LETTER TO DREW SUTHERLAND AND NINA ZUBRILINA ON MURMURATIONS AND ROOT NUMBERS.

AUGUST 2023

DEAR DREW AND NINA,

AFTER YOUR TALKS AT IAS LAST MAY AND THE MEETING AT ICERM IN THE SUMMER HERE IS MY TAKE ON THE MURMURATION PHENOMENON.

1) MACHINE LEARNING THE ELLIPTIC CURVE DATA BASE

THE L-FUNCTIONS AND MODULAR FORM DATABASE (SEE [LMFDB]) CONTAINS ALL ELLIPTIC CURVES $E/\mathbb{Q}$ OF CONDUCTOR N_E UP TO HALF A MILLION, AS WELL AS LARGE COLLECTIONS OF CURVES AND RELATED OBJECTS (STEIN-WATKINS, ...) OF CONDUCTORS AS LARGE AS 10^8. AS SUCH IT IS A WONDERFUL RESOURCE FROM WHICH TO MINE STATISTICAL CORRELATIONS BETWEEN THE SUBTLE ARITHMETICAL INVARIANTS ASSOCIATED WITH SUCH CURVES. FOR SMALL CONDUCTORS THERE ARE FEATURES THAT ARE SEEN WHICH I BELIEVE DISAPPEAR WHEN THE CONDUCTORS GET LARGE. MOST PROMINANT AMONG THEM IS "EXCESS RANK", SEE THE INSIGHTFUL DISCUSSIONS IN [B-M-S-W] AND [B-H-K-S-S-W].

A MORE STABLE THOUGH SMALLER CORRELATION IS THAT OF THE CO-EFFICIENTS $a_E(p)$ AT PRIMES p, WITH THE PARITY

OF THE RANK OF E, OR WHAT IS THE SAME ([D-D]) WITH THE ROOT NUMBER W_E. THE MACHINE LEARNING DISCOVERY IN [H-L-O-P], EXTENDED TO SOME OTHER FAMILIES AND GREATLY CLARIFIED IN [H-L-O-P-S] AND [SU] IS THAT FOR $p \sim N_E$, THE SIGN AND SIZE OF THESE CORRELATIONS IS APPROXIMATED BY A FUNCTION OF P/N CALLED A MURMURATION FUNCTION.

2) 1-LEVEL DENSITY IN THE TRANSITION RANGE

TO FORMULATE AND INVESTIGATE SUCH CORRELATIONS IT IS HELPFULL TO LOOK AT MORE GENERAL FAMILIES OF L-FUNCTIONS AND THEIR LINEAR STATISTICS. LET $\mathcal{F}$ BE SUCH A FAMILY (SEE [SA] AND [T-S-S] FOR DEFINITIONS). WE ASSUME THAT $\mathcal{F}$ IS ENDOWED WITH A NATURAL ORDERING COMING FROM A HEIGHT FUNCTION (USUALLY THE CONDUCTOR N_π OF $\pi \in \mathcal{F}$). $\mathcal{E}$ DENOTES THE FAMILY OF ELLIPTIC CURVES ORDERED BY CONDUCTOR AND IS THE CENTRAL EXAMPLE. THE FAMILY OF ELLIPTIC CURVES

$E_{a,b}$ IN WEIERSTRASS FORM ORDERED BY NAIVE HEIGHT IS ANOTHER EXAMPLE.

GIVEN A SMOOTH NONNEGATIVE WEIGHT FUNCTION $\Phi : (0,\infty) \to \mathbb{R}$ OF COMPACT SUPPORT AND $f : \mathcal{F} \to \mathbb{C}$, FOR $N \to \infty$ DEFINE

$$A_{\mathcal{F},\Phi,N}(f) = \sum_{\pi \in \mathcal{F}}{}' \, \Phi\left(\frac{N_\pi}{N}\right) f(\pi) \qquad (1)$$

AND THE EXPECTED VALUE OF f

$$\underset{\pi \in \mathcal{F}}{\mathrm{EXP}_{\Phi,N}}[f] := \frac{A_{\Phi,N}(f)}{A_{\Phi,N}(1)} \qquad (2).$$

THUS $\mathrm{EXP}_{\Phi,N}[f]$ IS THE AVERAGE VALUE OF $f(\pi)$ FOR π'S OF CONDUCTORS WHICH ARE MULTIPLICATIVELY CLOSE TO N.
THE BEHAVIOR OF THE NORMALIZATION FACTOR IN (2) DEPENDS ON THE DENSITY OF CONDUCTORS. WE ASSUME THAT THIS DENSITY HAS SOME SCALE INVARIANCE.
THE EXAMPLES THAT WE CONSIDER SATISFY

$$B_{\mathcal{Y}}(\tau) = \#\{\pi \in \mathcal{Y} : N_\pi \leq \tau\}$$
$$\sim \alpha \tau^\delta \quad \text{AS} \quad \tau \to \infty \qquad (3)$$

