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SURVEY OF DRINFEL'D MODULES

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INTRODUCTION

In Deligne [1971], two-dimensional ℓ-adic representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ are attached to "new" holomorphic modular forms of weight at least two on the Poincaré upper-half plane, or equivalently, to certain automorphic representations of the adèle group $GL_2(\mathbb{A}_{\mathbb{Q}})$. The correspondence preserves L-functions. This theory depends on the properties of moduli varieties of elliptic curves with given level structure. These varieties have a canonical structure over $\mathbb{Q}$ and the ℓ-adic representations are realized in the ℓ-adic H^1 of certain sheaves.

Drinfel'd [1973] transports the theory to the function field case by introducing the concept of elliptic module, which we call a Drinfel'd module, to replace elliptic curves.

Fixed notations throughout the article

C – absolutely irreducible projective and smooth curve over $\mathbb{F}_q$.

∞ – a closed point of C.

F – the function field $\mathbb{F}_q(C)$ of C over $\mathbb{F}_q$.

A – the ring $H^0(C-\infty,\mathcal{O}_C)$ of functions regular on $C-\infty$.

F_∞ – the completion of F at ∞ with valuation ring $\mathcal{O}_\infty$.

C_∞ – the completion of the algebraic closure of F_∞.

$\mathbb{F}_q$ – the finite field of q elements.

Relative to the function field F over $\mathbb{F}_q$, we will define Drinfel'd modules of rank r in the first chapter. Briefly, these are $A = H^0(C-\infty,\mathcal{O}_C)$ module structures on the additive group in characteristic p given by polynomials in Frobenius whose degree is a certain multiple of the rank r. The term elliptic module, which is Drinfel'd's original term, is used for Drinfel'd modules of rank 2 for these are objects which correspond closely to elliptic curves. In fact, we have the following dictionary:

elliptic curve	elliptic module (rank 2 Drinfel'd module)
$\mathbb{Q}$	$\mathbb{F}_q(C) = F$
infinite place	fixed place
$\mathbb{Z}$	$A = H^0(C-\infty, \mathcal{O}_C)$
scheme	scheme over A
n division point	I division point for I an ideal of A
n level structure	I level structure
moduli space	moduli space
lattice in $\mathbb{C}$	discrete A-modules

Most of the above dictionary is explained in chapter 1. Elliptic curves over the complex numbers $\mathbb{C}$ can be interpreted as classes of certain lattices in $\mathbb{C}$. In chapter 2 we describe Drinfel'd modules over C_∞ in terms of discrete A-modules in C_∞ defined over A. Drinfel'd modules defined over $\bar{F}_\infty$, the algebraic closure of F_∞, can be also described by lattices in $\bar{F}_\infty$, and the lattices with Galois invariance properties correspond to Drinfel'd modules over intermediate fields between F_∞ and $\bar{F}_\infty$.

For elliptic curves, indeed for more generally polarized abelian varieties, and for Drinfel'd modules there are moduli problems for families of these objects and moduli spaces. From the point of view of Shimura varieties, basic information about such moduli problems is collected in triples (F,G,h') consisting of:

- a reductive group G over a global field F,
- a conjugacy class of maps $h': G_m \longrightarrow G$.

The above dictionary extends further with the following examples of triples (F,G,h') for the corresponding moduli problems over F:

polarized abelian varieties (rank g) $\left(\mathbb{Q}, CSp_{2g}, \begin{pmatrix} \lambda I_g & 0 \\ 0 & I_g \end{pmatrix}\right)$

elliptic curves $\left(\mathbb{Q}, GL_2, \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\right)$

Drinfel'd modules (rank r) $\left(F, GL_r, \begin{pmatrix} \lambda I_1 & 0 \\ 0 & I_{r-1} \end{pmatrix}\right)$

elliptic module $\left(F, GL_2, \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\right)$

A critical step in the theory over $\mathbf{Q}$ was the calculation of the cohomology of the complex analytic variety of $\mathbf{C}$-valued points on the moduli scheme and the comparison with the ℓ-adic cohomology. In the function field case Drinfel'd puts a rigid analytic structure on the C_∞-valued points of the moduli scheme, calculates a rigid analytic H^1 for certain simple sheaves, and compares this H^1 with the ℓ-adic H^1. This rigid analytic space is the quotient of a C_∞-analogue of the Poincaré upper-half plane by a discrete group Γ, and the cohomology is in the middle of a short exact sequence with subgroup $H^1(\Gamma)$ and quotient the space of coclosed 1-cochains on the tree of $PGL(2,C_\infty)$ invariant under Γ. This in turn is interpreted in terms of representations. Finally, the various representations in H^1 are sorted out with a congruence formula which is analogous to the Eichler-Shimura congruence formula. These considerations are carried out in chapters 3 and 4.

In chapter 5 the global results on automorphic forms for $GL(2,\mathbb{A}_F)$ are applied to a local result conjectured by Langlands. For a local field K of equal characteristic p, there is a natural bijection between irreducible admissible representations of $GL(2,K)$ and two-dimensional representations of the Weil group $W(\bar{K}/K)$. The global theory of automorphic forms given by elliptic modules applies only to representations $\pi = \otimes_{v \in C} \pi_v$ where π_∞ is the special representation.

In the form of private letters Drinfel'd has a new theory of "shtuka" which handles all automorphic forms on $GL(1)$ and $GL(2)$ over a function field. Even so it seems that elliptic modules are still worth consideration because the theory for $r = 1$ and $r = 2$ may extend easier for general r than the corresponding theory of "shtuka" which depends on the Selberg trace formula. Another method to relate automorphic forms to ℓ-adic representations is in Drinfel'd [1983]. Recently, elliptic modules were used to describe modular forms in the function field case, see D. Goss [1977].

This article grew out of lectures by P. Deligne at the I.H.E.S. in the Winter-Spring of 1975 and a lecture in Bonn during the period of Karneval, 1975.

CHAPTER 1. ALGEBRAIC THEORY OF DRINFEL'D MODULES

In terms of the notations in the introduction, a Drinfel'd module is an A-module structure on the additive group G_a in characteristic p. There is a theory of division points and isogenies of Drinfel'd modules which parallels closely the corresponding theory for abelian varieties. Drinfel'd modules have a characteristic v, which is a valuation of $F = \mathbb{F}_q(C)$ over $\mathbb{F}_q$, and they exhibit a singular and sometimes supersingular behavior at v when $v \neq \infty$. Again, we are reminded of elliptic curves in characteristic p.

Drinfel'd formulated and solved a moduli problem for Drinfel'd modules with prescribed level structure. The constructions take place entirely with affine schemes making the existence of the moduli elementary compared with the existence question of the moduli scheme for elliptic curves with level structure. We introduce Drinfel'd's modification of the notion of level structure which allows for a clearer analysis of level structures at singular and supersingular points for the cases of both Drinfel'd modules and elliptic curves. See additional remarks 1.

§1. ENDOMORPHISMS OF THE ADDITIVE GROUP

The additive group (functor) G_a over a ring R is represented by the polynomial ring $R[X]$ in one variable with structure morphism $\alpha: R[X] \longrightarrow R[X] \otimes_R R[X]$ given by $\alpha(X) = X \otimes 1 + 1 \otimes X$. A morphism $\phi: G_a \longrightarrow G_a$ of the underlying schemes over R is given by a polynomial $\phi(X) \in R[X]$ where $\phi^*(a(X)) = a(\phi(X))$ for $a(X) \in R[X]$, and this morphism ϕ preserves the group scheme structure if and only if $\phi(X)$ is additive, i.e. $\phi(X+Y) = \phi(X) + \phi(Y)$. Composition $\phi\psi$ of morphisms ϕ and ψ is represented by substitution $\phi(\psi(X))$, and the sum of two morphisms $G_a \longrightarrow G_a$ is represented by the sum of two polynomials.

For example, $\phi(X) = aX^{p^i}$ is additive where p is a prime number and $pa = 0$ in R since the binomial coefficient $\binom{p^i}{m}$ is divisible by p for $0 < m < p^i$. For a field k of characteristic 0 the only additive polynomials are of the form cX, $c \in k$, but for a field k of characteristic $p > 0$ the additive polynomials are easily seen to be of the form

$$\phi(X) = \sum_{0\leqslant i} a_i X^{p^i} = a_0X + a_1X^p + a_2X^{p^2} + \cdots + a_nX^{p^n} \quad .$$

If $\psi(X)$ is a second additive polynomial, then

$$\psi(\phi(X)) = \psi(a_0X) + \psi(a_1X^p) + \cdots + \psi(a_nX^{p^n})$$

is also additive.

(1.1) DEFINITION. Let k be a field of characteristic $p > 0$. The twisted polynomial ring $k\{\tau\}$ is $k \otimes_{\mathbf{Z}} \mathbf{Z}[\tau]$ with the twisted tensor product algebra structure satisfying the commutation rule

$$(a \otimes \tau^i)(b \otimes \tau^j) = a(b)^{p^i} \otimes \tau^{i+j} \quad .$$

We denote $a \otimes \tau^i$ simply by $a\tau^i$ in $k\{\tau\}$, and then the commutation rule becomes $\tau a = a^p\tau$.

Let k denote a field of characteristic $p > 0$ in the remainder of this section.

(1.2) PROPOSITION. The function θ which assigns to an additive polynomial $\phi(X) = a_0X + a_1X^p + \cdots + a_nX^{p^n}$ the element $\tilde{\phi}(X) = a_0 + a_1\tau + \cdots + a_n\tau^n \in k\{\tau\}$ is an isomorphism $\theta\colon \mathrm{End}_k(G_a) \longrightarrow k\{\tau\}$ of rings.

The proof follows from the observation that the relation $(aX^{p^i})^{p^j} = a^{p^j}X^{p^{i+j}}$ becomes $\tau^j a\tau^i = a^{p^j}\tau^{i+j}$ under θ. Note that the multiplicative structure on $\mathrm{End}_k(G_a)$ is substitution of additive polynomials and on $k\{\tau\}$ the twisted polynomial multiplication.

We have two degree functions $\deg\colon \mathrm{End}_k(G_a) \longrightarrow \mathbf{Z}$ and $d\colon k\{\tau\} \longrightarrow \mathbf{Z}$ defined by the relations $\deg(a_0X + a_1X^p + \cdots + a_nX^{p^n}) = p^n$ and $d(a_0 + a_1\tau + \cdots + a_n\tau^n) = n$ where $a_n \neq 0$. The following relations hold for $\phi,\psi \in \mathrm{End}_k(G_a)$ and $a,b \in k\{\tau\}$

$$\deg(\phi\psi) = \deg(\phi)\deg(\psi) \quad , \quad \deg(\phi + \psi) \leqslant \max(\deg(\phi),\deg(\psi))$$

and

$$d(ab) = d(a) + d(b) \quad , \quad d(a+b) \leqslant \max(d(a),d(b)) \quad .$$

We have height functions $\mathrm{ht}\colon \mathrm{End}_k(G_a) \longrightarrow \mathbf{Z}$ and $\mathrm{ht}\colon k\{\tau\} \longrightarrow \mathbf{Z}$ defined by the relations $\mathrm{ht}(a_hX^{p^h} + \cdots + a_nX^{p^n}) = h$ and $\mathrm{ht}(a_h\tau^h + \cdots + a_n\tau^n) = h$ where $a_h \neq 0$. Clearly, $\mathrm{ht}(\phi) = \mathrm{ht}(\theta(\phi))$. The following relations hold $\mathrm{ht}(\phi\psi) = \mathrm{ht}(\phi) + \mathrm{ht}(\psi)$ and $\mathrm{ht}(ab) = \mathrm{ht}(a) + \mathrm{ht}(b)$ for $\phi,\psi \in \mathrm{End}_k(G_a)$ and $a,b \in k\{\tau\}$.

We have substitutions $\partial_0\colon \mathrm{End}_k(G_a) \longrightarrow k$ and $\partial\colon k\{\tau\} \longrightarrow k$ defined by the relations

$$\partial_0\left(\sum a_i X^{p^i}\right) = a_0 \qquad \text{and} \qquad \partial\left(\sum a_i \tau^i\right) = a_0$$

where ∂_0 is the derivative at the origin and ∂ the value at the origin. Clearly $\partial(\theta(\phi)) = \partial_0(\phi)$ for $\phi \in \mathrm{End}_k(G_a)$.

Finally, the following properties of additive polynomials seem to have been known for some time.

(1.3) PROPOSITION. Let $H \subset G_a(k)$ be a finite subgroup, and form the polynomial $P_H(X) = \prod_{h\in H}(X-h)$. Then $P_H(X)$ is an additive polynomial, so $P_H \in \mathrm{End}_k(G_a)$ with $\deg(P_H) = \mathrm{Card}(H)$, and we are able to recover $H = \ker(P_H)(k)$, the set of all $x \in k$ with $P_H(x) = 0$.

PROOF. To show that P_H is additive, consider $Q_Y(X) = P_H(X+Y) - P_H(Y)$ in $k(Y)[X]$. Since H is a subgroup, $Q_Y = 0$ on H and further $\deg(Q_Y) = \deg(P_H) = \mathrm{Card}(H)$. Thus $P_H(X)$ and $Q_Y(X)$ are monic polynomials of the same degree equal to $\mathrm{Card}(H)$ and each equal to zero on $H \subset k \subset k(Y)$. It follows that $P_H(X) = Q_Y(X) = P_H(X+Y) - P_H(Y)$, i.e. P_H is additive, and the other statements hold which proves the proposition.

(1.4) REMARK. As a kind of converse of (1.3) observe that for an additive polynomial $f\colon G_a \longrightarrow G_a$ over k, the set H of $x \in k$ with $f(x) = 0$ is a subgroup of $G_a(k)$ with $\mathrm{Card}(H)$ dividing $\deg(f)$. If k is algebraically closed, then $\deg(f) = p^{ht(f)}\cdot\mathrm{Card}(H)$, and that the group morphism $f\colon G_a(k) \longrightarrow G_a(k)$ is surjective for $f \neq 0$. Since $ht(f) = 0$ if and only if $\partial_0(f) \neq 0$, we see that $\partial_0(f) \neq 0$ implies that $\deg(f) = \mathrm{Card}(\ker(f))$ over an algebraically closed k.

§2. DEFINITION OF DRINFEL'D MODULE OVER A FIELD

We return to the basic notations of the introduction, in particular $A = H^0(C-\infty, \mathcal{O}_C)$, and let k denote a field of characteristic p. Before the definition we make some remarks which point out the natural limitations of the definition.

(2.1) REMARK. Since $\mathrm{End}_k(G_a)$ isomorphic to $k\{\tau\}$ is an integral domain, and since A is a Dedekind domain, any ring morphism $A \longrightarrow \mathrm{End}_k(G_a)$ is either injective or has image contained in the constants $k \subset k\{\tau\}$. The second case being relatively trivial means that we will be interested in morphisms which are injective (or faithful).

To the rational point ∞, we have associated an absolute value $|x|_\infty$ on the function field F with $\mathbb{F}_{q_\infty}$ as residue class field with q_∞ a power of q and the absolute value normalized such that $|a|_\infty = \mathrm{Card}(A/a)$ for $a \in A \subset F$.

(2.2) REMARK. For an injective $\phi: A \to \mathrm{End}_k(G_a)$ the composite $A \xrightarrow{\phi} \mathrm{End}_k(G_a) \xrightarrow{\deg} \mathbb{Z}$ denoted $\|a\| = \deg(\phi(a))$ satisfies $\|ab\| = \|a\|\|b\|$ and $\|a+b\| \leqslant \max(\|a\|,\|b\|)$ from properties of $\deg$. Since $\|a\| \geqslant 1$ for $a \neq 0$ and for some a, $\|a\| > 1$, the relation $\|a/b\| = \|a\|/\|b\|$ for $a/b \in F$ defines an extension of $\|a\|$ to F as an absolute value. Since $\|a\| \geqslant 1$ for all $a \in A$, $a \neq 0$, the absolute value $\|x\|$ on F must be equivalent to $|x|_\infty$, i.e. $\|x\| = |x|_\infty^r$ for some $r > 0$ and all $x \in F$.

