Mobius Randomness and Dynamics

by PETER SARNAK

$n \geq 1$

$$\mu(n) = \begin{cases} (-1)^t & \text{if } n = p_1 p_2 \cdots p_t \text{ distinct} \\ 0 & \text{if } n \text{ has a square factor} \end{cases}$$

1, -1, -1, 0, -1, 1, -1, 0, 0, 1, ...

is this sequence "random".

$$\frac{1}{\zeta(s)} = \prod_p (1 - p^{-s}) = \sum_{n=1}^{\infty} \mu(n) n^{-s}$$

so the zeros of $\zeta(s)$ are closely connected to

$$\sum_{n \leq N} \mu(n)$$

Prime number theorem $\overset{\text{(elementarily)}}{\Longleftrightarrow}$

$$\sum_{n \le N} \mu(n) = \sum_{n \le N} \mu(n) . 1 = o(N)$$

Riemann Hypothesis $\Longleftrightarrow$

for $\varepsilon > 0$

$$\sum_{n \le N} \mu(n) = O_{\varepsilon}\left(N^{\frac{1}{2}+\varepsilon}\right).$$

- usual randomness of $\mu(n)$ "square root cancellation"

<u>(Old heuristic)</u> "Mobius randomness law" <u>(I-K)</u>:

$$\sum_{n \le N} \mu(n) \xi(n) = o(N), \quad \text{"}\mu \text{ orthogonal to } \xi\text{"}$$

for any 'reasonable' independently defined bounded function $\xi(n)$.

[often used for primes via

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^e \\ 0 & \text{otherwise.} \end{cases}$$

What is reasonable?

computational complexity (?)

$\xi \in P$ if $\xi(n)$ can be computed in poly($\log n$) steps.

Perhaps $\xi \in P \Rightarrow \mu$ orthogonal to ξ

I DONT believe so since I believe factoring and hence μ is in P.

<u>Problem:</u> Construct $\xi \in P$ bounded s.t.

$$\frac{1}{N} \sum_{n \leq N} \mu(n)\xi(n) \to \alpha \neq 0.$$

Dynamical view (Furstenberg disjointness paper 1967):

· <u>Flow:</u> $F = (X, T)$

X compact topological space

$T : X \to X$ cts.

If $x \in X$ and $f \in C(X)$ the sequence

$$\xi(n) = f(T^n x) \quad \text{'return time'}$$

is realized in F.

To measure the complexity of $\xi(n)$ try realize $\xi(n)$ in a dynamical system F of low complexity.

Every sequence can be realized:

say $\xi(n) \in \{0,1\}$

$$\Omega = \{0,1\}^{\mathbb{N}}, \quad T: \Omega \to \Omega$$

$$(x_1, x_2, \dots) \to (x_2, x_3, \dots)$$

then if $\xi = (\xi(1), \xi(2), \dots) \in \Omega$

$f(x) = x_1$, $x = \xi$ realizes $\xi(n)$.

In fact it is realized in the perhaps simpler flow

$$F_\xi = (X_\xi, T), \quad X_\xi = \overline{\{T^j \xi\}_{j=1,2,\dots}}$$

The crudest measure of complexity of F is its topological entropy $h(F)$ (Adler et al 67), which measures the exponential growth rate of distinct orbits of length n.

<u>Definition</u> F is deterministic if $h(F)=0$. $\xi(n)$ is deterministic if it can be realized in a deterministic flow.

Process: is a flow together with an invariant measure

$$F_\nu = (X, T, \nu)$$

ν an invariant Borel probability measure

$$\nu(T^{-1}A) = \nu(A) \quad \text{for } A \subset X.$$

$h(F_\nu)$ = Kolmogorof-Sinai entropy

$h(F_\nu)=0$, "F_ν is deterministic" and it means that with ν-probability 1, $\xi(1)$ is determined from $\xi(2), \xi(3), \ldots$.

THEOREM 1:

$\mu(n)$ is not deterministic.

A much stronger form of this is that $\mu(n)$ cannot be approximated by a deterministic sequence.

<u>Definition:</u> $\mu(n)$ is <u>disjoint</u> from F if

$$\sum_{n \leq N} \mu(n) \xi(n) = o(N)$$

for every ξ belonging to F.