HERE $\alpha > 0$ AND $\delta = \delta(\mathcal{F}) \geq 0$, IS THE CONDUCTOR DIMENSION OF $\mathcal{Y}$. FOR $\mathcal{F}$ SATISFYING (3)

$$A_{\Phi,N}(1) \sim \alpha\delta N^\delta \int_0^\infty \Phi(\tau)\tau^\delta \frac{d\tau}{\tau}. \qquad (4)$$

THE p-TH COEFFICIENTS OF THE L-FUNCTION $L(s,\pi)$ ARE DENOTED BY $\lambda_\pi(p)$ AND ARE NORMALIZED SO THAT THE RAMANUJAN CONJECTURE FOR π WHICH IS AN AUTOMORPHIC CUSP FORM ON GL_n ASSERTS THAT

$$|\lambda_\pi(p)| \leq n. \qquad (5)$$

FOR THE LINEAR CORRELATIONS THAT WE EXAMINE SET

$$a_\pi(p) := \sqrt{p}\,\lambda_\pi(p) \qquad (6)$$

WE SPECIALIZE MOSTLY TO FAMILIES WHOSE MEMBERS ARE SELF DUAL ($\pi \cong \pi^*$) IN WHICH CASE THE NUMBERS $a_\pi(p)$ ARE REAL AND THE

GLOBAL ROOT NUMBER W_π IS 1 OR -1. FOR $W \in \{1,-1\}$ LET $\mathcal{F}^W$ CONSIST OF THE $\pi \in \mathcal{F}$ WITH $W_\pi = W$. THE LINEAR CORRELATIONS OF INTEREST ARE

$$\underset{\pi \in \mathcal{F}^W}{\mathrm{EXP}_{\Phi,N}} \left[a_\pi(p) \right] . \qquad \text{——————} (7)$$

IF ONE AVERAGES FURTHER OVER p'S WITH SUITABLE SMOOTH WEIGHTS, THEN SWITCHING THE ORDER OF AVERAGING SHOWS VIA THE EXPLICIT FORMULA THAT THESE DOUBLE AVERAGES ARE RELATED TO THE 1-LEVEL DENSITIES OF THE ZEROS OF $L(s,\pi)$, FOR $\pi \in \mathcal{F}^W$. THE CONJECTURED 1-LEVEL DENSITIES DEPEND ON THE SYMMETRY TYPE OF $\mathcal{F}([K-S])$ WHICH IS EITHER $O(\infty)$ OR $Sp(\infty)$:

FOR $a \neq 1$, AS $N \to \infty$

$$Sp(\infty): \quad \underset{p \sim N^a}{\mathrm{EXP}} \; \underset{\pi \in \mathcal{F}}{\mathrm{EXP}_{N,\Phi}} \left[a_\pi(p) \right] \longrightarrow \begin{cases} 0 & \text{IF } a<1 \\ -\frac{1}{2} & \text{IF } a>1 \end{cases} \qquad (7')$$

$$O(\infty): \quad \underset{p \sim N^a}{\mathrm{EXP}} \; \underset{\pi \in \mathcal{F}^W}{\mathrm{EXP}_{N,\Phi}} \left[a_\pi(p) \right] \longrightarrow \begin{cases} 0 & \text{IF } a<1 \\ \frac{W}{2} & \text{IF } a>1 . \end{cases}$$

THERE IS A PHASE TRANSITION WHEN $a=1$ AND THE MURMURATION CORRELATIONS ARE CONCERNED WITH THE BEHAVIOR OF (7) IN THIS THIS RANGE; THAT IS FOR $p \sim N$.

<u>DEFINITION:</u> A CONTINUOUS FUNCTION $M_{\Phi}:(0,\infty)\to\mathbb{R}$ IS A MURMURATION FUNCTION FOR $\mathcal{F}$ WITH WEIGHT Φ IF THERE IS $0\leq\delta<1$ SUCH THAT FOR $P\sim N$.

$$\underset{\pi\in\mathcal{F}}{\mathrm{EXP}_{\Phi,N}}\left[a_{\pi}(p)\right]=M_{\Phi}\left(\frac{P}{N}\right)+R(p,N) \qquad (8)$$

WHERE ~~THE LOCAL~~ OSCILLATING TERM R SATISFIES

$$\underset{P-H\leq p\leq P+H}{\mathrm{EXP}}\left[R(p,N)\right]=o(1)\ \text{, FOR } N^{\delta}\leq H=o(N). \qquad (9)$$

<u>REMARKS:</u>

(a) (9) SAYS THAT THE LOCAL p AVERAGES OF LENGTH AT LEAST N^{δ} AND LESS THAN N. OF R TEND TO ZERO. THIS DEFINITION WITH RECTANGULAR DOUBLE AVERAGING IS SIMILAR TO BOBER'S SUGGESTION; BOTH SHOULD GIVE THE SAME MURMURATION.