In fact, $r > 0$ is a natural number, but we defer the proof of this until the next section, see (3.3), and introduce the following basic concept which Drinfel'd called an elliptic module.

(2.3) DEFINITION. For a natural number $r > 0$, a Drinfel'd module over a field k of rank r for the pointed curve (C,∞) over $\mathbb{F}_q$ is a morphism of rings

$$\phi: A \longrightarrow \mathrm{End}_k(G_a)$$

such that G is isomorphic to G_a and $\|a\| = \deg(\phi(a)) = |a|_\infty^r = (\mathrm{Card}(A/a))^r$ for all nonzero $a \in A$.

Let Alg_k denote the category of commutative k-algebras R, and let Mod_A denote the category of A-modules. To a Drinfel'd module ϕ, we associate a functor

$$E: \mathrm{Alg}_k \longrightarrow \mathrm{Mod}_A$$

which assigns to a k-algebra R its underlying additive group $G_a(R)$ together with the A-module structure defined by requiring $\phi(a)(R)$ to be scalar multiplication by a on $G_a(R)$. Observe that the functor E determines ϕ, and frequently we refer to E as the Drinfel'd module instead of ϕ. We will denote $\phi(a)$ by just ϕ_a occasionally.

(2.4) DEFINITION. Let $\phi: A \to \mathrm{End}_k(G)$ and $\phi': A \to \mathrm{End}_k(G')$ be two Drinfel'd modules. A morphism $u: \phi \to \phi'$ is a morphism $u: G \to G'$ such that $\phi'_a u = u\phi_a$ for all $a \in A$. A nonzero morphism is called an isogeny.

Equivalently, u is a morphism of functors $E \to E'$ associated to ϕ and ϕ'. Thus Drinfel'd modules over k form a category with the composition of morphisms being evident.

(2.5) DEFINITION. Let $\phi: A \to \mathrm{End}_k(G)$ be a Drinfel'd module and form $\partial_0\phi: A \to k$ where $\partial_0: \mathrm{End}_k(G) \to k$ is the value of the derivative at the origin. The Drinfel'd module ϕ has characteristic ∞ provided $\partial_0\phi: A \to k$ is injective and has characteristic v, a valuation of F over $\mathbb{F}_q$ different from ∞, provided $A \cap m_v = \ker(\partial_0\phi)$.

Since $\ker(\partial_0\phi)$ is either zero or a maximal ideal of A, every Drinfel'd module has characteristic ∞ or some "finite" $v \neq \infty$.

(2.6) REMARK. If $u: \phi \to \phi'$ is a nonzero morphism (isogeny) between two Drinfel'd modules, then ϕ and ϕ' have the same rank and the same characteristic.

(2.7) REMARK. The definition (2.3) of a Drinfel'd module can be formulated in terms of a $(A, \mathrm{End}_k(G_a))$-bimodule N which is free of rank 1 over $\mathrm{End}_k(G_a)$ and satisfying the condition $\|a\| = (\mathrm{Card}(A/a))^r$ for $a \in A$, $a \neq 0$. The choice of a basis element for N identifies N with $\mathrm{End}_k(G_a)$.

§3. DIVISION POINTS

As with abelian varieties, division points of a Drinfel'd module play a basic role in structure and moduli problems. We continue with the notation $E: \mathrm{Alg}_k \to \mathrm{Mod}_A$ equal to the functor associated to a ring morphism $\phi: A \to \mathrm{End}_k(G_a)$.

(3.1) DEFINITION. For $a \in A$ (usually $a \neq 0$) the subfunctor $E_a \subset E$ of a-division points is $\ker(\phi_a)$. For an ideal $I \subset A$ the subfunctor $E_I \subset E$ of I-division points is $\bigcap_{a \in I} E_a$.

If $a_1, \ldots, a_r$ generate an ideal I in A, then $E_I = E_{a_1} \cap \cdots \cap E_{a_r}$. So in particular for a principal ideal (b) we have $E_{(b)} = E_b$. Since $E_a = E$ if and only if $a = 0$, the same holds for ideal $E_I = E$ if and only if $I = 0$, and essentially we consider E_I only for nonzero ideals I.

More explicitly, for any k-algebra R, the A-submodule $E_I(R)$ consists of all $x \in E(R)$ such that $ax = 0$ for any $a \in I$. This shows that we can view E_I as a functor defined $E_I: \mathrm{Alg}_k \to \mathrm{Mod}_{A/I}$.

Again as with abelian varieties, $E_I(\bar{k})$ will be a free A/I-module for I prime to the characteristic of E. For the case characteristic ∞ this is no restriction on I and for characteristic v corresponding to the maximal ideal P_v in A it means $I \subset P_v$. In order to prove this assertion we will use the following lemma on torsion modules over a discrete valuation ring V with local uniformizing parameter π. For a V-module L let $\ell(L)$ denote its length and $\pi_L : L \to L$ the action of the scalar π on L.

(3.2) LEMMA. Let L be a V/π^{2m}-module.

(a) We have $2\ell(\ker(\pi_L^m)) \geq \ell(L)$.

(b) For nonzero L, the equality holds in (a) if and only if L is a free V/π^{2m}-module, and in this case $\ker(\pi_L^m)$ is a free V/π^m-module.

PROOF. We can decompose L as the sum of modules N isomorphic to V/π^i with $0 < i \leq 2m$. For $N = V/\pi^i$ $(0 < i \leq 2m)$ observe that

$$\ker(\pi_L^m) = \begin{cases} V/\pi^i & \text{of length } i \text{ for } i \leq m \\ \pi^{i-m}(V/\pi^i) & \text{of length } m \text{ for } m \leq i . \end{cases}$$

In both cases $2\ell(\ker(\pi_N^m)) = 2\cdot\inf(i,m) \geq i = \ell(N)$ so that (a) holds. Also N is free, i.e. $i = 2m$ if and only if $2\ell(\ker(\pi_N^m)) = \ell(N)$, and $\pi^m(V/\pi^{2m})$ is isomorphic to V/π^m as V/π^m-modules which proves (b).

In the following basic structure theorem for $E_I(\bar{k})$ over the algebraic closure $\bar{k}$ of k, we also settle the question left at the end of (2.2). Note that the above definitions apply to any morphism $\phi : A \to \mathrm{End}_k(G_a)$ of rings.

(3.3) THEOREM. If $\phi : A \to \mathrm{End}_k(G_a)$ is a monomorphism of rings, then ϕ is a Drinfel'd module of rank $r > 0$ for an integer r. Moreover, for an ideal I relatively prime to the characteristic of ϕ, the A/I-module $E_I(\bar{k})$ is free of rank r.

PROOF. By (2.2) there exists a real number $r > 0$ with $\deg \phi_a = \mathrm{Card}(A/a)^r$ for all $a \in A$, $a \neq 0$. If $\partial_0 \phi_a = 0$, then by (1.4) $\deg \phi_a = \mathrm{Card}\, E_a(\bar{k})$ and also

$$\mathrm{Card}\, E_{a^2}(\bar{k}) = \deg(\phi_{a^2}) = \deg(\phi_a)^2 = \mathrm{Card}\, E_a(\bar{k})^2 .$$

For each irreducible element π of A prime to the characteristic of ϕ, which is a local uniformizing parameter of $A_p = V$, we apply lemma (3.2) to the

V/π^{2m}-module $E_{\pi^{2m}}(\bar{k})$ to prove that $E_{\pi^m}(\bar{k})$ is a free A/π^m-module of rank d where

$$\mathrm{Card}(E_{\pi^m}(\bar{k})) = \deg(\phi_{\pi^m}) = \mathrm{Card}(A/\pi^m)^r .$$

From this we deduce that r is an integer and so ϕ is a Drinfel'd module of rank r.

Next, for any nonzero $a \in A$ the primary components of $E_a(\bar{k})$ are of the form $E_{\pi^m}(\bar{k})$, so free A/π^m-module of rank r, from which we deduce that $E_a(\bar{k})$ is a free A/a-module of rank r. Finally, if I is an ideal of A prime to the characteristic of ϕ, then there exists another ideal J in A with $A = I+J$ and $IJ = (a)$, a principal ideal prime to the characteristic. Then $A/a = A/I \oplus A/J$, and $E_a = E_I \oplus E_J$ as functors. Since $E_a(\bar{k})$ is free of rank r over A/a, it follows that $E_I(\bar{k})$ is free of rank r over A/I, and this proves the theorem.

(3.4) REMARK. It remains only to consider for a Drinfel'd module E of rank r and characteristic $v \neq \infty$, the A/π^n-modules $E_{\pi^n}(\bar{k})$ where $v(\pi) = 1$. In this case $\partial_0 \phi_{\pi^i} = 0$ for all i, and by the discussion in (1.4), the endomorphism ϕ_π has a height h where $\deg(\phi_\pi) = p^h \cdot \mathrm{Card}\, E_\pi(\bar{k})$, or more generally

$$\deg(\phi_{\pi^n}) = (\deg(\phi_\pi))^n = (p^h \cdot \mathrm{Card}\, E_\pi(\bar{k}))^n = p^{nh} \cdot \mathrm{Card}\, E_{\pi^n}(\bar{k}) .$$

Hence again by applying lemma (3.2), we see that $E_{\pi^n}(\bar{k})$ is a free module of rank $r-h < r$ over A/π^n. As finite group schemes over k, we have a splitting when k is separable

$$E_{\pi^i} = E^0_{\pi^i} \times E^{et}_{\pi^i}$$

where $E^{et}_{\pi^i}$ is étale of rank $r-h$ and $E^0_{\pi^i}$ is infinitesimal of rank h.

(3.5) DEFINITION. The height of a Drinfel'd module E of characteristic $v \neq \infty$ is the height of ϕ_π where π is an irreducible with $v(\pi) = 1$.

Since (π) is uniquely determined by v, the height is well defined.

(3.6) REMARK. Let E be a Drinfel'd module of rank r and characteristic v. Let h be the height when $v = \infty$. For an irreducible π of A, let A_π be A localized at π. Then $\varinjlim_n E_{\pi^n}(\bar{k})$ is a divisible A_π-module which is isomorphic to

$$\begin{cases} (F/A_\pi)^r & \text{for } \pi \text{ prime to the characteristic,} \\ (F/A_\pi)^{r-h} & \text{for } v(\pi) = 1 \ . \end{cases}$$

§4. ISOGENIES

Recall that for a Drinfel'd A-module (ϕ, E) over k and any field extension k^* of k, the group $E(k^*)$ has a given A-module structure. For two Drinfel'd A-modules over k and $u \in k\{\tau\}$ with $u \neq 0$, the additive $u: \phi \to \phi'$ is an isogeny if and only if the additive $u: E(\bar{k}) \to E'(\bar{k})$ is A-linear. This follows from the fact that two polynomials are equal if and only if their associated polynomial maps $\bar{k} \to \bar{k}$ are equal. The kernel of $u: E(k^*) \to E'(k^*)$ is the A-module $\ker(u)(k^*)$ where $\ker(u)$ is the scheme kernel of $u: \phi \to \phi'$.

(4.1) Separable isogenies. An isogeny $u: \phi \to \phi'$ is separable when $u(X) = b_0X + b_1X^p + \cdots + b_sX^{p^s}$ where $b_0 \neq 0$. Then u is determined by $\ker(u)(\bar{k})$ a finite A-submodule of $E(\bar{k})$ up to constant factor. For a finite A-submodule $H \subset E(\bar{k})$, we form $P_H(X) = \prod_{h\in H}(X-h)$, and there is a unique Drinfel'd A-module ϕ' such that $\phi'_a P_H = P_H \phi_a$ for all $a \in A$.

If $a \cdot \ker(u)(\bar{k}) = 0$ for $a \neq 0$, then we can form the isogeny v with kernel $u(\ker(\phi_a)(\bar{k}))$ and $vu = \phi_a$. Note that such a nonzero $a \in A$ with $a \cdot \ker(u)(\bar{k}) = 0$ always exists for $u \neq 0$ since it is a finite A-module. When u is separable, we can always choose v to be separable too. By the same construction, if $u: \phi \to \phi'$ and $w: \phi \to \phi''$ are separable isogenies such that $w(\ker(u)(\bar{k})) = 0$, then there is a separable isogeny $v: \phi' \to \phi''$ with $w = vu$ where $\ker(v)(\bar{k}) = u(\ker(w)(\bar{k}))$.

(4.2) Purely inseparable isogenies. These are of the form $\tau^i \in k\{\tau\}$ or $X \mapsto X^{p^i}$. Since $\phi_a\tau^i = \tau^i\phi_a$ in $k\{\tau\}$ is equivalent to all coefficients of ϕ_a being in $\mathbb{F}_{p^i}$, it follows that a purely inseparable isogeny exists only when the characteristic v of ϕ is unequal to ∞ , and in this case p^i is a power of q_v where q_v is the cardinality of the residue class field of v . The finite group scheme $\ker(\tau^i) = \operatorname{Spec}(k[t]/(t^{p^i}))$. This case corresponds to a purely local kernel.

(4.3) REMARKS. Let H be a finite subgroup scheme of G_a , and let ϕ be a Drinfel'd module structure on G_a . Then H is the kernel of some isogeny $u: \phi \to \phi'$ if and only if H is stable under the action of A and

$$H^{\mathrm{loc}} = \begin{cases} 0 & \text{if the characteristic} = \infty \\ \mathrm{Spec}(k[t]/(t^{q^h})) & \text{if the characteristic} = v \text{ and} \\ & q = q_v = \mathrm{Card}(F(v)) \ . \end{cases}$$

For any isogeny $u\colon \phi \to \phi'$ there exists an isogeny $v\colon \phi' \to \phi$ and $a \in A$ with $\phi_a = vu$.

Let $\delta(u)$ denote the degree of the additive polynomial $u \in \mathrm{End}_k(E)$. The function $\delta\colon \mathrm{End}_k(E) \to \mathbb{Z}$ also prolongs under extension of scalars to

$$\begin{array}{ccccc} \mathrm{End}_k(E) & \longrightarrow & \mathrm{End}_k(E) \otimes_A F & \longrightarrow & \mathrm{End}_k(E) \otimes_A F \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q} \end{array} \ .$$

(4.4) REMARKS. We study the A-algebra $\mathrm{End}_k(E)$ with δ using the following properties of $\delta\colon \mathrm{End}_k(E) \otimes_A F_\infty \to \mathbb{Q}$:

(1) $\delta(u) \geqslant 0$ and $\delta(u) = 0$ if and only if $u = 0$.

(2) $\delta(au) = \|a\|\delta(u)$ for $a \in F$.

(3) $\delta(u+v) \leqslant \max(\delta(u),\delta(v))$.

(4) $\delta(vu) = \delta(v)\delta(u)$.

Moreover, $\mathrm{End}(E) \hookrightarrow \mathrm{End}_k(E) \otimes_A F_\infty$ will be seen to be a discrete A-module in this normed vector space over F_∞ , see (4.9)(2). The main step will be to show that $\mathrm{End}(E)$ is a finitely generated A-module. For this we use the following two lemmas.

(4.5) LEMMA. Let $A^n \subset X \subset F_\infty^n$ where X is a discrete A-module. Then X is finitely generated.