<u>Main Conjecture:</u> "MOBIUS RANDOMNESS LAW" μ is disjoint from any deterministic F, in particular μ is ~~disjoint~~ orthogonal to any deterministic ξ.

Why believe this?

<u>Chowla Conjecture</u> (self correlations)

$$0 < a_1 < a_2 < \dots < a_t$$

$$\sum_{n \leq N} \mu(n+a_1)\mu(n+a_2)\cdots\mu(n+a_t) = o(N)$$

<u>Proposition:</u> <u>Chowla</u> $\Rightarrow$ main Conj.

- Proof is purely combinatorial and is true for any uncorrelated sequence $\eta(n)$.

The point is one can make progress on the Main Conjecture thanks to methods of Vinogradov.

<u>Cases of M.C:</u>

(i) F a point $\Leftrightarrow$ Prime No Theorem

(ii) F finite $\Leftrightarrow$ Dirichlet's theorem.

(iii) $F = (\mathbb{R}/\mathbb{Z}, T_\alpha)$, $T_\alpha(x) = x + \alpha$
rotation of circle, Vinogradov / Davenport
1937.

(iv) Extends to any Kronecker flow
$\lceil F = (G, T_\alpha)$, G compact abelian
and ~~an~~ $T_\alpha g = \alpha + g \rfloor$

and any affine automorphism (of zero entropy) of such. Eg any deterministic affine automorphis of $\mathbb{R}^n/\mathbb{Z}^n$. (Liu-S)

(v) $F = (\Gamma \backslash N, T_\alpha)$
where N is a nilpotent group, Γ a lattice in N, $T_\alpha(\Gamma x) = \Gamma x \alpha$.
(Green-Tao)

.....

All of the above are "distal"
$\left[\inf_{n \geq 1} d(T^n x, T^n y) > 0 \text{ if } x \neq y \right]$ and deterministic.

More complex deterministic homogeneous dynamics:

$$F = (\Gamma\backslash G, T_\alpha), \quad T_\alpha(\Gamma g) = \Gamma g \alpha$$

G semi-simple.

$h(F) = 0$ iff T_α is ad-quasi unipotent

these have been studied Furstenberg, Dani, Ratner, Starkov, ...

- These are mixing of all orders (Mozes).

(vi) Ubis - S partial results for $SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$.

Dynamical system associated with μ and disjointness:

'simplest' realization of μ

$\{-1,0,1\}^{\mathbb{N}} = X$, T shift

$w = (\mu(1), \mu(2), \ldots\ldots) \in X$

$X_M = \overline{\{T^j w\}_{j=1}^{\infty}} \subset X$

$M = (X_M, T_M)$ the Mobius flow.

Look for factors and extensions

let $\eta = (\mu^2(1), \mu^2(2), \ldots\ldots) \in Y = \{0,1\}^{\mathbb{N}}$

Y_S = closure in Y of $T^j\eta$

$S := (Y_S, T_S)$

the square-free flow.

$\pi : X_M \to Y_S$.

$(x_1, x_2, \dots) \to (x_1^2, x_2^2, \dots)$

(Sato)

$\pi(\omega) = \eta$.

$$\begin{tikzcd} X_M \arrow[r, "T_M"] \arrow[d, "\pi"'] & X_M \arrow[d, "\pi"] \\ Y_S \arrow[r, dash, "T_S"'] & Y_S \end{tikzcd}$$

S is a factor of M.

Using an elementary square-free sieve one can investigate S!

<u>Definition:</u> $A \subset \mathbb{N}$ is admissible if the reduction $\overline{A}$ of A mod p^2 is not all of the residue classes mod p^2 for every prime p.

THEOREM 2:

(i) Y_S consists of all points y in Y whose support is admissible.

(ii) The flow S is not deterministic

$$h(S) = \frac{6}{\pi^2} \log 2 .$$

(iii) S is proximal

$$\inf_{n \geq 1} d(T^n x, T^n y) = 0 \text{ for all } x, y.$$

(iv) S has a nontrivial joining with the Kronecker flow

$$K = (G, T), \quad G = \prod_p \left(\mathbb{Z}/p^2\mathbb{Z}\right)$$

$$Tx = x + (1, 1, \ldots)$$

(v) S is not weak mixing.