(b) THE SMALLER δ THE MORE VISIBLE THE

MURMURATION FUNCTION M_{Φ} IN GRAPHS OF THE CORRELATIONS (7). IT IS VERY CLEAR FOR THE FAMILY $\mathcal{E}$ WITH CONDUCTORS UP TO 10^8 AND MUCH LESS VISIBLE FOR THE FAMILY OF ELLIPTIC CURVES ORDERED BY NAIVE HEIGHT [H-L-O-P-S], [SU].

(c) THE SMALLEST γ FOR WHICH (9) HOLDS IS THE (MURMURATION) SATURATION EXPONENT OF $\mathcal{F}$. IF $\gamma = 0$ THEN $R = 0$ IN (8), THAT IS NO P-AVERAGING IS NEEDED. A GUESS CONSISTENT WITH CASES WHERE THESE CAN BE COMPUTED IS THAT

$$\gamma + \delta > 1 \qquad \text{(10)}$$

AND THAT FOR $\delta > 1$, $\gamma = 0$.

THE RELATION BETWEEN M_{Φ}'S FOR DIFFERENT Φ'S IS LINEAR AND TAKES THE FORM

$$M_{\Phi}(x) = \int_0^\infty \Phi(t) M\left(\frac{x}{t}\right) t^{\delta} \frac{dt}{t} = x^{\delta} \int_0^\infty \Phi\left(\frac{x}{y}\right) y^{-\delta-1} M(y)\,dy \qquad \text{(11)}$$

IF A FUNCTION $M : (0,\infty) \to \mathbb{R}$ EXISTS SATISFYING (11) FOR ALL Φ'S THEN WE CALL IT THE ZUBRILINA MURMURATION DENSITY OF $\mathcal{F}$, DENOTED $Z_{\mathcal{F}}$. IN GENERAL $Z_{\mathcal{F}}$ MIGHT BE A DISTRIBUTION ON $C_0^\infty(\mathbb{R}_{>0})$, RATHER THAN A FUNCTION {THE RHS OF (11) ALLOWS FOR THIS) AND THIS HAPPENS IN SOME BASIC EXAMPLES.

SO THE MURMURATION FUNCTIONS $M_{\mathcal{F},\Phi}$ AND DENSITY $Z_{\mathcal{F}}$ DESCRIBE THE SUBTLE AND REFINED BEHAVIOR OF THE CORRELATIONS (7') IN THE TRANSITION RANGE $a=1$.

(3) SOME EXPLICIT MURMURATION DENSITIES

THERE ARE PRECIOUSLY FEW FAMILIES FOR WHICH THE 1-LEVEL DENSITY (7') HAS BEEN COMPUTED ANALYTICALLY FOR $a > 1$ ([O-S], [K-S 2], [I-L-S], [F-I]). ALL OF THESE HAVE CONDUCTOR DIMENSION AT LEAST 1. THESE ARE NATURAL CANDIDATES TO EXAMINE FOR A MURMURATION TRANSITION. THE FAMILY $\mathcal{H}$ OF WEIGHT TWO HOLOMORPHIC CUSP FORMS OF SQUARE-FREE LEVEL FAMOUSLY CONTAINS THE SEMI-STABLE E'S IN $\mathcal{E}$, AND IT DISPLAYS A $\gamma = 0$ MURMURATATION IN [H-L-O-P-S] AND [SU]. $\mathcal{H}$ HAS DIMENSION $S(\mathcal{H})$ EQUAL TO 2. NINA'S WORK [ZU] WHICH USES THE EICHLER-SELBERG TRACE FORMULA AS ITS STARTING POINT YIELDS THE EXACT MURMURATION DENSITY Z_H AND IT AGREES PERFECTLY WITH THE NUMERICS.

$$Z_H(y) = A\sqrt{y} + B \sum_{1 \le k \le 2\sqrt{y}} a(k)\sqrt{y - k^2/4} - \pi y. \qquad (12)$$

WHERE $A = \prod_p \left(1 + \frac{p}{(p+1)^2(p-1)}\right)$, $B = \prod_p \left(\frac{p^4 - 2p^2 - p + 1}{(p^2-1)^2}\right)$

AND

$$a(k) = \prod_{p \mid k} \left(1 + \frac{p^2}{p^4 - 2p^2 - p + 1}\right).$$

Z_H IS CONTINUOUS BUT NOT DIFFERENTIABLE AT $y = k^2/4$, $k \in \mathbb{Z}$. NEAR ZERO $Z_H(y) \sim A\sqrt{y}$, WHILE FOR y LARGE Z_H FLUCTUATES BETWEEN $y^{1/4}$ AND $-y^{+1/4}$.

THE SUBFAMILY $\mathcal{E}$ OF $\mathcal{H}$ HAS CONJECTURED DIMENSION 5/6 AND APPEARS TO HAVE A MURMURATION DENSITY $Z_{\mathcal{E}}$. GIVING A HEURISTIC DERIVATION OF $Z_{\mathcal{E}}$ WHICH AGREES WITH THE DATA, IS AN INTERESTING CHALLENGE.