PROOF. For all i observe that $m_\infty^i + A \subset F_\infty$ has finite index, e.g. $k[1/t] + t^i k[[t]] \subset k((t))$ has index $\mathrm{Card}(k)^i$. There exists an i with $X \cap (m_\infty^i)^n = 0$ since X is discrete, and thus X embeds in $(F_\infty/m_\infty^i)^n$ and X/A^n embeds in $(F_\infty/m_\infty^i + A)^n$ which is finite. Since $(X : A^n)$ is finite and A^n is finitely generated, it follows that X is a finitely generated A-module.

(4.6) PROPOSITION. For a finite dimensional subspace V over F of $\mathrm{End}(E) \otimes_A F$, it follows that $V \cap \mathrm{End}(E)$ is a finitely generated A-module which is projective.

PROOF. For $V_\infty = F_\infty \otimes_F V$, it follows that $X = \mathrm{End}(E) \cap V = \mathrm{End}(E) \cap V_\infty$, and we can assume that S generates V and so V_∞ or replace V by a smaller subspace. Let $x_1, \ldots, x_n$ be a basis of V_∞ with $x_i \in X$. Then with this basis $A^n \subset X \subset F_\infty^n \simeq V_\infty$, and we are reduced to the previous lemma since $\mathrm{End}(E)$ is a discrete subspace of $\mathrm{End}(E) \otimes_A F_\infty$. To see that $X = V \cap \mathrm{End}(E)$ is projective, we have only to remark that it is flat since it is torsion free over a Dedekind domain and finitely generated flat modules are projective over a Noetherian ring.

(4.7) COROLLARY. The A-module $\mathrm{End}(E)$ is projective.

PROOF. Let $W = \coprod_i W_i$ where $W = \mathrm{End}(E) \otimes_A F$ and $\dim_F W_i$ is finite. Then $X_i = \mathrm{End}(E) \cap W_i$ is projective by (4.6) and the restrictions of the projections $W \to W_i$ to $f_i\colon \mathrm{End}(E) \to X_i$ define a morphism $f\colon \mathrm{End}(E) \to \coprod_i X_i$ onto a projective module with $\ker(f) = 0$. Hence $\mathrm{End}(E)$ is projective.

For estimates on the rank we use the following lemma.

(4.8) LEMMA. Let $a \in A$ be prime to the characteristic of E. Then $\mathrm{End}(E) \otimes A/a \to \mathrm{End}(E_a)$ is injective.

PROOF. If $w \in \mathrm{End}(E)$ and $w(\ker(\phi_a)) = 0$, then as in (4.1), it follows that $w = v\phi_a$, and $w|E_a = 0$ implies $w \in \mathrm{End}(E)a$.

Now we summarize all the basic results in the following theorem and remarks.

(4.9) THEOREM. Let E be a Drinfel'd module over an algebraically closed field k of rank r. The

(1) $\mathrm{End}(E)$ is a projective A-module of rank $\leqslant r^2$, and
(2) $\mathrm{End}_k(E) \otimes_A F_\infty$ is a field in which $\mathrm{End}(E)$ embeds as a discrete A-module of this normed space over F_∞.

PROOF. (1) The fact that $\mathrm{End}(E)$ is projective is contained in (4.7) and the injectivity of $\mathrm{End}(E) \otimes A/a \to \mathrm{End}(E_a)$ coming from (4.8) bounds the rank by r^2 since E_a is an A/a-module of rank r for a prime to the characteristic.

(2) The existence of δ on $\mathrm{End}_k(E) \otimes_A F$ proves that it is a field, and since F_∞/F is a separable extension $\mathrm{End}_k(E) \otimes_A F_\infty$ is also a field.

The subspace where $\delta = 0$ on $\mathrm{End}_k(E) \otimes_A F_\infty$ is zero since $\dim_{F_\infty} \mathrm{End}(E) \otimes_A F_\infty = \mathrm{rank}(\mathrm{End}_k(E))$.

(4.10) REMARK. With the notations of the previous theorem, the ring $\mathrm{End}_k(E)$ is commutative for E of characteristic ∞ and further its rank $\leq r$.

(4.11) REMARK. For a place v of F we denote by $D_v(E) = \varinjlim_n E_{(v)^n}$ and then the Tate module would be by definition

$$T_v(E) \quad = \quad \mathrm{Hom}(F_v/A_v, D_v(E)) \quad .$$

Then $\mathrm{End}(E) \otimes_A A_v \hookrightarrow \mathrm{End}\, T_v(E) = \mathrm{End}\, D_v(E)$, and the cokernel is without torsion as an A_v-module.

§5. DRINFEL'D MODULES OVER A BASE SCHEME

Recall that locally free sheaf of rank 1 and invertible sheaf are the same notions, for which we use also the term line bundle.

(5.1) DEFINITION. Let S be a scheme in characteristic p . A Drinfel'd module over S of rank r is an invertible sheaf L and a morphism or rings $\phi\colon A \longrightarrow \mathrm{End}(L)$ such that locally over open sets where L is trivial $\phi_a(x) = \sum_{i=0}^m a_i x^{p^i}$ where $p^m = \|a\|^r$ and a_m is a unit. We say that ϕ_a is strictly of degree p^m .

In order to analyze the condition that a_m is a unit, we use the following lemma.

(5.2) LEMMA. Let R be a ring with $p = 0$ and $\phi, \psi, f \in R\{\tau\}$ where $f\phi = \psi f$, $\phi = \sum_{i=0}^N a_i \tau^i$ with a_N a unit, $\psi = \sum_{i=0}^N b_i \tau^i$ with b_N a unit, and $f = \sum_{i=0}^M c_i \tau^i$. Then the leading coefficient c_M is either a unit or zero, and so f is either zero or strictly of degree $\leq M$.

PROOF. Comparing the coefficients of τ^{M+N} in the relation $\psi f = f\phi$, we deduce $b_N c_M^N = c_M a_N^M$ or $c_M(b_N c_M^{N-1} - a_N^M) = 0$. Since b_N is a unit, this can be written

$$c_M\left(c_M^{N-1} - a_N^M b_N^{-1}\right) \quad = \quad 0 \quad .$$

Thus either $c_M = 0$ or $c_M =$ a unit with $c_M^{N-1} = a_N^M/b_N$.

The next lemma is a version of the Weierstrass preparation theorem.

(5.3) LEMMA. Let R be a ring with $p = 0$. If $\phi = \sum_{i=0}^N a_i\tau^i$ where a_M is a unit and $a_{M+1},\ldots,a_N$ are nilpotent, then after a change of coordinates with $\alpha(\tau) = 1 + \sum_{i=1}^N c_i\tau^i$ such that

$$(\alpha^{-1}\phi\alpha)(\tau) = \sum_{i=0}^{M} a_i'\tau^i = \phi'(\tau)$$

with a_M' a unit.

The coefficients $c_N, c_{N-1},\ldots,c_{M+1}$ are chosen by decreasing induction.

(5.4) REMARKS on the definition (5.1). An alternative form of the definition of a Drinfel'd module over a scheme S in characteristic p is to give E/S a group scheme locally isomorphic to G_a and a morphism of rings $\phi\colon A \longrightarrow \mathrm{End}_S(E)$ with degree $\phi_a = \|a\|_\infty^r$ locally over S.

Using lemma (5.3), we can put an $\mathcal{O}_S$-linear structure on E such that the action of some ϕ_a for $\|a\|_\infty > 1$ is locally given by a polynomial expression with highest coefficient a unit and of degree $\|a\|_\infty^r$. Using lemma (5.2) and the relation $\phi_a\phi_b = \phi_{ab} = \phi_b\phi_a$, it follows that all ϕ_b are given locally by polynomials of strict degree $\|b\|_\infty^r$. Questions of $\mathcal{O}_S$-linearity need only be checked over closed subschemes $\mathrm{Spec}(R) \hookrightarrow S$ where R is a local Artin ring.

(5.5) <u>Analogue of characteristic</u>. The function $a \longmapsto \partial_0\phi_a$ defines a morphism $A \longrightarrow \mathcal{O}_S$ of rings and hence a morphism of schemes $S \longrightarrow \mathrm{Spec}(A)$. For a closed point $s \in S$, the Drinfel'd module L_s over the field $F(s)$ has characteristic $\theta(s) \in \mathrm{Spec}(A)$.

§6. LEVEL STRUCTURE AND THE MODULI SPACE

Let I be an ideal in A, let $V(I)$ denote the set of prime ideals of A containing I, and let E/S be a Drinfel'd module of rank r over S with characteristic morphism $\theta\colon S \longrightarrow \mathrm{Spec}(A)$. Then form the finite flat group scheme $E_I/S = \bigcap_{a\in I} \ker_S(E \xrightarrow{a} E)$ of rank equal to $\mathrm{Card}(A/I)^r$. The scheme E_I/S is étale outside the characteristics which divide I, i.e.

over $\theta^{-1}(\mathrm{Spec}(V) - V(I)) \subset S$.

In the most elementary sense an I-level structure on a Drinfel'd module E should be an isomorphism $\alpha\colon (I^{-1}/A)^r \to E_I$ over S , and in fact, this definition works very well away from characteristics dividing I . In order to deal smoothly with characteristics dividing I, we are led to the following definition of Drinfel'd which has also an analogue for elliptic curves with level structures.

(6.1) DEFINITION. An I-level structure on a Drinfel'd module E/S of rank r is an A-linear morphism

$$\alpha\colon (I^{-1}/A)^r \longrightarrow E_I$$

such that for all i in I^{-1}/A the corresponding sections $\alpha(i)$ of E_I have the property that as divisors on E

$$\sum_{i \in (I^{-1}/A)^r} (\alpha(i)) = (E_I) \quad .$$

Locally E is isomorphic to G_a , and in this case E_I is defined as the kernel of a polynomial map $P\colon G_a \to G_a$ where

$$P(X) = \prod_{i \in (I^{-1}/A)^r} (X - \alpha(i)) \quad .$$

Let F_I^r denote the contravariant functor from schemes to sets which assigns to a scheme S the set of isomorphism classes of Drinfel'd modules over S of rank r with I-level structure.

(6.2) THEOREM. Let I be an ideal in A with $\mathrm{Card}\, V(I) > 1$. The functor F_I^r is representable by an affine scheme of finite type over A .

PROOF. For $x \in V(I)$ it suffices to show that the functor restricted to the category of schemes over $\mathrm{Spec}(A) - \{x\}$ is representable. Then for a scheme S over $\mathrm{Spec}(A) - \{x\}$, and a Drinfel'd module E over S with an I-level structure, the choice of nonzero elements for the I-level structure gives a trivialization of E . Over these local pieces E is given by coordinates of ϕ_a for each $a \in A$ and elements $\alpha(i)$ for $i \in (I^{-1}/A)^r$ subject to the relations:

(a) $\phi_a \phi_b = \phi_b \phi_a$.

(b) The leading coefficient of ϕ_a is invertible.

(c) $\phi_a(\alpha(i)) = 0$.

(d) $P(X) = \prod(X - \alpha(i))$.

All of these relations are affine in nature and can be represented by an affine scheme.

The book of Katz-Mazur [1985] carries the construction of moduli spaces of elliptic curves using Drinfel'd's definition of level structure.

CHAPTER 2. ANALYTIC THEORY OF THE AFFINE MODULES OF DRINFEL'D

The group of points of an elliptic curve over the complex numbers is of the form $\mathbf{C}/L$ where L is a lattice over $\mathbf{Z}$ (so of rank 2) in $\mathbf{C}$. There is a similar assertion for Drinfel'd modules of rank r. The group of C_∞-valued points is of the form C_∞/L where L is an A-lattice (discrete A-module) of rank r in C_∞.

The description of Drinfel'd modules in terms of lattices gives a calculation of the moduli space of Drinfel'd modules over C_∞ in terms of a quotient $GL(r,A)\backslash \mathrm{Mon}_{F_\infty}(F_\infty^r, C_\infty)$. This is similar to the quotient $GL(2,\mathbf{Z})\backslash \mathrm{Mon}_{\mathbb{R}}(\mathbb{R}^2,\mathbf{C})$ which classifies elliptic curves plus a differential form. In the last section an adèlic description of the points of this moduli space is given.

§1. EXPONENTIAL FUNCTION ASSOCIATED TO A LATTICE

Let X be a subset of a complete nonarchimedean value field K such that $0 \in X$ and $B(a,r) \cap X$ is finite for any ball $B(a,r) \subset K$ around a of radius $r > 0$. Then the Euler product

$$e_X(t) = t \prod_{\gamma \in X - \{0\}} (1 - \frac{t}{\gamma})$$

defines an entire function $e_X\colon K \to K$ with zeros on X, because the hypothesis implies $\lim_{\gamma \in X - \{0\}} t/\gamma = 0$ from which we deduce that $e_X(t)$ converges uniformly on any ball.

By changing f and t by scalar factors, we can assume that the power series expansion of an entire function has the form $f(t) = \sum_{0 \leq n} a_n t^n$ where $|a_i| \leq 1$, $|a_i| < 1$ for $i < r$, and $|a_r| = 1$; when f is nonconstant, also $r > 0$. By solving congruences modulo powers of the maximal ideal of F, we can factor

$$f(t) = (t^r + c_1 t^{r-1} + \cdots + c_r)(b_0 + b_1 t + \cdots) .$$

If, in addition, K is algebraically closed, then it follows that every nonconstant entire function has at least one zero. Thus the Euler product $e_X(t)$

is the unique entire function, up to a constant factor, with one simple zero at each point of X and no other zeros.

(1.1) DEFINITION. A subgroup L of the additive group of a complete value field K is called a lattice provided the intersection $L \cap B(a,r)$ is finite for any ball $B(a,r)$.

When K is of characteristic p, then $pL = 0$ and the torsion group L is a limit of finite subgroups H, i.e.

$$L = \varinjlim_{H \text{ finite} \subset L} H \quad .$$

If the above Euler expansion $e_X(t) = t \prod_{\gamma \in X - \{0\}} (1 - \frac{t}{\gamma})$ is additive, i.e. $e_X(x+y) = e_X(x) + e_X(y)$ for any $x,y \in F$, then X is clearly a lattice and $e_X(t)$ is its exponential function.

(1.2) PROPOSITION. Let L be a lattice in a complete value field K of characteristic p. Then $e_L(x+y) = e_L(x) + e_L(y)$ for any $x,y \in K$.

PROOF. Since $L = \varinjlim_{H \text{ finite} \subset L} H$, it follows that uniformly on any ball

$$e_L(t) = \varinjlim_{H \text{ finite} \subset L} e_H(t) \quad .$$

Each $e_H(t)$ is additive since it equals $c \cdot \prod_{h \in H} (t-h)$ which is additive by 1(1.3). Thus $e_L(x+y) = e_L(x) + e_L(y)$ since it is a limit of functions $e_H(t)$ satisfying this property.

(1.3) REMARK. If L is a lattice in a complete value field K, and if $\lambda \in K^*$, then λL is a lattice also and the exponential function $e_L(t)$ of L determines the exponential function of λL by $e_{\lambda L}(t) = \lambda e_L(\lambda^{-1} t)$. The lattice λL is called the dilation of L by $\lambda \in K^*$.

Two lattices L and L' are in the same dilation class provided $L' = \lambda L$ for $\lambda \in K^*$. We always normalize the exponential function so that

$$e_L(t) = t + (\text{higher order terms}) \quad .$$

§2. CHARACTERIZATION OF DRINFEL'D MODULES OVER C_∞

Lattices $\Gamma \subset C_\infty$ such that $a\Gamma \subset \Gamma$ for $a \in A \subset C_\infty$ with the induced scalar action by $a \in A$ are also A-modules, which we call A-lattices. A dilation $\lambda\Gamma$ of an A-lattice Γ is again an A-lattice for $\lambda \in C_\infty^*$, and we can speak of dilation classes of A-lattices.

