Letter to J. Mozzochi on Linking Numbers of Modular Geodesics

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from

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Dear Jeff,

Here are some remarks in connection with Ghys' spectacular ICM 2006 talk [1]. I indicate what the tools from the spectral theory of hyperbolic surfaces and especially the Selberg Trace Formula provide in this direction. The technicalities with multiplier systems and weights are quite delicate and I did not check them in detail.

In this write-up [1] Ghys shows that $M = PSL_2(\mathbb{R})/\Gamma$ where $\Gamma = PSL_2(\mathbb{Z})$ is homeomorphic to S^3/τ where τ is the trefoil knot. Furthermore, if C_A is the closed (primitive) geodesic in M corresponding to the primitive hyperbolic element A in Γ, then the linking number $\mathsf{link}(C_A, \tau)$ of C_A with τ is equal to $\Psi(A)$ the Rademacher function of A (see [2] page 54, equation (63)). The pictures from his talk that can be found on the web, beg one to study the behavior of these knots C and their linking with τ as one varies over the closed geodesic C. Let $\ell(C_A)$ denote the length of the closed geodesic C. If A is conjugate in $SL_2(\mathbb{R})$ to $\begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ with $\lambda > 1$, then $\ell(C_A) = \log N(A)$ where $N(A) = 2\log\lambda$. We order the geodesics by their lengths.

For $y \geq 2$ let

$$\pi(y) := |\{C : \ell(C) \leq y\}| \tag{1}$$

The prime geodesic theorem for the modular surface in its strongest form [3] asserts that

$$\pi(y) = Li(e^y) + O(e^{\frac{7y}{10}}) . \tag{2}$$

Here

$$Li(x) = \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x} . \tag{3}$$

For $n \in \mathbb{Z}$ set

$$Li(x; n) = \int_2^x \frac{\log t}{(\log t)^2 + \left(\frac{\pi n}{3}\right)^2} \, dt . \tag{4}$$

Define $\pi(y;n)$ by

$$\pi(y;n) \,:=\, |\{C : \ell(C) \,\leq\, y\,,\, \mathsf{link}(C,\tau) \,=\, n\}|\,. \tag{5}$$

I outline a proof of

$$\sum_{\substack{\ell(C)\,\leq\, y \\ \mathsf{link}(C,\tau)\,=\,n}} \ell(C) \,=\, \frac{1}{3}\, Li(e^y;n) \,+\, O(e^{\frac{3y}{4}})\,. \tag{6}$$

A key point in (6) is the uniformity that is the implied constant is absolute, that is it is independent of n and y. Thus (6) gives the main term in the count (5) for $|n|$ as large as $\exp(y/8)$. In particular, it follows that for n fixed

$$\pi(y;n) \,\sim\, \frac{\pi(y)}{3y} \left(1 \,+\, \frac{2(1-(\frac{\pi n}{3})^2}{y^3} \,+\, \cdots\right)\,. \tag{7}$$

Thus to leading order the number of geodesics with a given linking number is independent of n. However, the next order term endures that $\pi(y;n) > \pi(y;m)$ if $|n| < |m|$ and y is large. So the most common linking number is zero. Among these closed geodesics C whose linking number is zero are the reciprocal geodesics (see [4]) of which there are $3/4e^{y/2}$ asymptotically.

Summing over n in (6) leads to the distribution of the values of $\mathsf{link}(C,\tau)$ as we vary C:

If $-\infty \,\leq\, a \,\leq\, b \,\leq\, \infty$ are fixed then as $y \to \infty$,

$$\frac{1}{\pi(y)} \left|\left\{C :\, \ell(C) \,\leq y\,,\, a \,\leq\, \frac{\mathsf{link}(C,\tau)}{\ell(C)} \,\leq\, b\right\}\right| \longrightarrow\, \frac{\mathsf{arctan}\left(\frac{b\pi}{3}\right) - \mathsf{arctan}\left(\frac{a\pi}{3}\right)}{\pi} \tag{8}$$

The analogues of (7) and (8) are known for $\mathbb{H}/\Gamma$ a compact hyperbolic surface and where instead Ψ we have a group homomorphism Φ of Γ to $\mathbb{Z}$, that is we are counting the winding of geodesics in homology (see [5],[6],[7],[8] for example). There are notable differences. Firstly in that case $\pi(y;n) \sim c(\Phi)\,\pi(y)/\sqrt{y}$ as $y \to \infty$ and secondly the corresponding normal order of $\Phi(C)$ is $\sqrt{\ell(C)}$ with the limiting distribution being Gaussian. If $\mathbb{H}/\Gamma$ is not compact but is of finite area then the analogue of (7) is proven in [9] and for certain Φ's the noncompactness changes things with the normal order of Φ being much larger due to the winding around the cusp. According to (6), (7) and (8) this effect persists for our Ψ (the trefoil in S^3 corresponds to the cusp) which is a quasi morphism (see [10]). This non-local Cauchy Distribution in (8) has appeared before in related contexts. In [11] it comes up in connection with questions involving Dedekind sums, while in [4] it appears in connection with the winding in homology of a generic geodesic on the unit tangent bundle of a hyperbolic surface (noncompact) and in [15] in the similar problem on S^3/τ. The fluctuations of certain quasi-morphisms (including our Ψ) when ordered combinatorially by word length is always Gaussian ([16]). The reason is that the combinatorial ordering does not make the cusp singular.