THE SIMPLEST FAMILY OF SELF-DUAL L-FUNCTIONS IS THAT OF QUADRATIC DIRICHLET CHARACTERS $\mathcal{D} = \{ \chi_d \ ; \ d \in \mathcal{D} \text{ FUND. DISCRIMINANT} \}$. IT HAS DIMENSION 1. THE Φ-SMOOTHING, POISSON SUMMATION AND GAUSS RECIPROCITY ALLOW ONE TO EXAMINE (7') FOR $a > 1$ (SEE [IW], [O-S], [K-S2], [So]) USING THIS ONE CAN SHOW THAT (8) AND (9) HOLD WITH ANY $\gamma > 0$ (ASSUMING THAT THERE ARE PRIMES IN SHORT INTERVALS TO AVERAGE)

AND TO COMPUTE THE MURMURATION DENSITY FOR THIS FAMILY (RESTRICTING TO ODD d's). MIKE RUBINSTEIN AND I ARRIVE AT:

$$Z_{\mathcal{D}}(y) = -\frac{1}{2} + \frac{\pi^2}{8} \sum_{n=1}^{\infty} b(n) \sqrt{y} \cos\left(\frac{\pi}{2} n^2 y\right) \qquad (13)$$

WHERE $\quad b(n) = \sum'_{\substack{\alpha | n \\ (\alpha,2)=1}} \frac{\mu(\alpha)}{\alpha}$.

SO IN THIS CASE $Z_{\mathcal{D}}$ IS NOT CONTINUOUS, IT IS NOT A MEASURE EITHER (ACCORDING TO THE IDEMPOTENT THEOREM [CO]). IT IS THE DISTRIBUTIONAL DERIVATIVE OF A FUNCTION. FOR ANY Φ THE CORRESPONDING MURMURATION FUNCTION $M_{\Phi,\mathcal{D}}$ IS SMOOTH AND INTERPOLATES BETWEEN $M_{\Phi}(0) = 0$ AND $M_{\Phi}(\infty) = -\frac{1}{2}$, REFLECTING THE TRANSITION IN (7') BETWEEN $a<1$ AND $a>1$ ($\mathcal{D}$ HAS AN $Sp(\infty)$ SYMMETRY).

ANOTHER FAMILY OF DIMENSION BIGGER THAN 1 IS THAT OF HOLOMORPHIC MODULAR FORMS OF WEIGHT 1. ACCORDING TO [D-S] AND [K-W] THESE CORRESPOND TO FINITE ODD 2-DIMENSIONAL GALOIS REPRESENTATIONS AND THEIR L-FUNCTIONS ARE CLASSICAL ARTIN L-FUNCTIONS. WHILE COUNTING MEMBERS OF THIS FAMILY ACCORDING TO CONDUCTOR IS PROBLEMATIC [SE], IT IS EXPECTED THAT THE MAJORITY OF THESE ARE DIHEDRAL CORRESPONDING TO NON-TRIVIAL CHARACTERS ψ OF THE IDEAL CLASS GROUP $C(D)$ OF THE IMAGINARY QUADRATIC FIELD $\mathbb{Q}(\sqrt{-D})$, $D>3$ $D \equiv 3\,(4)$ AND SQUARE FREE. WE DENOT THIS DIHEDRAL FAMILY BY Δ. THE CLASS NUMBER $h(D)=|C(D)|$ ARE OF SIZE $\sqrt{D}$ AND THE CONDUCTORS $N_\psi = D$, SO THAT THE DIMENSION $\delta(\Delta)$ IS EQUAL TO $3/2$. THE ROOT NUMBERS W_ψ FOR $\psi \in \Delta$ ARE EQUAL TO 1 AND Δ HAS A SYMPLECTIC SYMMETRY [F-I]. ASSUMING THE RIEMANN HYPOTHESIS FOR DIRICHLET L-FUNCTIONS THEY ESTABLISH (7') FOR Δ WITH $1<a<4/3$.

THE ORTHOGONALITY OF CHARACTERS ψ YIELDS THE MURMURATION SUMS FOR Δ:

$$\mathop{\mathrm{A}_{\Phi,N}}_{\psi\in\Delta}\left[a_\psi(p)\right] = \frac{\sqrt{p}}{4}\sum_{D\in\Delta}\left[h(D)-1\right]\Phi\!\left(\frac{D}{N}\right)\left|\left\{x^2+Dy^2=4p\right\}\right|. \tag{13'}$$

FOR NNP THE ASYMPTOTICS IN (13')
CAN BE STUDIED USING THE METHODS IN [F-I]
TO HANDLE THE SUMS

$$\sum_{x^2+dy^2=4p} \Phi\left(\frac{d}{N}\right)$$

AND THOSE IN [ZU] TO DEAL WITH $h(p)$.

THIS SHOULD LEAD TO (14) BEING ASYMPTOTIC
TO A FUNCTION $M_{\Delta,\Phi}(p/N)$ AND ALSO
TO THE CORRESPONDING MURMURATION DENSITY $Z_{\Delta}(y)$.