Given an A-lattice Γ in C_∞, we form the ring morphism $\phi^\Gamma: A \to \mathrm{End}_{C_\infty}(G_a)$ where ϕ_a^Γ for $a \in A$ is defined by the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma & \longrightarrow & C_\infty & \xrightarrow{e_\Gamma} & C_\infty & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow a & & \downarrow \phi_a^\Gamma & & \\ 0 & \longrightarrow & \Gamma & \longrightarrow & C_\infty & \xrightarrow{e_\Gamma} & C_\infty & \longrightarrow & 0 \end{array} .$$

For $\lambda\Gamma$, the relation $\phi_a^{\lambda\Gamma}(e_{\lambda\Gamma}(t)) = e_{\lambda\Gamma}(at)$ becomes in view of (1.3) simply $\phi_a^{\lambda\Gamma}(\lambda e_\Gamma(\lambda^{-1}t)) = \lambda e_\Gamma(a\lambda^{-1}t)$ or $\lambda^{-1}\phi_a^{\lambda\Gamma}(\lambda e_\Gamma(t)) = e_\Gamma(at)$. This implies that $\phi_a^\Gamma = \phi_a^{\lambda\Gamma}$ for $\lambda \in C_\infty$.

(2.1) THEOREM. The function, which assigns to each A-lattice Γ, the ring morphism $\phi^\Gamma: A \to \mathrm{End}_{C_\infty}(G_a)$ defined above, is a bijection from the set of A-lattice dilation classes in C_∞ determined by Γ which are projective of rank r onto the set of isomorphism classes of affine A-modules ϕ_a over C_∞ of rank r with $\partial\phi_a = a$.

PROOF. First we calculate $\ker(\phi_a^\Gamma)$. This is isomorphic to $\ker(\phi_a^\Gamma e_\Gamma)/\Gamma = \ker(e_\Gamma a)/\Gamma$, and $\ker(e_\Gamma a)/\Gamma$ is isomorphic to $a^{-1}\ker(e_\Gamma)/\Gamma = a^{-1}\Gamma/\Gamma$. Hence $\ker(\phi_a^\Gamma)$ is isomorphic to $(A/a)^r$ since Γ is projective of rank r, and moreover $\deg(\phi_a^\Gamma) = \mathrm{Card}(A/a)^r$. Thus $\Gamma \to \phi^\Gamma$ is a well defined function from dilation classes of A-lattices projective of rank r to isomorphism classes of affine A-modules over C_∞ of rank r. The property $\partial\phi_a = a$ follows from $e_\Gamma(t) = t + (\text{higher terms})$, and $\phi_a^{\lambda\Gamma} = \lambda\phi_a^\Gamma\lambda^{-1}$ from the calculation preceding the statement of the theorem.

To show $\Gamma \mapsto \phi^\Gamma$ is a bijection, we consider an affine module $\phi: A \to \mathrm{End}_{C_\infty}(G_a)$ of rank r and construct an additive entire function $e: C_\infty \to C_\infty$ such that $\phi_a(e(X)) = e(aX)$ and $e(t) = t + (\text{higher terms})$. Then the corresponding A-lattice Γ is the set of $x \in C_\infty$ with $e(x) = 0$.

The first step is to show that there is a unique formal solution $e(\tau) \in k\{\{\tau\}\}$ such that $\phi_a(e(x)) = e(ax)$ for a given $a \in A$ with $\phi_a(\tau) = a_0 + \cdots + a_s\tau^s$ and $a = a_0 \neq 0,1$. The condition $\phi_a(e(x)) = e(ax)$ is equivalent to $\phi_a(e(a^{-1}x)) = e(x) = \sum_{0 \leq i} e_i x^{p^i}$. Equating coefficients of τ, we derive from the relation

$$\left(\sum_i a_i\tau^i\right)\left(\sum_j e_j(\tau a^{-1})^j\right) = \sum e_n\tau^n$$

the formula $e_n = \sum_{i+j=n} a_i e_j^{p^i} a^{-p^n}$, and hence the inductive definition for the coefficients e_n starting with $e_0 = 1$

$$e_n(1 - a^{1-p^n}) = \sum_{i=1}^{n} a_i e_{n-i}^{p^i} a^{-p^n} .$$

This shows both the existence and uniqueness of $e(\tau)$.

The second step is to show that for this $e(\tau)$, the relation $\phi_b(e(x)) = e(bx)$ holds for any $b \in A$. Consider

$$(\phi_b e b^{-1})(x) = (\phi_b(\phi_a e a^{-1})b^{-1})(x) = (\phi_a(\phi_b e b^{-1})a^{-1})(x) .$$

From the expression it follows that e and $\phi_b e b^{-1}$ both satisfy the relation $\phi_a e a^{-1} = e$, and from the uniqueness assertion in step one, it follows that $e_b e = eb$ for any $b \in A$.

Next we have to show that $e(x) = \sum_{0 \leq n} e_n x^{p^n}$ is an entire function. Assume in the inductive definition of $e(x)$ that $|a| > 1$ and write the above recurrence formula for e_n as

$$e_n(a^{p^n} - a) = \sum_{i=1}^{s} a_i e_{n-i}^{p^n} \quad \text{for } n \geq s .$$

Then $|a| \cdot |e_n|^{p^{-n}} \leq \max_{1 \leq i \leq s} \{|a_i|^{p^{-n}} r_{n-i}\}$ where $r_j = |e_j|^{p^{-j}}$. For θ such that $1/|a| < \theta < 1$ and any $n \geq$ fixed n_0, each term $|a_i|^{p^{-n}} \leq 1 + \varepsilon$, and it follows that

$$r_n \leq \theta \cdot \max_{1 \leq i \leq s} r_{n-i} .$$

Since $\sum_{0 \leq n} r_n \leq \frac{1}{1-\theta} \max\{r_0, \ldots, r_s\}$, we see that $r_n = |e_n|^{p^{-n}} \to 0$, which proves that $e(x)$ is an entire function.

Finally, we calculate ϕ_a for this $e(x)$ where Γ equals the subset of $x \in C_\infty$ with $e(x) = 0$. We have

$$\phi_a(x) = a \cdot x \prod_{\gamma \in a^{-1}\Gamma/\Gamma - \{0\}} \left(1 - \frac{x}{e(\gamma)}\right)$$

since $\phi_a(x) = 0$ on $a^{-1}\Gamma/\Gamma$. Thus the degree of ϕ_a equals $\mathrm{Card}(a^{-1}\Gamma/\Gamma) = \mathrm{Card}(\Gamma/a) = (\mathrm{Card}(A/a))^r$. This proves the theorem.

For our future considerations I-level structures will play a basic role where I is a nonzero ideal in A. This leads to the following definition.

(2.2) DEFINITION. Let Y be a projective A-module of rank r. For a nonzero ideal $I \subset A$ an I-level structure is an isomorphism $\alpha: Y/IY \to (A/I)^r$ of A-modules (or free A/I-modules).

An isomorphism $Y/IY \xrightarrow{u} (A/I)^r$ is equivalent to an isomorphism $I^{-1}Y/Y \xrightarrow{v} (I^{-1}/A)^r$ since $v = I^{-1} \otimes_A u$. In the case $Y = \Gamma$ an A-lattice of rank r and the affine A-module $E = \phi^\Gamma: A \to \mathrm{End}_{C_\infty}(G_a)$ corresponding to Γ by (2.1), we calculated $\ker(\phi_a^\Gamma) = a^{-1}\Gamma/\Gamma$ and thus the subfunctor $E_I = \bigcap_{a \in I} \ker(\phi_a) \subset E$ has the form

$$E_I(C_\infty) = \bigcap_{a \in I} a^{-1}\Gamma/\Gamma = I^{-1}\Gamma/\Gamma .$$

Thus we immediately deduce the following assertion.

(2.3) PROPOSITION. Using $E_I(C_\infty) = I^{-1}\Gamma/\Gamma$, we have that an I-level structure on the affine A-module $E = \phi^\Gamma$ is the same as on I-level structure on the projective A-module Γ.

§3. DISCRETE MODULES IN A VECTOR SPACE OVER A LOCAL FIELD

The following preliminaries are needed in the next section in order to parametrize A-lattices in C_∞ as homogeneous spaces.

(3.1) NOTATIONS. Let A be a discrete subring of a local field K with field of fractions $F \subset K$. We assume that K/A is a compact abelian group.

(3.2) EXAMPLES. (1) $A = \mathbb{Z} \subset F = \mathbb{Q} \subset K = \mathbb{R}$.

(2) $A = \mathbb{Z} + fR \subset K = \mathbb{Q}(\sqrt{-d})$ for $d > 0$, $d \subset \mathbb{Z}$ square free and $F \subset K = \mathbb{C}$.

(3) $A = \mathbb{F}_q[t] \subset F = \mathbb{F}_q(t) = \mathbb{F}_q(t^{-1}) \subset K = \mathbb{F}_q((t^{-1}))$.

(4) $A = \mathbb{F}_q[C - \infty] \subset F = \mathbb{F}_q(C - \infty) = \mathbb{F}_q(C) \subset K = F_\infty$ where C is an affine curve and F_∞ is the completion of F at ∞.

Of course (3) is a special case of (4), and (4) is the case of interest in this part.

For any finite dimensional vector space V over K, the topology is well defined and given by a norm. A subgroup H in V is discrete provided there exists a neighborhood N' of 0 in V with $N' \cap H = 0$. If $N + N \subset N'$, then $N + x \cap N + y = \emptyset$ for $x, y \in H$ if and only if $x = y$.

For an A-module H, the rank of H, denoted $rk_A H$, is $\dim_K(K \otimes_A H)$.

(3.3) PROPOSITION. Let H be a discrete A-module contained in a finite dimensional K-vector space V. Then

$$rk_A(H) \leq \dim_K(V) .$$

PROOF. Let $x_1, \ldots, x_n$ denote a set of elements H forming a basis of the vector space $W = K \cdot H \subset V$. Then

$$L = Ax_1 \oplus \cdots \oplus Ax_n \subset H \subset W \subset V .$$

Since H is discrete in V, there exists a neighborhood N' of $0 \in V$ with $H \cap N' = 0$ and a neighborhood N with $N + N \subset N'$. Now $N+L$ is a neighborhood of 0 in V/L intersecting H/L only at 0. Thus H/L is a discrete subgroup of V/L and of the compact abelian group

$$W/L = Kx_1 \oplus \cdots \oplus Kx_n / Ax_1 \oplus \cdots \oplus Ax_n .$$

Thus H/L is finite and $n = \dim_F(F \otimes_A L) = \dim_F(F \otimes_A H) = \dim_K W$ which in turn implies $rk_A(H) = \dim_F(F \otimes_A H) \leq \dim_K V$. This proves the proposition.

(3.4) PROPOSITION. Let H be a projective A-submodule contained in a finite dimensional K-vector space V. Then H is discrete in V if and only if $\theta \colon K \otimes_A H \longrightarrow V$ is injective.

PROOF. Assume H is discrete in V. Then $H \subset \mathrm{im}(\theta) \subset V$ is discrete and so $rk_A(H) \leq \dim_K(\mathrm{im}(\theta))$ by (3.3). On the other hand, for any H, we have

$$rk_A(H) = \dim_K(K \otimes_A H) \geq \dim_K(\mathrm{im}(\theta)) .$$

Hence $\dim_K(K \otimes_A H) = \dim_K(\mathrm{im}(\theta))$ and so θ is injective.

Conversely, assume that θ is injective. There exists H' with $H \oplus H'$ free, and a free module $A^n \subset K^n$ is a discrete A-submodule. Hence H is discrete in $K \otimes_A H$, and since θ is injective, H is discrete in V. This proves the proposition.

(3.5) REMARK. Let C be the completion of the algebraic closure of K. Then for every projective A-module P with $rk_A(P) \leq [C : K]$ there exists an A-monomorphism

$$P \longrightarrow C$$

such that P is discrete in C, i.e. by (3.4), the induced $K \otimes_A P \to C$ is injective.

In example (1), $2 = [C:K]$ and free modules P of rank ≤ 2 over $\mathbf{Z}$ can be embedded as discrete subgroups of $\mathbf{C} = C$. In examples (3) and (4), $[C_\infty : F_\infty] = \infty$ and all projective modules embed as A-lattices of C_∞.

§4. MODULI SPACES AS HOMOGENEOUS SPACES (LOCAL THEORY)

Now we return to the notations of the article where $F = \mathbb{F}_q(C)$, $A = \mathbb{F}_q[C-\infty]$, F_∞ completion of F at ∞, and C_∞ the completion of the algebraic closure $\bar{F}_\infty$ of F_∞. Let K denote F_∞ a local field as referred to in §3 where $A \subset K$ is discrete and K/A compact. We denote points in projective space $\mathbb{P}^{r-1}(C_\infty)$ by their homogeneous coordinates $y_1 : \cdots : y_r$. We begin by parametrizing F-vector subspaces or equivalently free A-lattices in C_∞.

(4.1) PROPOSITION. The function $f\colon C_\infty^*\backslash(\mathrm{Hom}_{F_\infty}(F_\infty^r, C_\infty) - \{0\}) \to \mathbb{P}^{r-1}(C_\infty)$ given by $f(u) = u(e_1) : \cdots : u(e_n)$ where $e_i = (0,\ldots,1,\ldots,0)$ is a bijection which restricts to a bijection

$$C_\infty^* \backslash \mathrm{Mon}_{F_\infty}(F_\infty^r, C_\infty) \longrightarrow \mathbb{P}_{r-1}(C_\infty) - \bigcup (F_\infty\text{-rational hyperplanes}) \quad .$$

PROOF. A nonzero element u of $\mathrm{Hom}_{F_\infty}(F_\infty^r, C_\infty)$ is determined by its values $(u(e_1),\ldots,u(e_r))$ on the canonical basis elements $e_i \in F_\infty^r$. A dilation of u yields a dilation of the r-tuple $(u(e_1),\ldots,u(e_r)) = (x_1,\ldots,x_r)$. Next, $(a_1,\ldots,a_r) \in \ker(u) \subset F_\infty^r$ if and only if $\sum_{i=1}^d a_i x_i = 0$, i.e. for $(a_1,\ldots,a_r) \neq 0$, $(x_1,\ldots,x_r)$ is on the F_∞-rational hyperplane with equation

$$a_1 x_1 + \cdots + a_r x_r = 0 \quad .$$

This proves the proposition.

For an A-module Y projective of rank r, we have embeddings and isomorphism $Y \to F_\infty \otimes_A Y \simeq F_\infty^r$ and $GL_A(Y) \to GL_F(F_\infty \otimes_A Y) \simeq GL(r, F_\infty)$ coming from the tensor product. By taking the image of Y or $F_\infty \otimes_A Y$, and using (3.4), the following horizontal arrows are well defined functions.

(4.2) PROPOSITION. Let Y be a projective module of rank r over A. The following diagram is commutative and the horizontal arrows are bijections

$$
\begin{tikzcd}[column sep=small]
\mathrm{Mon}_{F_\infty}\backslash(F_\infty \otimes_A Y, C_\infty)/GL_A(Y) \arrow[rr] \arrow[dr] \arrow[dd] & & \left\{\begin{array}{l}\text{A-sublattices of } C_\infty \\ \text{isomorphic to } Y\end{array}\right\} \arrow[dr] \arrow[dd] & \\
& C_\infty^*\backslash \mathrm{Mon}_F(F \otimes_A Y, C_\infty)/GL_A(Y) \arrow[rr] \arrow[dd] & & \left\{\begin{array}{l}\text{A-sublattices of } C_\infty \\ \text{isomorphic to } Y \\ \text{modulo dilations in } C_\infty^*\end{array}\right\} \arrow[dd] \\
\mathrm{Mon}_{F_\infty}\backslash(F_\infty \otimes_A Y, C_\infty)/GL(r,F) \arrow[rr] \arrow[dr] & & \left\{\begin{array}{l}\text{r-dimensional} \\ \text{F-vector subspaces} \\ \text{of } C_\infty\end{array}\right\} \arrow[dr] & \\
& C_\infty^*\backslash \mathrm{Mon}_{F_\infty}(F_\infty \otimes_A Y, C_\infty)/GL(r,F) \arrow[rr] & & \left\{\begin{array}{l}\text{r-dimensional F-vector} \\ \text{subspaces of } C_\infty \\ \text{modulo dilations in } C_\infty^*\end{array}\right\} .
\end{tikzcd}
$$

Now putting (4.1) and (4.2) together along with (2.1), we obtain the following assertion. Let P_A^r denote the set of isomorphism classes of projective A-modules of rank r.