I outline a proof of (6) and (8). First one needs to connect Ψ to the spectral theory of $L^2(\mathbb{H}/SL_2(\mathbb{Z}), v, r)$ where v is a multiplier system for $SL_2(\mathbb{Z})$ of weight r (here r is any real number). For this we use the defining relation for the allied function $\Phi : \Gamma \to \mathbb{Z}$ (page 49, equation 60 of [2])

$$\log \eta \left(\frac{az+b}{cz+d}\right) - \log \eta(z) \;=\; \frac{1}{2}\,\mathrm{sgn}^2(c)\,\log\left(\frac{cz+d}{i, \mathrm{sgn}(c)}\right) + \frac{\pi i}{12}\,\Phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \tag{9}$$

and $\eta(z)$ is the Dedekind eta function. Using (9) we can relate Φ to the multiplier system $v_{1/2}$ of the eta function i.e.,

$$\eta(Az) \;=\; v_{1/2}(A)\,(cz+d)^{1/2}\,\eta(z)\,,\; A \in SL_2(\mathbb{Z}) \tag{10}$$

(note that $v_{1/2}$ is defined on $SL_2(\mathbb{Z})$ and not $PSL_2(\mathbb{Z})$, unlike Φ and Ψ). From this and the relation between Φ and Ψ (page 54) we have; for $A \in SL_2(\mathbb{Z})$ and $\mathsf{trace}(A) > 0$

$$v_{1/2}(A) \;=\; e^{i\pi\Psi(A)/12}\,. \tag{11}$$

Hence for any $r \in \mathbb{R}$ the multiplier system of weight r given by $(v_{1/2})^{r/2}$ satisfies:

For $A \in SL_2(\mathbb{Z})$ with $\mathsf{trace}(A) > 0$

$$v_r(A) \;=\; e^{i\pi r\Psi(A)/6}\,. \tag{12}$$

Next consider the spectral problem for the Laplacian $\triangle_r$ on $L^2(\mathbb{H}/SL_2(\mathbb{Z})\,,\, v_r, r)$, (see Chapter 9 of [12] i.e. functions transforming by

$$f(\gamma z) \;=\; v_r(\gamma)\,\left(\frac{cz+d}{|cz+\alpha|}\right)^r\, f(z), \gamma \in SL_2(\mathbb{Z})\,. \tag{13}$$

We restrict to $-6 < r \leq 6$ with the critical interval being $-1 \leq r \leq 1$. In this range the bottom eigenvalue is $\lambda_0(r) = \frac{|r|}{2}\,(1 - \frac{|r|}{2})$ and it is the only eigenvalue in $[0, \frac{1}{4})$, see [13]. In fact in [13] it is shown that for $-6 < r \leq 6$ there are no exceptional eigenvalues (not accounted for by known holomorphic forms). With this and the precise trace formula that is derived in Hejhal [12] for this space and a lengthy and detailed analysis of the uniformity as $r \to 0$, one shows:

Uniformly for $6 \leq r \leq 6$ and $x \geq 5$

$$\sum_{\substack{\{\gamma\}_{SL_2(\mathbb{Z})} \\ \mathsf{trace}(\gamma) > 2 \\ N(\gamma) \leq x \\ (\gamma\ \mathsf{primitive})}} \log N(\gamma)\, v_r(\gamma) \;=\; \begin{cases} \frac{x^{1-|r|/2}}{1-\frac{|r|}{2}} + O\left(x^{\frac{3}{4}}(\log \frac{1}{|r|} + 1)\right), & \mathsf{if}\ |r| \leq \frac{1}{2} \\ \\ O(x^{\frac{3}{4}}) & \mathsf{otherwise} \end{cases} \tag{14}$$

To get (6) integrate both sides of (14) against $e^{-\pi\ell nr/6}$ w.r.t. r over (6,6]. Using (12) the left hand side becomes

$$12 \sum_{\substack{\{\gamma\}_\Gamma \\ N(\gamma)\,\leq\,x,\Psi(\gamma)\,=\,n}} \log N(\gamma). \tag{15}$$

The right hand side equals

$$\int\limits_{-1/2}^{1/2} \frac{x^{1-|r|/2}}{1-|r|/2}\, e^{-i\pi nr/6}\, dr \,+\, O(x^{3/4}) \tag{16}$$

$$= \,4\, Li(x;n) \,+\, O(x^{3/4})\,, \ \text{ uniformly for } x\, \geq 2\,,\, n \in \mathbb{Z}\,. \tag{17}$$

Or

$$\sum_{\substack{\ell(c)\,\leq\,y \\ \Psi(c)\,=\,n}} \ell(c) \,=\, \frac{1}{3}\, Li(e^y;n) \,+\, O(e^{3y/4})\,, \tag{18}$$

this is (6). (7) is an elementary consequence of (6) while (8) follows from (6) by summing over n in the indicated range.

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