4) ARITHMETICAL WEIGHTS AND LANDAU SIEGEL ZEROS

FOR L-FUNCTIONS OF SELF-DUAL PGL_2 MODULAR FORMS THERE IS AN ARITHMETICAL WEIGHT THAT CAN BE INCORPORATED IN (1) AND WHICH COMES NATURALLY WHEN APPLYING THE PETERSON OR KUZNETZOV SUMMATION FORMULA (SEE [I-K]).

$$A^{arith}_{\mathcal{F},\Phi,N}[f] := \sum_{\pi\in\mathcal{F}} \Phi\left(\frac{N_\pi}{N}\right)\frac{f(\pi)}{L(1,\mathrm{sym}^2\pi)} \qquad (14)$$

HERE $L(s,\mathrm{sym}^2\pi)$ IS THE (FINITE PART) OF THE SYMMETRIC SQUARE L-FUNCTION ASSOCIATED TO π. IT'S VALUE AT 1 IS POSITIVE AND VARIES MILDLY WITH THE CONDUCTOR N_π.

THE ABOVE SUMMATION FORMULA RENDERS KLOOSTERMAN SUMS IN PLACE OF CLASS NUMBERS (WHICH ARE TO THE TRACE FORMULA) ON THE GEOMETRIC SIDE. THESE LEAD TO SMOOTHER OSCILLATIONS AND CLEANER EXPRESSIONS, HOWEVER IN PRINCIPLE ONE CAN GO BACK AND FORTH BETWEEN THESE DIFFERENT AVERAGING PROCEDURES.

WE FOCUS ON THE FAMILY $\mathcal{H}OL$ OF HOLOMORPHIC CUSP FORMS f OF GROWING WEIGHT AND LEVEL ONE. SUCH AN f HAS EVEN WEIGHT k AN ITS CONDUCTOR $N_f = k^2$ WHILE ITS ROOT NUMBER W_f DEPENDS ONLY ON k; $W_f = (i)^k$.

SINCE $\#\{f \text{ OF WEIGHT } k\} = \frac{k}{12} + O(1)$ IT FOLLOWS THAT $\mathcal{H}OL$ HAS DIMENSION $\delta(\mathcal{H}OL) = 1$.

THIS FAMILY IS STUDIED IN DEPTH IN [I-L-S] WHERE IT SHOWN THAT FOR $W = \pm 1$

$$\underset{f \in \mathcal{H}OL^{W}}{\mathrm{EXP}}{}^{\text{arith}}_{\Phi, N}\left[a_f(p)\right] = W \frac{4\pi p}{N} \sum_{c=1}^{\infty} \frac{S(1,p;c)}{c^2} \Phi\left(\frac{16\pi^2 p}{c^2 N}\right)$$

$$+ \text{SMALL}. \qquad (15)$$

HERE $S(m;n;c) = \sum_{x\bar{x} \equiv 1 (c)} e\left(\frac{mx + n\bar{x}}{c}\right)$ IS THE KLOOSTERMAN SUM.

IT FOLLOWS THAT FOR $p \sim N$ AND AN ARBITRARY SMALL AVERAGING OF p NEAR P THAT WE GET A MURMURATION FUNCTION WITH ANY $\delta > 0$:

$$M^{arith}_{HOL^W, \Phi}(x) = 4\pi W \sum_{c=1}^{\infty} \frac{\mu^2(c)}{c^2 \phi(c)} \Phi\left(\frac{16\pi^2 x}{c^2}\right) \quad (16)$$

WHERE μ AND ϕ ARE MÖBIUS AND EULER'S FUNCTIONS RESPECTIVELY.

THIS GIVES THE MURMURATION DENSITY

$$Z^{arith}_{HOL^W}(y) = 4\pi W \sum_{c=1}^{\infty} \frac{\mu^2(c) c^2}{\phi(c)} \delta_{c^2}(y) \quad (17)$$

WHERE $\delta_\xi(y)$ IS THE DELTA MASS AT ξ.

SO IN THIS CASE THE MURMURATION DENSITY IS A MEASURE AND IT HAS ONE SIGN, NAMELY W. THE CORRESPONDING MURMURATION FUNCTIONS $M^{arith}_{HOL^W, \Phi}(x)$ ARE ONE SIGNED TRANSITIONING FROM 0 AT 0, TO $\frac{W}{2}$ AT INFINITY.

(15) ALLOWS ONE TO AVERAGE OVER p IN (7′) BY OPENING THE KLOOSTERMAN SUM AND SUMMING OVER PRIMES IN ARITHMETIC PROGRESSIONS MOD c. ASSUMING GRH FOR DIRICHLET L-FUNCTIONS THIS ALLOWS ONE TO EXECUTE (7′) FOR $1 < a < 2$.