(4.3) PROPOSITION. Using the previous notations, we have the following bijections

$$
\begin{tikzcd}
(\mathbb{P}_{r-1}(C_\infty) - \bigcup(F_\infty\text{-rational hyperplanes}))/GL_A(Y) & \\
C_\infty^*\backslash \mathrm{Mon}_{F_\infty}(F_\infty \otimes_A Y, C_\infty)/GL_A(Y) \arrow[u] \arrow[r] & \left\{\begin{array}{l}\text{A-sublattices of } C_\infty \\ \text{isomorphic to } Y \\ \text{modulo dilations in } C_\infty^*\end{array}\right\} ,
\end{tikzcd}
$$

and taking disjoint unions

$$
\begin{tikzcd}
\coprod_{Y \in P_A^r} (\mathbb{P}_{r-1}(C_\infty) - \bigcup(F_\infty\text{-rational hyperplanes}))/GL_A(Y) & \\
\coprod_{Y \in P_A^r} C_\infty^*\backslash \mathrm{Mon}_{F_\infty}(F_\infty \otimes_A Y, C_\infty)/GL_A(Y) \arrow[u] \arrow[r] & \left\{\begin{array}{l}\text{A-sublattices of } C_\infty \text{ of rank } r \\ \text{modulo dilations in } C_\infty^*\end{array}\right\} \arrow[d] \\
& \left\{\begin{array}{l}\text{isomorphism classes of affine} \\ \text{A-modules } \phi_a \text{ over } C_\infty \text{ of} \\ \text{rank } r \text{ with } \partial\phi_a = a\end{array}\right\} .
\end{tikzcd}
$$

(4.4) REMARK. In the case $r = 1$, $P_0(C_\infty) - (F_\infty\text{-rational hyperplanes})$ is a point, and (4.3) is a statement about bijections between sets with one point. We will be interested particularly in the case $r = 2$ in latter parts. If $a_1x_1 + a_2x_2 = 0$ where $a_1, a_2 \in F_\infty$, then $x_1 : x_2 = -a_2 : a_1 \in \mathbb{P}_1(F_\infty)$. Thus we have

$$\mathbb{P}_1(C_\infty) - (F_\infty\text{-rational lines}) = \mathbb{P}_1(C_\infty) - \mathbb{P}_1(F_\infty) \quad .$$

The next step is to modify (4.3) to include the case of A-lattices with I-level structure. For a projective A-module Y of rank r the projection $Y \to Y/IY$ includes a group morphism $GL_A(Y) \to GL_A(Y/IY)$ which has kernel $GL_A(Y,I)$. In the following two statements we have a restatement of (4.2) and a modification taking into account I-level structures. Fix an I-level structure

$$\alpha_0 : Y/IY \longrightarrow (A/I)^r \quad .$$

(1) Each monomorphism $u: F_\infty \otimes_A Y \to C_\infty$ determines a lattice $u(Y) \subset C_\infty$ and an I-level structure $\alpha_0((u|Y)^{-1} \bmod I) = \alpha$ on $u(Y)$ where $u|Y: Y \to u(Y) \subset C_\infty$. Moreover, all A-sublattices of C_∞, isomorphic to Y together with an I-level structure, come by this construction.

(2) Two monomorphisms $u, u': F_\infty \otimes_A Y \to C_\infty$ determine the same lattice in C_∞ if and only if there exists $h \in GL_A(Y)$ with $u' = uh$. They determine the same lattice in C_∞ with I-level structure if and only if there exists $h \in GL_A(Y,I)$ with $u' = uh$.

This leads to the following modification of the previous proposition.

(4.5) THEOREM. Let I be a nonzero ideal in A, and use the previous notations. We have the following bijections:

$$\begin{tikzcd}
\coprod_{Y\in P^r_A}(\mathbb{P}_{r-1}(C_\infty) - \cup(F_\infty\text{-rational hyperplanes}))/GL_A(Y,I) & \\
\coprod_{Y\in P^r_A} C_\infty^*\backslash \mathrm{Mon}_{F_\infty}(F_\infty\otimes_A Y, C_\infty)/GL_A(Y,I) \arrow[u, "\wr"', "{\begin{array}{c}\alpha:\ I^{-1}Y/Y \to (I^{-1}/A)^r \\ \text{modulo Aut of } Y\end{array}}"] \arrow[r, "\sim"] & \left\{\begin{array}{l}\text{A-sublattices of } C_\infty \\ \text{of rank } r \text{ with an I-level} \\ \text{structure modulo dilations} \\ \text{in } C_\infty^*\end{array}\right\} \arrow[d, "\wr"', "{\begin{array}{c}\alpha:\ I^{-1}Y/Y \to (I^{-1}/A)^r \\ \text{modulo Aut of } Y\end{array}}"] \\
& \left\{\begin{array}{l}\text{isomorphism classes of affine A-modules } \phi_a \\ \text{over } C_\infty \text{ with I-level structure and of} \\ \text{rank} = r \text{ with } \partial\phi_a = a\end{array}\right\}
\end{tikzcd}$$

or more briefly the bijection

$$\boxed{\coprod_{(\alpha,Y)\in P^r_A(I)} \Omega^r(C_\infty)/GL_A(Y,I) \xrightarrow{\sim} M^r_I(C_\infty)} \quad .$$

Here $\Omega^r(C_\infty) = \mathbb{P}_{r-1} = \mathbb{P}_{r-1}(C_\infty) - \cup(F_\infty\text{-rational hyperplanes})$, also M^r_I is the moduli functor for affine A-modules rank r with I-level structure, and $P^r_A(I)$ is the set of ordered pairs (α,Y) with $Y \in P^r_A$ and $\alpha\colon I^{-1}Y/Y \to (I^{-1}/A)^r$ is an isomorphism.

§5. MODULI SPACES AS HOMOGENEOUS SPACES (ADÈLIC THEORY)

The basic bijection describing $M^r_I(C_\infty)$ in (4.4) involves only the local field at ∞. In this section we will describe $M^r_I(C_\infty)$ using all the primes of F together with $M^r_I(C_\infty)$. This adèlic description is closely related to the adèlic description of vector bundles on a curve which we recall now.

(5.1) REMARKS ON VECTOR BUNDLES. Over any ring space $X = (X,\mathcal{O}_X)$ a vector bundle E of rank n is a (locally free) sheaf on X locally isomorphic to $\mathcal{O}_X^n$. If X is a nonempty open set of the curve C over $\mathbb{F}_q$, for example $X = C - \{\infty\}$, then a vector bundle of rank n can be described as a family $(V,L_v)_{v\in X}$ where X is an n-dimensional vector space over $F = \mathbb{F}_q(C)$ and each L_v is a free module of rank n contained in V over the local ring $\mathcal{O}_{(v)}$ at v. We assume that there exists a basis $x_1,\dots,x_n$

of V over F and a finite set $S \subset X$ with

$$L_v = \mathcal{O}_{(v)}x_1 + \cdots + \mathcal{O}_{(v)}x_n$$

for all $v \in X - S$. The basis $x_1,\ldots,x_n$ is called a trivialization of V on $X - S$. If $y_1,\ldots,y_n$ is any other basis of V, then

$$y_j = \sum_i a_{i,j}x_i \qquad \text{and} \qquad x_i = \sum_j b_{j,i}y_j$$

where $(a_{ij}),(b_{ij}) \in GL(n,\mathcal{O}_{(v)})$ for all but a finite number of v. Hence every basis of V is a trivialization of V over some $X - T$ where $T \subset X$ is finite.

To an $\mathcal{O}_X$-sheaf E, locally isomorphic to $\mathcal{O}_X^n$, we assign $V = E_\eta$ where η is the general point of C. The $\mathcal{O}_{(v)}$-submodule $L_v \subset V$ is defined by trivializing $E|(X-S)$ on an open set $X-S$ around v with $x_1,\ldots,x_n \in V$ and requiring for all $w \in X - S$ with S finite that

$$L_w = \mathcal{O}_{(w)}x_1 + \cdots + \mathcal{O}_{(w)}x_n \quad .$$

These lattices which come up in the above description of a vector bundle can be identified with certain homogeneous spaces. Let F_v denote the completion of F at $v \in X$ with valuation ring $\mathcal{O}_v$. Then using matrix transform between basis of a lattice in a vector space F^n or F_v^n, we have the following commutative diagram of bijections:

$$\begin{array}{ccc} GL(n,F)/GL(n,\mathcal{O}_{(v)}) & \xrightarrow{\sim} & \{\mathcal{O}_{(v)}\text{-lattices in } F^n\} \\ \downarrow\wr & & \downarrow\wr \\ GL(n,F_v)/GL(n,\mathcal{O}_v) & \xrightarrow{\sim} & \{\mathcal{O}_v\text{-lattices in } F_v^n\} \end{array} \quad .$$

(5.2) <u>Homogeneous space description of trivialized vector bundles</u>

A vector bundle E with a trivialization is described by (F^n,L_v) for $v \in X$ where $L_v = \mathcal{O}_{(v)}^n \subset F^n$ for all $v \in X - S$ where S is finite. Thus the vector bundle trivialized on $X - S$ is determined by a finite set of lattices $L_v \subset F^n$ for $v \in S$. This leads to the following bijections:

$$\begin{array}{ccc}\prod_{v\in S} GL(n,F_v)/GL(n,\mathcal{O}_v) & \xrightarrow{\ \sim\ } & \left\{\begin{array}{l}\text{trivialized over } X-S\\ \text{n-dimensional vector bundles on } X\\ \text{all up to isomorphism}\end{array}\right\}\\ \big\downarrow & & \\ \mathrm{Vec}_S(X) = \prod_{v\in S} GL(n,F_v)\times\prod_{v\in X-S} GL(n,\mathcal{O}_v)\Big/\prod_{v\in X} GL(n,\mathcal{O}_v) & & \end{array}$$

where the vertical arrow is a homeomorphism of locally compact spaces.

At points of S we can consider a Δ-level structure on the vector bundle E relative to a positive divisor Δ supported on S. This is just an isomorphism of the sheaf $E/E(-\Delta) = E\otimes_{\mathcal{O}}(\mathcal{O}/\mathcal{O}(-\Delta)) \to (\mathcal{O}/\mathcal{O}(-\Delta))^n$. In terms of the description of E with the data (F^n, L_v) this is an isomorphism $L_v/\pi^{s(v)}L_v \to (\mathcal{O}_{(v)}/\pi^{s(v)}\mathcal{O}_{(v)})^n$ where $s(v) = \mathrm{ord}_{(v)}(\Delta)$ and π_v is a local uniformizing parameter of $\mathcal{O}_{(v)}$ and hence $\mathcal{O}_v$. If we denote by

$$GL(n,\mathcal{O}_v,\Delta_v) = \ker(GL(n,\mathcal{O}_v) \to GL(n,\mathcal{O}_v/\pi_v^{s(v)}\mathcal{O}_v))$$

and $K(n,\Delta) = \prod_{v\in X} GL(n,\mathcal{O}_v,\Delta_v)$, then the previous diagram becomes modified as follows for $\mathrm{supp}(\Delta)\subset S$:

$$\begin{array}{ccc}\prod_{v\ S} GL(n,F_v)/GL(n,\mathcal{O}_v,\Delta_v) & \xrightarrow{\ \sim\ } & \left\{\begin{array}{l}\text{trivialized over } X-S\\ \text{n-dimensional vector bundles}\\ \text{on } X \text{ with } \Delta\text{-level structure}\\ \text{all up to isomorphism}\end{array}\right\}\\ \big\uparrow\wr & & \\ \mathrm{Vec}_S(X;\Delta) = \prod_{v\in S} GL(n,F_v)\times\prod_{v\in X-S} GL(n,\mathcal{O}_v)\Big/K(n,\Delta) & & \end{array}$$

Now we consider two special cases for X namely $X = C$ and $X = C - \{\infty\}$ and remove the condition that the vector bundle is trivialized.

(5.3) Vector bundles on C.

The product in the numerator of $\mathrm{Vec}_S(X)$ or $\mathrm{Vec}_S(X;\Delta)$ is one of the terms in the inductive limit which defines the adèle group $GL(n,\mathbb{A}_F)$ where

$$GL(n,\mathbb{A}_F) = \varinjlim_S \prod_{v\in S} GL(n,F_v)\times\prod_{v\in C-S} GL(n,\mathcal{O}_v) .$$

Hence considering vector bundles with Δ-level structure and trivialization over some open set $C-S$ of C where $\mathrm{Supp}(\Delta)\subset S$, we obtain the bijection

$$GL(n,\mathbb{A}_F)/K(n,\Delta) \longrightarrow \left\{\begin{array}{l}\text{vector bundles of dimension } n\\ \text{with a } \Delta\text{-level structure and}\\ \text{trivialization off the } \mathrm{Supp}(\Delta)\\ \text{all up to isomorphism}\end{array}\right\}.$$

Since the various trivializations are all related by the action of $GL(n,F)$ on basis, we have the following quotient of the previous bijection

$$\boxed{GL(n,F)\backslash GL(n,A_F)/K(n,\Delta) \longrightarrow \left\{\begin{array}{l}\text{isomorphism classes of}\\ n\text{-dimensional vector bundles}\\ \text{on } C \text{ with } \Delta\text{-level structure}\end{array}\right\}}.$$

(5.4) Vector bundles on $C-\{\infty\}$.

The product in the numerator of $\mathrm{Vec}_S(X)$ or $\mathrm{Vec}_S(X;\Delta)$ for $X = C - \{\infty\}$ is one of the terms in the $\varinjlim$ which defines the finite adèle group $GL(n,\mathbb{A}_F^f)$ where $\mathbb{A}_F^f = \hat{A}\otimes_A F$ and $\hat{A} = \varprojlim_J A/J$, the limit being taken over ideals $J \subset A$. A divisor Δ can be described as an ideal $I \subset A$, and the group $K(n,\Delta)$ is given by

$$K(n,\Delta) = \ker\left(GL(n,\hat{A}) \rightarrow GL(n,\hat{A}/\hat{A}I)\right)$$

which we also denote by $GL(n,\hat{A},I)$ to avoid confusion with the previous case discussed in (5.3). As before we obtain in the injective limit the bijection

$$GL(n,\mathbb{A}_F^f)/GL(n,\hat{A},I) \xrightarrow{\ \sim\ } \left\{\begin{array}{l}\text{vector bundles of dimension } n \text{ on}\\ C-\{\infty\} \text{ with a } \Delta\text{-level structure}\\ \text{and trivialization off the } \mathrm{Supp}(\Delta)\\ \text{all up to isomorphism}\end{array}\right\}.$$

On the affine $C-\{\infty\}$ with coordinate ring $A = H^0(C-\{\infty\},\mathcal{O})$ a vector bundle of dimension n is just a projective module Y of rank n over A.

