm{loc}}(\phi) \geq n^{1/12}.$$

Conclusion: For quantizations of strongly chaotic systems, there appears to be no evidence of chaotic features in the semiclassical limit. The eigenstates appear to behave like random waves. The fine features of individual eigenstates are so approachable only via number theory. Perhaps that is not so surprising since quantizing makes quantities discrete — which is number theory — the final frontier!

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