AN APPLICATION GIVEN IN [I-L-S] IS A COMPUTATION OF THE 1-LEVEL DENSITY OF THE ZEROS OF $L(s,f)$ FOR SUPPORT IN $(-2,2)$. PRECISELY, LET $h \in \mathcal{S}(\mathbb{R})$ THEN THE 1-LEVEL DENSITY OF ZEROS OF $L(s,f)$ IS

$$D(h,f) := \sum_{L(\frac{1}{2}+i\gamma_f, f)=0} h\left(\frac{\log N_f}{2\pi} \gamma_f \right) \qquad \text{——(18)}$$

ASSUMING GRH, FOR SUPPORT $\hat{h} \subset (-2,2)$, AS $N \to \infty$

$$\underset{f \in \mathrm{HOL}^+}{\overset{\text{arith.}}{\mathrm{EXP}}}{}_{\Phi, N} \left[D(h,f) \right] \longrightarrow \int_{-\infty}^{\infty} h(x) \left[1 + \frac{\sin 2\pi x}{2\pi x} \right] dx$$

$$= \int_{-\infty}^{\infty} h(x)\,dx + \frac{1}{2} \int_{-\infty}^{\infty} \hat{h}(\xi)\,d\xi - \frac{1}{2} \int_{|\xi| \geq 1} \hat{h}(\xi)\,d\xi$$

——(19).

THE THIRD TERM IN (19) IS THE MURMURATION TERM. WITH IT ONE CAN OPTIMIZE THE CHOICE OF h AND SHOW THAT

$$\lim_{N\to\infty} \frac{\#\{f\in HOL^{+} : L(\tfrac{1}{2},f)\neq 0,\ N_f\sim N\}}{\#\{f\in HOL^{+} : N_f\sim N\}} > \frac{9}{16} \tag{20}$$

WITHOUT IT THIS OPTIMIZATION YIELDS 1/2 ON THE RHS OF (20). THE SIGNIFICANCE OF BEING BIGGER THAN 1/2 IS THAT IN [I-S] THE LOWER BOUND IN (20) OF 1/2 IS PROVED UNCONDITIONALLY. MOREOVER IT SHOWN THAT (ESSENTIALLY) ANY PROPORTION BIGGER THAN 1/2 IMPLIES THAT THERE ARE NO "LANDAU-SIEGEL" ZEROS FOR MEMBERS OF $\mathcal{D}$ AND EFFECTIVELY SO!

THE MURMURATION TERM IN (19) EXPLAINS THIS PHONOMENON CLEARLY. IF WE ASSUME GRH FOR DIRICHLET L-FUNCTIONS EXCEPT FOR SOME $\chi\in\mathcal{D}$ OF CONDUCTOR b WHICH HAS A "SIEGEL" TYPE ZERO NEAR 1, WE CAN TRACE ITS IMPACT IN THE AVERAGING OVER PRIMES ARITHMETIC PROGRESSIONS TO MODULI $c \doteq b$

AND LARGER. FOR $N = q^A$ THIS LEADS TO THE MODIFICATION OF THE MURMURATION TERM IN (19) YIELDING

$$\int_{-\infty}^{\infty} h(x)\,dx + \frac{1}{2}\int_{-\infty}^{\infty} \hat{h}(\xi)\,d\xi - \frac{1}{2}\int_{1 \le |\xi| \le 1+\frac{1}{A}} \hat{h}(\xi)\,d\xi \qquad (20)$$

SO FOR A LARGE, WHICH IMPACTS CORRESPONDING RANGE $N = q^A$, WE FIND THAT WE REVERT IN (20) BACK TO $1/2$. THIS MECHANICS OF ASSUMING ALL BUT ONE BAD ZERO IN DIAGNOSING SUCH ZERO DENSITIES APPROACHES TO THE LANDAU-SIEGEL ZERO PROBLEM OFTEN CRYSTALIZES WHERE THE KEY ISSUES LIE. FOR HOL, THE PROPORTION OF NONVANISHING AND THE ONE-LEVEL DENSITY HAVE A SHARP RELATION TO LANDAU-SIEGEL ZEROS.

(5) LEARNING THE ROOT NUMBER

CAN THE STATISTICAL CORRELATIONS BETWEEN THE $a_E(p)$'S FOR AN ELLIPTIC CURVE E AND ITS ROOT NUMBER W_E, BE EXPLOITED TO GIVE AN EFFICIENT COMPUTATION OF W_E? BY EFFICIENT WE MEAN POLYNOMIAL TIME IN THE INPUT. GIVEN E IN WEIESTRASS FORM ONE CAN COMPUTE DISCRIMINANT(E) AND FACTOR IT INTO PRIMES. THIS ALLOWS US TO COMPUTE $W_E(p)$ FOR p'S DIVIDING THE DISCRIMINANT AND HENCE THE ROOT NUMBER $W_E = -\prod_{p \mid \mathrm{disc}(E)} W_E(p)$ $\left(W_E(\mathbb{R}) = -1 \right)$,

FOR ANY p, $W_E(p)$ AND $a_E(p)$ CAN BE COMPUTED EFFICIENTLY (FOR THE LATTER THERE IS [SCH]). SO THE PROBLEMATIC STEP IN THE ABOVE IS FACTORING, WHOSE EFFICIENCY HAS RESISTED ALL EFFORTS. THE BEST KNOWN ALGORITHMS FOR THIS TASK ARE PROBABILISTIC IN THEIR RUNNING TIME WHICH IS SUBEXPONENTIAL, THAT IS $N^{o(1)}$ STEPS IN THE CONDUCTOR N.