Now we combine the previous bijection with

$$C_\infty^*\backslash \mathrm{Mon}_{F_\infty}(F_\infty^r, C_\infty) = \Omega^r(C_\infty) = \mathbb{P}_{r-1}(F_\infty) - \cup\{F_\infty\text{-rational hyperplanes}\}.$$

We have a bijection

$$\begin{array}{c}(GL(r,A_F^f)/GL(r,\hat{A},I)) \times \Omega^r(C_\infty)\\ \big\downarrow\wr\\ \left\{\begin{array}{l}\text{vector bundles } Y \text{ on } C-\infty \text{ of}\\ \text{rank } r \text{ with a trivialization}\\ \text{and a } \Delta\text{-level structure all up}\\ \text{to isomorphism}\end{array}\right\} \times \left\{\begin{array}{l}\text{monomorphisms } F_\infty^r \rightarrow C_\infty\\ \text{over } F_\infty \text{ up to dilations in}\\ C_\infty^*\end{array}\right\}\end{array}.$$

The trivialization of Y is a basis of $Y \otimes_A F$ over F which defines an embedding $Y \longrightarrow F_\infty^r$ as a discrete A-submodule. When this embedding is composed with a monomorphism, $F_\infty^r \longrightarrow C_\infty$ yields an A-sublattice of C_∞ of rank r with an I-level structure modulo dilations in C_∞^*. Hence by (4.5) we have a map

$$(GL(r,\mathbb{A}_F^f)/GL(r,\hat{A},I)) \times \Omega^r(C_\infty) \longrightarrow M_I^r(C_\infty)$$

where the fibres correspond to the various bases of $Y \otimes_A F$ over F. Hence factoring out by the action of $GL(r,F)$ yields the adèlic description of $M_I^r(C_\infty)$.

(5.6) THEOREM. The above map induces a bijection

$$GL(r,F)\backslash GL(r,\mathbb{A}_F^f) \times \Omega^r(C_\infty)/GL(r,\hat{A},I) \xrightarrow{\approx} M_I^r(C_\infty) \quad .$$

(5.7) REMARK. The quotient space descriptions in (4.5) and (5.6) generalize to any open compact subgroup $H \subset GL(r,A)$ to give the C_∞-valued points $M_H^r(C_\infty)$ of a moduli scheme M_H :

$$M_H^r(C_\infty) = GL(r,F)\backslash GL(r,\mathbb{A}_F^f) \times \Omega^r(C_\infty)/H \qquad \text{adèlic version}$$

$$= \coprod_{xH \in GL(r,\mathbb{A}_F^f)/H} \Omega^r(C_\infty)/(xHx^{-1} \cap GL(r,F)) \qquad \text{local version} \quad .$$

The moduli scheme M_H^r arises from H-level structures on Drinfel'd modules.

CHAPTER 3. THE RIGID ANALYTIC MODULI SPACE

Now our aim is to describe the rigid analytic structure on the C_∞-valued points of the moduli space. These points were parametrized by a quotient of $\Omega^r(C_\infty)$ by a discrete group in 2(4.5). The rigid analytic structure on the moduli space is a quotient of the rigid analytic structure on $\Omega^r(C_\infty)$.

In order to define the rigid analytic structure on $\Omega^r(C_\infty)$, we make use of a natural mapping

$$\lambda\colon \Omega^r(C_\infty) \longrightarrow I(F_\infty^r)_{\mathbf{Q}} \subset I(F_\infty^r)_{\mathbb{R}}$$

onto the rational points $I(F_\infty^r)_{\mathbf{Q}}$ of the geometric realization $I(F_\infty^r)_{\mathbb{R}}$ of the building $I(F_\infty^r)$ of the group $PLG(r)$ over the local field F_∞. The admissible open sets of the rigid analytic structure are inverse images by λ^{-1} of certain open neighborhoods of skeltons in $I(F_\infty^r)_{\mathbb{R}}$.

In this discussion we indicate how both $\Omega^r(C_\infty)$ and the building $I(F_\infty^r)$ are p-adic analogues of real symmetric spaces.

§1. NORMS ON VECTOR SPACES OVER A LOCAL FIELD

(1.1) NOTATIONS. Let K be a local field with valuation ring R, maximal ideal $p = R\pi$, and $q = \mathrm{Card}(k)$ where $k = R/R\pi$. We normalize $|a|$ on K as $|a| = q^{-\mathrm{ord}(a)}$ so that $|\pi| = q^{-1}$ and $|K^\times| = q^{\mathbf{Z}}$. We consider finite dimensional vector spaces over K, and thus the closed unit ball and unit sphere are compact. If V is of dimension m over K, then a lattice is any R-submodule M free of rank m.

(1.2) DEFINITION. A norm on a vector space V over K is a function $\alpha\colon V \longrightarrow \mathbb{R}$ satisfying:

(a) $\alpha(x) \geqslant 0$ and $\alpha(x) = 0$ if and only if $x = 0$.

(b) $\alpha(ax) = |a|\alpha(x)$ for $a \in K$ and $x \in V$.

(c) $\alpha(x+y) \leqslant \sup(\alpha(x),\alpha(y))$ for $x,y \in V$.

A norm is called integral provided $\alpha(V) = |K| = \{0\} \cup q^{\mathbf{Z}}$ and rational provided $\alpha(V) \subset \mathbf{Q}$.

If $\alpha(x) < \alpha(y)$, then axiom (c) can be strengthened to the equality $\alpha(x+y) = \alpha(y) = \sup(\alpha(x),\alpha(y))$.

If α is a norm and $t > 0$, then the dilation $t\alpha$ is a norm, and we denote by $N(V)$ the set of dilation classes of norms on V .

(1.3) EXAMPLE (I). For a basis $x_0,\ldots,x_m$ of V and real numbers $r_0 > 0 ,\ldots, r_m > 0$ the function

$$\alpha(a_0x_0 + \cdots + a_mx_m) = \sup(r_0|a_0| ,\ldots, r_m|a_m|)$$

is a norm on V . In fact, every norm can be described by this formula. This is proved by induction on m by considering a nonzero linear form $\lambda: V \to F$ and choosing $x_0 \neq 0$ in V such that $x \mapsto |\lambda(x)|/\alpha(x)$ defined on the compact projective space $\mathbb{P}(V)$ takes its maximum at x_0 . Then $V = Fx_0 \oplus \ker(\lambda)$ for which on $\ker(\lambda)$ the inductive hypothesis applies. See Goldman and Iwahori for the details.

(1.4) EXAMPLE (II). The norm α_M associated to a lattice M in V is given by the formula $\alpha_M(x) = \inf\{1/|a| : ax \in M\}$.

Observe that M is the unit α_M-ball in V of all $x \in V$ with $\alpha_M(x) \leqslant 1$, and for $x \neq 0$ we have $\alpha_M(x) = q^{-m}$ where $p^m = \{a \in F: ax \in M\}$. For two lattices L and M the inclusion $L \subset M$ holds if and only if $\alpha_L(x) \geqslant \alpha_M(x)$ for all $x \in V$.

For nonzero $c \in F$ we have $\alpha_{cM}(x) = (1/|c|)\alpha_M(x)$ for all $x \in V$, and in particular, $\alpha_{\pi M}(x) = q\alpha_M(x)$. Finally, if $x_0,\ldots,x_m$ is a basis of a lattice M over R , then

$$\alpha_M(a_0x_0 + \cdots + a_mx_m) = \sup\{|a_0| ,\ldots, |a_m|\} .$$

Thus this example is a special case of (1.4).

Going back to $\alpha(a_0x_0 + \cdots + a_mx_m) = \sup(r_0|a_0| ,\ldots, r_m|a_m|)$, we see that replacing x_i by cx_i for $c \neq 0$ replaces r_i by $r_i/|c|$. In particular, x_i replaced by $\pi^s x_i$ leads to r_i replaced by $q^s r_i$. Hence, by rescaling the basis vectors, we can always require the constants r_i to be in an interval of the form $[r,qr)$ or $(r,qr]$ for some given $r > 0$. So when $\alpha(V) = \{0\} \cup q^{\mathbb{Z}}$, we can choose the x_i with each $r_i = 1$, and this gives the next proposition.

(1.5) PROPOSITION. The following are equivalent for $t > 0$ and a norm α on a vector space V :

(1) $\alpha = t\cdot\alpha_M$ for some lattice M ,

(2) $\alpha(V) = |F|\cdot t$, and

(3) $\alpha(a_0x_0 + \cdots + a_mx_m) = t\cdot\sup(|a_0|, \ldots, |a_m|)$ for some basis $x_0,\ldots,x_m$ of V .

Now we wish to study to what extent the representation

$$\alpha(a_0x_0 + \cdots + a_mx_m) = \sup(r_0|a_0|, \ldots, r_m|a_m|)$$

for a norm α is unique by renormalizing and reordering so that

$$q \geqslant r_0 \geqslant r_1 \geqslant \cdots \geqslant r_{m-1} \geqslant r_m \geqslant 1 \ .$$

Then the basis elements are unique up to multiplication by $\pi^{\pm 1}$ and cyclic permutation. The requirement that $r_m > 1$ removes this ambiquity, and further after a dilation of α we can assume that $r_0 = q$. The set of values $\alpha(V) = \{0\} \cup q^{\mathbf{Z}}r_0 \cup \cdots \cup q^{\mathbf{Z}}r_m$, and thus the numbers r_i are uniquely determined by α . They make up the set $\alpha(V) \cap (1,q]$ when we require $r_m > 1$. With these notations, consider the lattices L_i for $i = m, m-1, \ldots, 0$ where

$$L_i = \begin{cases} Rx_0 + \cdots + Rx_i + R\pi^{-1}x_{i+1} + \cdots + R\pi^{-1}x_m \ , & \text{for } r_i > r_{i+1} \\ L_{i+1} \ , & \text{for } r_i = r_{i+1} \ , \end{cases}$$

and $L_m = Rx_0 + \cdots + Rx_m$.

(1.6) LEMMA. The lattice L_i is the open ball $B(0,qr_i)$.

PROOF. For $x = \sum a_jx_j \in V$ clearly $x \in L_i$ if and only if $|a_j| \leqslant 1$ if $j \leqslant i$ and $|a_j| \leqslant q$ if $j > i$, or equivalently, $r_j|a_j| \leqslant r_j$ for $j \leqslant i$ and $r_j|a_j| \leqslant qr_j$ for $j > i$. Now assume that $r_i > r_{i+1}$, and we see that both inequalities are equivalent to $r_j|a_j| < r_iq$. Hence, $x \in L_i$ if and only if $\alpha(x) < qr_i$. If $r_i = r_{i+1}$, then $L_i = L_{i+1}$, and so the result holds by induction from m to 0 .

Finally, we obtain the following structure theorem.

(1.7) THEOREM. Let α be a norm on a vector space V with $\alpha(V) \cap (1,q] = S$, and for $r \in S$, let $L(r) = B(0,qr)$ an open ball. Then $\dim V \geqslant \mathrm{Card}(S)$ and $\alpha = \sup_{r\in S}(r\alpha_{L(r)})$.

PROOF. As in (1.3), we can choose a basis $x_0,\dots,x_m$ of V with

$$\alpha(a_0,x_0 + \cdots + a_mx_m) = \sup(r_0|a_0|,\dots,r_m|a_m|)$$

and $q \geqslant r_0 \geqslant r_1 \geqslant \cdots \geqslant r_m \geqslant 1$. The r_i exhaust the set S with possible repetitions and thus $\dim V \geqslant \mathrm{Card}(S)$.

By (1.6) the lattice $L_i = L(r_i)$ for $r_i > r_{i+1}$ is given by $L_i = Rx_0 + \cdots + Rx_i + R\pi^{-1}x_{i+1} + \cdots + R\pi^{-1}x_m$. The norm $\alpha_i = \alpha_{L(r_i)}$ is given by

$$\alpha_i(a_0x_0+\cdots+a_mx_m) = \sup(|a_0|,\dots,|a_i|, q^{-1}|a_{i+1}|,\dots,q^{-1}|a_m|) ,$$

and since $rq^{-1}|a_{i+1}| \leqslant r_{i+1}|a_{i+1}|$, it follows that

$$\alpha(x) = \sup(r_0\alpha_0(x),\dots,r_m\alpha_m(x)) = \sup_{r\in S}(r\cdot\alpha_{L(r)}(x)) .$$

This proves the theorem.

§2. THE BUILDING FOR PGL(V) OVER A LOCAL FIELD

We continue with the notations (1.1) in this section. The dilation (or homothety) class $\{L\}$ of a lattice L is the set of all lattices λL where $\lambda \in F^{\times}$ in the vector space V. Observe that $\{L\}$ is the set of all π^iL for i in the integers.

Let $PGL(V)$ denote $GL(V)/(\text{scalars})$ for a vector space V.

(2.1) DEFINITION. Let V be a vector space over the local field K. The building $I(V)$ for the group $PGL(V)$ over K is the simplicial complex whose vertices are dilation classes $\{L\}$ of lattices L in V, and whose simplexes $\{v_0,\dots,v_n\}$ are sets of vertices where after reordering $v_i = \{L_i\}$ with $L_0 > L_1 > \cdots > L_n > \pi L_0$.

Observe that the ordering of the vertices $v_0,\dots,v_n$ such that representatives $L_i \in v_i$ can be chosen with $L_0 > \cdots > L_n > \pi L_0$ is unique up to the action of the cyclic group on $n+1$ elements inside the symmetric group.

The simplicial complex $I(V)$ has dimension equal to $\dim V - 1$, and each simplex is contained in a top dimensional simplex (one whose dimension equals

$\dim I(V)$). For $\dim V = 2$, the building $I(V)$ is one dimensional, a graph. In fact, $I(V)$ is simply connected and hence a tree. Each vertex $v = \{L\}$ is contained in $(m+1)$ 1-simplexes $\{v,v'\}$ corresponding to the $m+1$ lattices L' with $L > L' > \pi L$.

This building $I(V)$ is a special case of the buildings (immeubles) which have been associated to general semisimple groups over a local field by Bruhat and Tits. These simplicial complexes are contractible, and the vertices are in natural bijective correspondence with the cosets of $PGL(V)/$(maximal compact subgroup K). Here K is the image of $GL(L)$ in $PGL(V)$ for a lattice L of V. This description together with other features suggest that $I(V)$ is an analogue of the symmetric space for real Lie groups.

Now recall some generalities on geometric realizations as applied to $I(V)$. The geometric realization $I(V)_{\mathbb{R}}$ is the subset of $(t_v) \in \prod_{v\in I(V)}[0,1]$ such that $\{v: t_v \neq 0\}$ is a simplex of $I(V)$ and $\sum_v t_v = 1$. For each simplex $\sigma = \{v_0,\dots,v_n\}$ of $I(V)$ its geometric realization $|\sigma| \subset I(V)_{\mathbb{R}}$ is the subset of $(t_v) \in I(V)_{\mathbb{R}}$ with $t_v = 0$ for $v \notin \sigma$. As a subset of $\mathbb{R}^{n+1} \supset |\sigma|$ is compact, we give $I(V)$ the inductive (weak) topology where M is closed in $I(V)_{\mathbb{R}}$ if and only if $M \cap |\sigma|$ is closed in $|\sigma|$ for each simplex σ of $I(V)$. We will make use of the dense subset $I(V)_{\mathbb{Q}} \subset I(V)_{\mathbb{R}}$ consisting of (t_v) with each $t_v \in \mathbb{Q}$. The set $I(V)_{\mathbb{Z}} \subset I(V)_{\mathbb{Q}}$ of (t_v) with each $t_v \in \mathbb{Z}$ can be identified with the set of vertices of $I(V)$.

The function which links the considerations of this section with those of the previous section is $\theta: I(V)_{\mathbb{R}} \to N(V)$ from the geometric realization of the building to the dilation classes of norms on V as follows: Let $\sigma = \{v_0,\dots,v_n\}$ be a simplex and $t = (t_v) \in |\sigma|$ be a point. We can choose an ordering $\sigma = (v_0,\dots,v_n)$ such that $t_n > 0$ and lattices $L_i \in v_i$ with $L_0 > L_1 > \cdots > L_n > \pi L_0$. Then $\theta(t) = \alpha$ where $\alpha = \sup\left(q^{t_i+\cdots+t_n}\cdot\alpha_{L_i}\right)$.