At the ergodic level there is an important invariant measure for S.

On cylinder sets C_A, $A \subset N$ finite

$$C_A = \{ y \in Y : y_a = 1 \text{ for } a \in A \}$$

let

$$\nu(C_A) = \prod_p \left(1 - \frac{t(\bar{A}, p^2)}{p^2} \right)$$

ν extends to a T-invariant ~~probability~~ measure on Y whose support is Y_S.

<u>THEOREM 3:</u> $S_\nu = (Y_S, T_S, \nu)$ satisfies

(i) η is generic for ν, that is the sequence $T^n \eta \in Y$ is ν-equidistr.

(ii) S_ν is ergodic.

(iii) S_ν is deterministic as a ν-process.

(iv) S_ν has $K_\mu = (K, T, d\mu)$ as a Kronecker factor, $d\mu$ is Haar measure.

- Since S is a factor of M

$$h(M) \geq h(S) > 0 \implies h(M) > 0$$

$\implies \mu$ is not deterministic.

- One can form a process N_v which conjecturally describes M and from which the main Conjecture can (at least in part) be seen as a disjointness statement as in Furstenbergs theory.

Vinogradov's Method:

$$F = (X, T),$$

~~need~~ to examine

$$\sum_{n \leq N} \mu(n) f(T^n x) \quad \text{or} \quad \sum_{p \leq N} f(T^p x)$$

$x \in X$, $f \in C(X)$,

need quantitative equidistribution on <u>progressions</u> for X

$$\sum_{n \leq N} f(T^{dn} x) \quad , \quad \text{type I sums}$$

and similarly for sums connected with joinings of X with itself

$f_1, f_2 \in C(X)$, $x_1, x_2 \in X$

$$\sum_{n \leq N} f_1(T^{d_1 n} x_1) f_2(T^{d_2 n} x_2) \quad \text{type II bilinear sums}$$

<u>Definition</u>; Level of distribution for a uniquely ergodic F (i.e. one for which there is only one invariant measure μ). $F = (X, T, \mu)$ has level α, $0 \leq \alpha < 1$ if for every x and f with $\int f \, d\mu = 0$;

$$\sum_{d \leq D} \left| \sum_{n \leq N/d} f(T^{nd} x) \right| \ll \frac{N}{(\log N)^A}$$

for D as large as N^{α}.

Consider $G = SL_2(\mathbb{R})$

$\Gamma = SL_2(\mathbb{Z})$

$u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ unipotent

$F = (X, T)$, $X = \Gamma \backslash G$, $T(\Gamma x) = \Gamma x u$.

$h(F) = 0$.

<u>Dani:</u> a point $x \in X$ is either
(i) periodic
(ii) equidistributed in a closed horocycle.
(iii) equidistributed in X w.r.t. dg.

<u>Ubis-S (2010):</u>
give an effective version of Dani ~~[illegible]~~ with a level of distribution $1/5$ for the Birkhoff sums.

It is conjectured (Margulis ?) that in case (iii) the sequence

$\Gamma x u^p$, $p = 2, 3, 5, 7, 11, \ldots$

is equidistributed in X w.r.t. dg.

for x as above

$$\nu_x^{(N)} := \frac{1}{\pi(N)} \sum_{p \leq N} \delta_{\Gamma x u^p}$$

<u>Theorem</u> (Ubis-S):
x as above, ν_x a limit of $\nu_x(N)$ as $N \to \infty$ then ν_x is absolutely continuous;

$$\nu_x \leq 10\, dg.$$

$\Rightarrow$ $U \subset X$ is open $Vol(U) > \frac{9}{10}$ then for a positive density of primes

$$\Gamma x u^p \in U.$$

There are difficulties with the bilinear sums as we don't know how to effectivize equidistribution on $F \times F$, let alone get a good enough level of distribution.

For special points okay
key bilinear sums (treated by spectral technique of Sarnak, Blomer/Harcos)

$$\sum_{an+bm=h} \lambda(n)\lambda(m)$$

for a, b, h fixed and n, m varying

and λ Fourier coefficients
of modular forms.

Use sharpest λ bounds towards Ramanujan/Selberg Conjecture.