THE ABOVE SHOWS THAT COMPUTING W_E IS IN NP AND THAT THE BENCHMARK RUNNING TIME IS SUBEXPONENTIAL.

TO SIDESTEP FACTORING ONE CAN COMPUTE $a_E(p)$ FOR $p \in P_X$ A SET OF PRIMES OF CARDINALITY X, WHERE X MUCH SMALLER THAN N, AND USE THESE TO COMPUTE OR AT LEAST PREDICT W_E. WE EXPECT THAT THERE IS NO BIAS FOR E'S HAVING SPECIFIC ROOT NUMBERS, THAT IS $\underset{E \in \mathcal{E}}{\mathrm{PROB}}_{\Phi,N}[W_E = 1]$ SHOULD BE VERY CLOSE TO 1/2. NOTE HOWEVER THAT MARTIN ([MA]) USING THE TRACE FORMULA HAS SHOWN AND HIGHLIGHTED THAT FOR THE LARGER FAMILY $\mathcal{H}$, THERE IS ALWAYS AN EXCESS OF π's WITH $W_\pi = 1$. IN FACT

$$\underset{\pi \in \mathcal{H}}{\mathrm{PROB}}_{\Phi,N}\left[W_\pi = 1\right] = \frac{1}{2} + \left(1+o(1)\right) c_\Phi N^{-1/2},$$

WITH $c_\Phi > 0$. ——— (21)

SUPRISINGLY (TO ME) ACCORDING TO DREW'S COMPUTATIONS FOR SMALL CONDUCTORS

$$\underset{E \in \mathcal{E}}{\mathrm{PROB}}_{[0,N]}\left[W_E = 1\right] \leq \frac{1}{2} - \frac{1}{\sqrt{N}} \qquad (22)$$

WHETHER THIS EXCESS OF ROOT NUMBER -1 FOR ELLIPTIC CURVES IS AN ARTIFACT OF SMALL CONDUCTOR AND EXCESS RANK AND DISSOLVES FOR LARGE N, IS AN INTERESTING QUESTION.

ON THE OTHER HAND THE CORRELATIONS OF W_π AND $a_\pi(p)$ FOR p SMALL (BUT NOT "EQUAL TO 1") ARE APPARENTLY OF THE SAME SIGN FOR $\mathcal{E}$ AND $\mathcal{H}$. THEIR MURMURATION FUNCTIONS (AS WELL AS FOR THE OTHER FAMILIES CONSIDERED BY DREW) VANISH FOR $y=0$ AND ARE OF THE SAME SIGN AS W FOR SMALL y (AS APTLY PUT BY LOWRY-DUDA [L-D] AS FAR AS DREW'S PICTURES "BLUE IS ALWAYS ON TOP --- UNTIL IT'S NOT").

RETURNING TO PREDICTING W_E GIVEN THE $a_E(p)$'s $p \in \mathcal{P}_X$, THE FIRST QUESTION IS TO UNDERSTAND IS THE CONDITIONAL PROBABILITY

$$\underset{E \in \mathcal{E}}{\mathrm{PROB}_{\mathcal{E},N}} \left[W_E = 1 \mid a_E(p),\ p \in \mathcal{P}_X \right] \qquad (23)$$

IF $\mathcal{P}_X = \{p : p \leq X\}$ AND $X \gg (\log N)^2$ THEN AT LEAST UNDER GRH FOR THE RANKIN-SELBERG L-FUNCTIONS $L(s, E \times E')$ FOR ELLIPTIC CURVES E AND E', THE PROBABILITY IN (23) IS 0 OR 1. INDEED AS IS WELL KNOWN UNDER THIS ASSUMPTION, THE $a_E(p)$'s DETERMINE E AND HENCE W_E, ALBEIT WITH A PROCEDURE OF VERY HIGH COMPLEXITY.

SO THE LEARNING CHALLENGE IS WHETHER THERE IS A LOW COMPLEXITY STATISTIC OF THE $a_E(p)$'s, $p \in \mathcal{P}_X$ FOR WHICH THE PROBABILITY IN (21) GOES TO 0 OR 1 AS $N \to \infty$. AS DREW NOTES CHOOSING THE p's NEAR THE PEAKS OF THE MURMURATION FUNCTIONS $M_{\mathcal{E},\Phi}$ MIGHT HAVE SOME ADVANTAGES OVER THE INITIAL SEGMENT.