(2.2) THEOREM. The function $\theta: I(V)_{\mathbb{R}} \to N(V)$ is a well-defined bijection which carries the vertices $I(V)_{\mathbb{Z}}$ onto the set of classes containing integral norms and $I(V)_{\mathbb{Q}}$ onto the set of classes containing rational norms.

PROOF. The function θ is a bijection by (1.10), the structure theorem for norms. In that theorem we proved that each norm α has a unique representation $\alpha = \sup(r_0\alpha_{L_0},\dots,r_m\alpha_{L_m})$ where $q = r_0 \geqslant r_1 \geqslant \cdots \geqslant r_m > 1$ up to dilation. Let $r_i = q^{t_i+\cdots+t_m}$ or $t_i = \log_q(r_i/r_{i+1})$. Then $(t_0,\dots,t_m)$ determines the unique point in $|(\{L_0,\dots,L_m\})|$ which maps to α under θ. The remaining statements are clear from the formulas relating the t_i's and

and r_i's. This proves the theorem.

§3. METRIC ON THE BUILDING

We continue with the notations of (1.1) in this section.

(3.1) DEFINITION. Let V be a vector space over the local field K. For two norms α,β on V, we define the distance $\rho(\alpha,\beta)$ between α and β by the following equivalent formulas:

$$\begin{aligned}\rho(\alpha,\beta) &= \log_q\Big(\sup_{x\in V,\, x\neq 0} \alpha(x)/\beta(x)\Big) + \log_q\Big(\sup_{x\in V,\, x\neq 0} \beta(x)/\alpha(x)\Big)\\ &= \log_q\Big(\sup_{x\in V,\, x\neq 0} \alpha(x)/\beta(x)\Big) - \log_q\Big(\inf_{x\in V,\, x\neq 0} \alpha(x)/\beta(x)\Big)\end{aligned}$$

(3.2) REMARK. From the first formula for $\rho(\alpha,\beta)$ we see that $\rho(\alpha,\beta) = \rho(\beta,\alpha)$, and from the second form it follows that $\rho(\alpha,\beta) \geqslant 0$, and $\rho(\alpha,\beta) = 0$ if and only if $\beta = t\cdot\alpha$ for some $t > 0$. Moreover, for $t_1 > 0$ and $t_2 > 0$ we have $\rho(t_1\alpha,t_2\beta) = \rho(\alpha,\beta)$, and thus

$$\rho(\{\alpha\},\{\beta\}) = \rho(\alpha,\beta)$$

is well defined on dilation classes of norms on V. Finally, it is easy to check the triangle inequality

$$\rho(\alpha,\gamma) \leqslant \rho(\alpha,\beta) + \rho(\beta,\gamma)$$

using the relations of the form

$$\sup_{x\in V,\, x\neq 0} \alpha(x)/\beta(x) = \sup_{x\in V,\, x\neq 0} \alpha(x)/\beta(x). \sup_{x\in V,\, x\neq 0} \beta(x)/\gamma(x) .$$

Hence ρ is a metric on the space $N(V)$ of dilation classes of norms on V.

Note from the definition of ρ that if $t_1\alpha \leqslant \beta \leqslant t_2\alpha$ on V, then $\rho(\alpha,\beta) \leqslant \log_q(t_1/t_2)$.

(3.3) EXAMPLE. Let M be a lattice in V. Then the integral norm α_M is defined as in (1.4), and

$$M = \{x \in V\colon \alpha_M(x) \leqslant 1\} \quad\text{and}\quad \pi M = \{x \in V\colon \alpha_M(x) < 1\} .$$

Thus $M - \pi M$ is the unit α_M-sphere of all x with $\alpha_M(x) = 1$ in V. Moreover, since every $x \in V$, $x \neq 0$, is proportional to some x' with

$\alpha_M(x') = 1$, we deduce that

$$\rho(\alpha,\alpha_M) = \log_q\left(\sup_{x\in M-\pi M}\alpha(x)\right) - \log_q\left(\inf_{x\in M-\pi M}\alpha(x)\right) .$$

For two lattices M and N in V , there exists a natural number r with $M \supset \pi^r N$ and $N \supset \pi^r M$ so that $\alpha_M \leqslant q^r\alpha_N$ and $\alpha_N \leqslant q^r\alpha_M$. We obtain $\rho(\alpha_M,\alpha_N) \leqslant 2r$. For a more precise calculation, we reorganize the hypothesis in the next proposition.

(3.4) PROPOSITION. Let $M \supset N \supset \pi^r M$ be lattices in V where $\pi M \not\supset N \not\supset \pi^{r-1}M$. Then $r = \rho(\alpha_M,\alpha_N)$.

PROOF. From $\alpha_M \leqslant \alpha_N \leqslant q^r\alpha_M$ we deduce that $\rho(\alpha_M,\alpha_N) \leqslant r$. Since $\pi M \not\supset N$, we have $x \in (M-\pi M) \cap N$ and therefore $\alpha_N(x) \leqslant 1$. Thus $-\log_q(\inf_{x\in M-\pi M}\alpha_N(x)) \geqslant 0$. Since $\pi^{-r+1}N \not\supset M$ and $\pi^{-r+1}N \supset \pi M$, there exists $x \in M-\pi M$ with $x \in \pi^{-r+1}N$ and so $\alpha_N(x) \geqslant q^r$. Thus $\log_q(\sup_{x\in M-\pi M}\alpha_N(x)) \geqslant r$. By the example (3.3) it follows that $\rho(\alpha_N,\alpha_M) \geqslant r$ and hence $\rho(\alpha_N,\alpha_M) = r$.

(3.5) COROLLARY. A set of lattices $M_0,\ldots,M_r$, or equivalently, a set of integral norms $\alpha_0,\ldots,\alpha_r$ (for example $\alpha_i = \alpha_{M_i}$) , determine a simplex in the building $I(V)_{\mathbb{Z}}$ or $N(V)$ if and only if $\rho(\alpha_i,\alpha_j) = 1$ for $i \neq j$.

PROOF. Arrange, after reordering and dilation, the lattices as follows: $M_0 \supset M_1 \supset \cdots \supset M_{r-1} \supset M_r \supset \pi^i M_0$ with $M_r \not\supset \pi^{i-1}M_0$. Then $1 = \rho(\alpha_{M_0},\alpha_{M_r}) \geqslant i$ by (3.4), so $i = 1$, and the classes $\{M_i\}$ form a simplex. Conversely, $\rho(\alpha_i,\alpha_j) = 1$ for $i \neq j$ when the classes determine a simplex again by (3.4).

§4. THE MAPPING FROM THE p-ADIC SYMMETRIC SPACE TO THE BUILDING

Now we return to the basic situation of the function field $F = \mathbb{F}_q(C)$ of the smooth curve $C/\mathbb{F}_q$, the local field F_∞ at ∞ on the curve, and C_∞ the completion of the algebraic closure of F_∞ .

The simple critical observation is the following: For $z = (z_1,\ldots,z_r) \in C_\infty^r$ the function on F_∞^r

$$a = (a_j) \longmapsto a_z(a) = |z_1a_1 + \cdots + z_ra_r|$$

is a norm on the F_∞-vector space F_∞^r provided $|z_1a_1 + \cdots + z_ra_r| = 0$ implies that $a = (a_1,\ldots,a_r) = 0$. This is the case exactly for

$$z \in C_\infty^r - \{\text{all } F_\infty\text{-rational hyperplanes}\} .$$

Further, for $c \in C_\infty$ and $z \in C_\infty^r$ the relation $\alpha_{cz} = |c|\alpha_z$, which is a dilation of norms, holds and this leads to the following definition.

(4.1) DEFINITION. The building map λ defined on the p-adic symmetric space $\Omega^r(C_\infty)$ to the building $N(F_\infty^r) = I(F_\infty^r)_{\mathbb{R}}$ of the group $PGL(r,F_\infty)$ is given by

$$z = (z_j) \in \Omega^r \longmapsto \lambda(z) = \text{dilation class of } \alpha_z .$$

For a representative r-tuple $(z_1,\ldots,z_r) = z$ of $z \in \Omega^r$, we represent $\lambda(z)$ as the norm α_z, i.e.

$$\lambda(z)(a_1,\ldots,a_r) = |z_1a_1 + \cdots + z_ra_r| .$$

Note, since $|C_\infty^\times| = q^{\mathbb{Q}}$, in fact $\lambda\colon \Omega^r(C_\infty) \longrightarrow I(F_\infty^r)_{\mathbb{Q}}$.

(4.2) PROPOSITION. The building map $\lambda\colon \Omega^r(C_\infty) \longrightarrow I(F_\infty^r)_{\mathbb{Q}}$ is $GL(r,F_\infty)$-equivariant for right actions. In particular, it is also $GL(r,F)$-equivariant.

PROOF. For $s \in GL(r,F_\infty)$ we view the matrix as acting on the left and a^ts the action on the right. The norm $\lambda(z)$ acted on by s on the right is $\lambda(z)s$, or for $a \in F_\infty^r$, it is $\lambda(z)(s(a)) = |\langle z|sa\rangle| = |\langle z^ts|a\rangle| = \lambda(z^ta)(a)$. Thus $\lambda(z)s = \lambda(z^ts)$ which proves the proposition.

For a subgroup $\Gamma \subset GL(r,F_\infty)$ we have a quotient building mapping

$$\lambda_\Gamma\colon \Omega^r(C_\infty)/\Gamma \longrightarrow I(F_\infty^r)_{\mathbb{Q}}/\Gamma \subset I(F_\infty^r)_{\mathbb{R}}/\Gamma .$$

In the case Γ is a certain subgroup of $GL(r,A)$ this is a mapping of the corresponding moduli space associated with Γ to a quotient of the building by the discrete group Γ.

(4.3) REMARKS. The building map λ is useful for several purposes. First, the sets λ^{-1} (ball around a vertex or simplex) can be used to describe the rigid analytic structure on the p-adic symmetric space. In the special case $r = 2$ so that $I(F_\infty^2)$ is a tree T, we will describe a topological model for $\Omega^2(C_\infty) = C_\infty - F_\infty$ and the map $\lambda: C_\infty - F_\infty \longrightarrow T$. This is used to calculate the cohomology of $\Omega^2(C_\infty)/\Gamma$. There are some coverings of T/Γ which induce back to admissible covers of $\Omega^2(C_\infty)/\Gamma$, and these give rise to a spectral sequence which could be thought of as the Leray spectral sequence of the map $\lambda_\Gamma: \Omega^2(C_\infty)/\Gamma \longrightarrow T/\Gamma$. In this case $H^*(T/\Gamma)$ is just the cohomology of the group Γ.

In order to illustrate further what is involved in defining the rigid analytic structure, we must calculate $\rho(\lambda(z),\alpha_\Lambda)$ where Λ is the standard lattice $\mathcal{O}_\infty^r \subset F_\infty^r$. Since any other lattice is Λ up to the action of $GL(r,F_\infty)$ and since both ρ and λ are $GL(r,F_\infty)$-invariant, this calculation leads to $\rho(\lambda(z),\alpha_L)$ for any vertex α_L.

By (3.3) we have

$$\rho(\lambda(z),\alpha_\Lambda) = \log_q\Big(\sup_{a\in S(\Lambda)} |a_1z_1+\cdots+a_rz_r|\Big) - \log_q\Big(\inf_{a\in S(\Lambda)} |a_1z_1+\cdots+a_rz_r|\Big)$$

where $S(\Lambda) = \Lambda - \pi\Lambda$, the set of all $(a_1,\ldots,a_r) \in F_\infty^r$ with all $|a_i| \leqslant 1$ and at least one $|a_i| = 1$.

Thus we see that $\rho(\lambda(z),\alpha_\Lambda) \leqslant s$ if and only if for all $a,b \in S(\Lambda)$

$$\frac{1}{q^s} \leqslant \frac{|a_1z_1+\cdots+a_rz_r|}{|b_1z_1+\cdots+b_rz_r|} \leqslant q^r .$$

In the case $|z_1| > |z_2| > \cdots > |z_r|$ with all ratios $|z_i|/|z_j| \notin q^{\mathbf{Z}}$ for $i = j$, we have

$$\rho(\lambda(z),\alpha_\Lambda) = \log_q(|z_1|/|z_r|)$$

by an easy straightforward calculation. When some of the ratios $|z_i|/|z_j| \in q^{\mathbf{Z}}$ the calculation of $\rho(\lambda(z),\alpha_\Lambda)$ is more complicated, and in fact, it is at the basis for the structure of $\Omega^r(C_\infty)$ as a rigid analytic space.

§5. FILTRATION OF THE 1-DIMENSIONAL p-ADIC SYMMETRIC SPACE

In this section we study the case $r = 2$ in detail where $\Omega^2(C_\infty) = \mathbb{P}_1(C_\infty) - \mathbb{P}_1(F_\infty) = C_\infty - F_\infty$. We use the notation Ω for $C_\infty - F_\infty = \Omega^2(C_\infty)$ where $u \in \Omega$ is $u \in C_\infty$ and $T = I(F_\infty^2)$ is the building which in this case is a tree (a contractible 1-dimensional simplicial complex). The building map $\lambda: \Omega \longrightarrow T$ is given by

$$\lambda(u)(a,b) = |a + bu| \quad ,$$

and we wish to study

$$\rho(\lambda(u),\alpha_\Lambda) = \log_q \frac{\sup_{(a,b)\in S(\Lambda)}|a+bu|}{\inf_{(a,b)\in S(\Lambda)}|a+bu|} = \rho(\lambda(1/u),\alpha_\Lambda)$$

in terms of congruence properties of u relative to elements of F_∞. This is formulated by using the following notion.

(5.1) DEFINITION. For $u \in C_\infty$ we define the irrational absolute value of u to be $|u|_{ir} = \inf_{a\in F_\infty}|u-a|$.

Observe that this is just the distance from u to $F_\infty \subset C_\infty$. The following properties are easily deduced from the definition.

(5.2) PROPOSITION. The irrational norm satisfies the following:

(1) For $u \in C_\infty$, $|u|_{ir} = 0$ if and only if $u \in F_\infty$.

(2) For $u \in C_\infty$ and $c \in F_\infty$ we have $|cu|_{ir} = |c|\cdot|u|_{ir}$.

(3) For $|u| \notin q^{\mathbf{Z}}$ and $u \in C_\infty$ it follows that $|u|_{ir} = |u|$.

(4) For $|u| = 1$ with residue class $\bar{u} \in \overline{\mathbb{F}}_q$ the irrational norm $|u|_{ir} = |u| = 1$ if and only if $\bar{u} \in \overline{\mathbb{F}}_q - \mathbb{F}_q$.

From the previous section we see for $|u| \notin q^{\mathbf{Z}}$ that $\rho(\lambda(u),\alpha_\Lambda) = \log_q(\max(|u|,1/|u|))$ which for $|u| < 1$ becomes simply $\rho(\lambda(u),\alpha_\Lambda) = -\log_q|u|$.