IN ONGOING WORK WITH RUBINSTEIN ([R-S]) WE GIVE SUBEPONENTIAL TIME ALGORITHM TO COMPUTE W_E (WITHOUT FACTORING N_E). THE $a_E(p)$'s WITH $p \leq X$ AND $X \sim N^{o(1)}$ ARE COMPUTED AND PUT INTO MOMENT SUMS FOR THE THE EXPLICIT FORMULA GIVING MOMENT SUMS FOR THE ZEROS OF $L(s,E)$. WHILE THIS IS AN UNDERDETERMINED MOMENT PROBLEM WE ARE ABLE TO GIVE AN EXACT COUNT FOR THE NUMBER OF ZEROS IN CERTAIN INTERVALS. IN PARTICULAR THE PARITY OF THE NUMBER OF ZEROS IN SUCH INTERVALS UNCOVERS W_E. THE ALGORITHM GIVES THE CORRECT ANSWER WHEN IT TERMINATES, WHAT IS PROBABILISTIC IS THE RUNNING TIME WHICH IS DICTATED BY THE n-LEVEL DENSITY CONJECTURES ([K-S]) FOR THE FAMILY $\mathcal{E}$. THIS PASSAGE THROUGH THE ZEROS HAS ITS LIMITATIONS AS THE VERY STRONG SZEGO LIMIT THEOREMS FOR THE UNDERDERMINED MOMENT PROBLEM ([L-J]) LIMIT THIS APPROACH TO GIVING LITTLE MORE THAN SUBEXPONENTIAL.

BOOKER, HIARY AND KEATING [B-H-K] GIVE A SUBEXPONENTIAL TIME ALGORITHM TO DETERMINE IF A LARGE INTEGER N IS SQUARE-FREE. LIKE THE ABOVE, IT DOES SO WITHOUT FACTORING N, APPEALING INSTEAD TO THE EXPLICIT FORMULA FOR DIRICHLET L-FUNCTIONS AND THEIR ZEROS AND RELATED RANDOM MATRIX THEORY.

WITH BEST REGARDS

PETER S.

REFERENCES

[B-H-K] A. BOOKER, G. HIARY, J. KEATING; DUKE MATH JNL. 164 (2015) 235-275.

[B-M-S-W] B. BEKTEMIROV, B. MAZUR, W. STEIN, M. WATKINS BAMS 44 (2007) 233-254.

[B-H-K-S-S-W] J. BALAKRISHAN, W. HO, N. KAPLAN, S. SPICER, W. STEIN, J. WEIGANDT LMS. J. COMPUT. MATH 19 (2016) 351-370.

[CO] P. COHEN AM. JOUR. MATH 82 (1960) 191-212.

[D-S] P. DELIGNE AND J.P. SERRE ANN. ECOL NORM SUP 7 (1974) 507-530

[D-D] T. DOKCHISTER, V. DOKCHISTER, ANNALS 172 (2010) 567-596.

[F-I] E. FOUVRY, H. IWANIEC; DMJ VOL 116 (2) 189-217 2001

[H-L-O-P] Y. HE, K. LEE, T. OLIVER, A. POZDNYAKOV arXiv 2204.10140 (2022).

[H-L-O-P-S] Y. HE, Y. LEE, T. OLIVER, A. POZDNYAKOV, A. SUTHERLAND ; IN PREPARATION.

[IW] H. IWANIEC SEM. THEOR. NOM. BORDEAUX 2 (1990) 365-376.

[I-K] H. IWANIEC AND E. KOWALSKI AMS COLLOQ. PUBL. 53 2004.

[I-L-S] H. IWANIEC, W. LUO, P. SARNAK. PUBL. I.H.E.S 91 (2000) 55-131.

[I-S] H. IWANIEC, P. SARNAK ISRAEL MATH JNL. 120 (2000) 155-177.

[J-L] K. JOHANSSON, G. LAMBURT ANN. PROB 49 (2021) 2961-3010

[K-S] N. KATZ, P. SARNAK BAMS 36 (1999) 1-26

[K-S2] N. KATZ, P. SARNAK APPENDIX TO ''

[K-W] C. KHARE AND J.P. WINTENBERGER, ANNAL 169 (2009) 222-253.

[L-D] D. LOWRY-DUDA //DAVIDLOWRYDUDA.COM/MAASS-MURMURATIONS/

[LMFDB] www.lmfdb.org

[MA] K. MARTIN JNL OF NUMBER THEORY 188 (2018) 1-17.

[O-S] A. OZLUK AND C. SNYDER ACTA ARITH. XCI.3 (1999) 209-228

[SA] P. SARNAK "DEFINITION OF FAMILIES OF L-FUNCTIONS" PUBLICATIONS.IAS.EDU/SARNAK/PAPER/507

[SCH] R. SCHOOF MATH COMP 44 (1985) 483-494.

[SE] J.P. SERRE ; IN ALG. NUMBER FIELDS "L-FUNCTIONS AND GALOIS PROPERTIES" DURHAM 1975.

[SO] K. SOUNDARARAJAN ANNALS VOL 152, 447-488 (2000)

[SU] A. SUTHERLAND, MATH.MIT.EDU/~DREW/RUBINSTEIN-SARNAKLETTER.PDF

[T-S-S] N. TEMPLIER, S.W. SHIN, P. SARNAK SIMONS SYMP SPRINGER 2016 531-578

[ZU] N. ZUBRILINA ; "MURMURATIONS" IN PREPARATION.