(5.3) PROPOSITION. The distance from $\lambda(u)$ to the standard vertex α_Λ in the tree T is given by

$$\rho(\lambda(u),\alpha_\Lambda) = \begin{cases} -\log_q|u|_{ir} & \text{for } |u| \leq 1 \\ -\log_q|1/u|_{ir} & \text{for } |u| \geq 1 . \end{cases}$$

PROOF. For the case $|u| \leq 1$, we have $\sup_{(a,b)\in S(\Lambda)}|a+bu| = 1$ since $|a+ub| \leq \max(|a|,|b|\cdot|u|) \leq 1$ and $|1+0\cdot u| = 1$. Further, $\inf_{(a,b)\in S(\Lambda)}|a+bu| = \inf_{|a|\leq 1,\,|b|=1}|a+bu| = \inf_{|e|\leq 1}|e+u| = |u|_{ir}$ since $|u| \leq 1$. Hence, we have

$$\rho(\lambda(u),\alpha_\Lambda) = \log_q(1/|u|_{ir}) = -\log_q|u|_{ir} \quad .$$

For the case $|u| \geq 1$ observe that $\rho(\lambda(u),\alpha_\Lambda) = \rho(\lambda(1/u),\alpha_\Lambda) = -\log_q|1/u|$ by the first part. This proves the proposition.

Now we are in a position to describe the inverse image in Ω of closed balls around the standard lattice vertex $* = \alpha_\Lambda$. First, some notation for residue class reduction $r: \mathcal{O}_{C_\infty} \longrightarrow \overline{\mathbb{F}}_q$ or $r: \mathbb{P}_1(C_\infty) \longrightarrow \mathbb{P}_1(\overline{\mathbb{F}}_q)$. We choose a cross section $s: \mathbb{P}_1(\overline{\mathbb{F}}_q) \longrightarrow \mathbb{P}_1(C_\infty)$ with $s(0) = 0$ and $s(\infty) = \infty$ such that rs is the identity on $\mathbb{P}_1(\overline{\mathbb{F}}_q)$.

(1) Clearly $u \in \lambda^{-1}(*)$ from (2.2) if and only if $|u|_{ir} = 1$. This gives the following relations for $\lambda^{-1}(*) \subset \mathcal{O}^{x}_{C_\infty}$,

$$\begin{aligned}
\lambda^{-1}(*) &= r^{-1}(\overline{\mathbb{F}}^{x}_q - \mathbb{F}^{x}_q) \\
&= \{u \in C_\infty : |u| = 1\} - \bigcup_{\xi \in \mathbb{F}^{x}_q} \{u \in C : |u - s(\xi)| < 1\} \\
&= \mathcal{O}_{C_\infty} - \bigcup_{\xi \in \mathbb{F}_q} B(s(\xi),1) \\
&= \mathbb{P}_1(C_\infty) - \bigcup_{\eta \in \mathbb{P}_1(\mathbb{F}_q)} B(s(\eta),c)
\end{aligned}$$

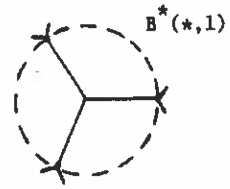
where $B(v,c)$ is the open ball of radius c around v in C_∞ for $v \neq \infty$ and $B(\infty,c) = \{\infty\} \cup \{u \in C: |u| > 1/c\} \subset \mathbb{P}_1(C_\infty)$.

(2) For the closed ball $B^*(*,c)$ with $0 \leq c < 1$ observe that $B^*(*,c) \subset$ open star of the vertex $* = \alpha_\Lambda$ in the tree T. In this case

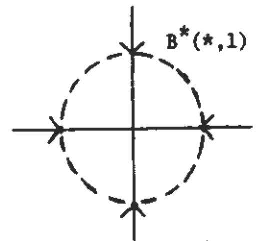
$$\lambda^{-1}(B^*(*,c)) = \mathbb{P}_1(C_\infty) - \bigcup_{\eta \in \mathbb{P}_1(\mathbb{F}_q)} B(s(\eta),q^{-c}) \quad ,$$

and again the inverse image is $\mathbb{P}_1(C_\infty)$ minus $(q+1)$ balls, but this time of slightly smaller radius $q^{-c} \leq 1$. Now we see the relation between these balls and the edges of the tree coming out from the vertex $* = \alpha_\Lambda$. All the points of $B(s(\eta),1) - B(s(\eta),q^{-c})$ project to the edge corresponding to $\eta \in \mathbb{P}_1(C_\infty)$.

(3) As the radius c of the closed ball $B^*(*,c)$ approached 1 in the tree, we come to $(q+1)$ new vertices each with q new edges coming out as c increases in $1 < c < 2$. In the building we have the pictures



(for $q = 2$)



(for $q = 3$) .

In terms of $\lambda^{-1}(B^*(*,c))$ this means that as c increases to 1 and through $[1,2)$, that $B(s(\eta),q^{-c})$ will decrease in size and at the moment $c = 1$, it will split into q balls of radius q^c for $c \in [1,2)$ each parametrized by $\mathbb{F}_q$ around the points $s(\eta) + s(\xi_1)$ where $(\eta,\xi_1) \in \mathbb{P}_1(\mathbb{F}_q) \times \mathbb{F}_q$ so that for $c \in [1,2)$

$$\lambda^{-1}(B^*(*,c)) = \bigcup_{(\eta,\xi_1) \in \mathbb{P}_1(\mathbb{F}_q) \times \mathbb{F}_q} B(s(\eta) + s(\xi_1)\pi, q^{-c}) \quad .$$

Note for $c < 1$ that $B(s(\eta),q^{-c}) = B(s(\eta) + s(\xi_1)\pi,q^{-c})$.

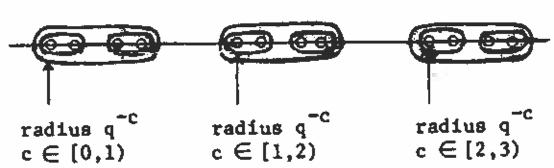
The general result follows by the same considerations as above and proved using $\rho(\lambda(u),*) = -\log_q|u_{ir}|$ for $|u| \leqslant 1$ and the relation between $|u|$ and $|u|_{ir}$.

(5.4) THEOREM. With the above notations associated with $\lambda\colon \Omega^2(C_\infty) \longrightarrow T$ we have, for $c \in [m,m+1)$ where m is an integer > 0 , the following

$$\lambda^{-1}(B(*,c)) = \mathbb{P}_1(C_\infty) - \bigcup_{\xi \in \mathbb{P}_1(\mathbb{F}_q) \times \mathbb{F}_q^m} B(s(\xi_0) + \cdots + s(\xi_m)\pi^m, q^{-c})$$

where $\xi = (\xi_0,\xi_1,\ldots,\xi_m)$ with $\xi_0 \in \mathbb{P}_1(\mathbb{F}_q)$ and $\xi_i \in \mathbb{F}_q$ for $1 \leqslant i \leqslant m$.

Thus for $m \leqslant c < m+1$ it is the projective line minus $(q+1)q^m$ balls of radius q^{-c} . We have the following picture of $\mathbb{P}_1(C_\infty)$ - {these balls} for $q = 2$:



.

(5.5) REMARK. Each $\lambda^{-1}(B(*,c))$ has the natural structure of a rigid analytic space. In this way the increasing union of the $\lambda^{-1}(B(*,c))$, which is $\mathbb{P}_1(C_\infty) - \mathbb{P}_1(F_\infty)$ has the structure of a rigid analytic space.

(5.6) REMARK. We have the following intuitive picture of the "topological" space $\mathbb{P}_1(C_\infty) - \mathbb{P}_1(F_\infty)$. For each vertex of the tree T we take a copy of

$$\mathbb{P}_1(C_\infty) - \{q+1 \text{ open discs indexed by the edges with that vertex}\}$$

and for each edge we take a copy of an annulus

$$\mathbb{P}_1(C_\infty) - \{2 \text{ open discs}\} \ .$$

Now for each edge we glue the two boundary circles of the associated annulus onto the two boundary circles in the spaces

$$\mathbb{P}_1(C_\infty) - \{q+1 \text{ open discs}\}$$

associated to the vertices of the edge respectively. The result is similar to the boundary of a tubular neighborhood of the tree T embedded in Euclidean space.

CHAPTER 4. COHOMOLOGY OF THE MODULI SPACE

The aim of this chapter is to calculate the étale cohomology of the moduli spaces M_I^2. This is done in two steps. First, we describe the rigid analytic cohomology of $\Omega^2(C_\infty)$ in terms of coclosed cochains on the building, which in this case is a tree. This consists in determining the rigid cohomology of $\mathbb{P}_1$ minus a finite union of discs and then using a patching argument over the edges of the tree with a compatibility condition at each vertex. The étale cohomology of the moduli space, which by comparison is isomorphic to the rigid analytic cohomology $H^1(\Omega^2(C_\infty)/\Gamma)$ of the analytic moduli space, is the middle term of a short exact sequence

$$0 \longrightarrow H^1(\Gamma) \longrightarrow H^1(\Omega^2(C_\infty)/\Gamma) \longrightarrow H^2(\Omega^2(C_\infty))^\Gamma \longrightarrow 0$$

where Γ is the discrete subgroup of $GL(2,F)$ corresponding to the I-level structure. Finally, the action of the inertia subgroup of $\mathrm{Gal}(F_{\infty,s}/F_\infty)$, provides an isomorphism of $H^2(\Omega^2(C_\infty))^\Gamma$ onto $H^1(\Gamma)$.

§1. GENERALITIES ON THE COHOMOLOGY OF RIGID ANALYTIC SPACES

For a complete nonarchmedian field K with separable algebraic closure K_s we make the following definitions of $H^i = H^i_{\mathrm{rigid}}$ for a rigid analytic space X over K and $i = 0,1$. We do this using the étale cohomology groups for the coefficients $\mathbb{Z}/n$ and μ_n where n^{-1} is in K.

(1.1) DEFINITION OF H^0. In both cases of coefficients $H^0(X,\mathbb{Z}/n) = H^0_{et}(X_s,\mathbb{Z}/n)$ and $H^0(X,\mu_n) = H^0_{et}(X_s,\mu_n)$ where $X_s = X \otimes_K K_s$.

Recall that $H^1_{et}(X,\mu_n)$ can be described as pairs (L,ϕ), up to an evident isomorphism, of an invertible sheaf L on X and an isomorphism $\phi\colon \mathcal{O}_X \xrightarrow{\sim} L^{n\otimes}$ with the group structure given by tensor product. Further $H^1_{et}(X_s,\mu_n) = \varinjlim_L H^1_{et}(X\otimes L,\mu_n)$ for $K \subset L \subset K_s$ and $[L:K]$ finite. These definitions have meaning for rigid analytic spaces.

(1.2) DEFINITION OF H^1. The group $H^1(X,\mu_n)$ is the group under tensor product of pairs (L,ϕ), up to isomorphism, of an invertible sheaf L on X and an isomorphism $\phi\colon \mathcal{O}_X \xrightarrow{\sim} L^{n\otimes}$. Moreover, $H^1(X_s,\mu_n) = \varinjlim_{K'} H^1(X\otimes K',\mu_n)$ where $K \subset K' \subset K_s$ and $[K' : K]$ is finite.

Both H^0 and H^1 are clearly functors under rigid morphisms.

This cohomology has the following properties which we state without any complete proofs.

(1.3) PROPOSITION. If $X = W_{an}$ for W a projective variety or an affine curve over K, then $H^i(W_s) = H^i(X_s)$ for $i = 0,1$, and coefficients μ_n and $\mathbb{Z}/n$.

This follows from the GAGA-type theorems of Kiehl. The projective comparison theorem implies the affine curve comparison theorem since a covering of $0 < |z| < r$ of order n prime to p extends to a ramified covering of $|z| < r$.

(1.4) PROPOSITION. If $f\colon X \to Y$ is a finite étale morphism of rigid analytic spaces with Galois group G of order prime to n, then $H^1(Y,\mu_n) \to H^1(X,\mu_n)^G$ is an isomorphism.

This is easy from the definitions.

(1.5) PROPOSITION (Kummer sequence). We have an exact sequence over a rigid analytic space:

$$0 \to H^0(\mu_n) \to H^0(\mathcal{O}_X^*) \xrightarrow{n} H^0(\mathcal{O}_X^*) \to H^1(\mu_n) \to H^1(\mathcal{O}_X^*) \xrightarrow{n} H^1(\mathcal{O}_X^*) .$$

(1.6) PROPOSITION. Let $\{X_i\}_{i\in I}$ be an admissible open covering of a rigid analytic space X with nerve of dimension ≤ 1, then there is an exact sequence

$$0 \to H^0(X,\mu_n) \to \prod_i H^0(X_i,\mu_n) \to \prod_{i\neq j} H^0(X_i\cap X_j,\mu_n) \to H^1(X,\mu_n) \to \prod_i H^1(X_i,\mu_n) \to \prod_{i\neq j} H^1(X_i\cap X_j,\mu_n) .$$

§2. COHOMOLOGY OF $\Omega^2(C_\infty)$

Using the generalities of the previous section and some specific information, we are able to make the following calculation.

(2.1) PROPOSITION. Let $D_0,\ldots,D_m$ be $m+1$ pairwise disjoint open discs in $\mathbb{P}_1(C_\infty)$ of radius in $|C_\infty|$. Then for the rigid analytic space $X = \mathbb{P}_1(C_\infty) - (D_0\cup\cdots\cup D_m)$ we have for n prime to p :

(a) $H^0(X_s,\mathbb{Z}/n) = \mathbb{Z}/n$

(b) $H^1(X_s,\mu_n) = (\mathbb{Z}/n)^m$

for $X_s = X \otimes_{F_\infty} F_{\infty,s}$.

PROOF. (a) This is a question of connectedness of discs in $\mathbb{P}_1(C_\infty)$ over any finite extension of F_∞ . Since $X\cup D_1\cup\cdots\cup D_m = \mathbb{P}_1(C_\infty) - D_0$ is a disc and hence connected, and since the spherical boundaries of D_i are connected, it follows that X is connected.

(b) For this, first observe that the rational fuctions without poles on X are dense in $H^0(\mathcal{O}_X)$. Next, if f is a rational function without zeros or poles on X , then there exists $c \in F_\infty$ with $\sup_{x\in X}|cf(x) - 1| < 1$, and hence, $cf = g^n$ for some $g \in H^0(\mathcal{O}_X^*)$. For S_i equal to the boundary of D_i , we have an isomorphism

$$H^1(X,\mu_n)/\mathrm{im}(F_\infty^*/(F_\infty^*)^n) \longrightarrow \bigoplus_{i=1}^{m} H^1(S_i,\mu_n)/\mathrm{im}(F_\infty^*/(F_\infty^*)^n)$$

since $H^1(\mathcal{O}_X^*) = 0$ in the Kummer sequence where every divisor on X is principal. Now S_i is a special case of the more general X , and $H^1(S_i,\mu_n)/\mathrm{im}(F_\infty^*/(F_\infty^*)^n) \simeq \mathbb{Z}/n$. This reduces to the assertion that z^j is not an nth power for $n \nmid j$ which follows by writing $z^j = f^n$, $\|f\| = 1$, and reducing modulo the maximal ideal. Hence we have the short exact sequence

$$0 \longrightarrow F_\infty^*/(F_\infty^*)^n \longrightarrow H^1(X,\mu_n) \longrightarrow (\mathbb{Z}/n)^m \longrightarrow 0 \quad .$$

Now pass to the separable algebraic closure of F_∞ through finite extensions to see that $H^1(X_s,\mu_n) = (\mathbb{Z}/n)^m$. This proves the proposition.

Now we can describe the cohomology of $\Omega = \Omega^2(C_\infty)$ using the following notion.

(2.2) DEFINITION. Let M be an abelian group, and let B be a graph (1-dimensional simplicial complex) with B_e the set of oriented edges. The group of 1-cochains $C^1(B,M)$ is the subgroup of $c \in \prod_{e \in B_e} M$ such that $c(-e) = -c(e)$, and the group of coclosed (harmonic) 1-cochains $\underline{H}^1(B,M)$ is the subgroup of $c \in C^1(B,M)$ such that $\sum_{e \in e(b)} c(e) = 0$ where the sum is over $e(b)$ the set of edges ending at b, and this holds for each vertex b of B.

The group of 1-chains $C_1(B,M)$ is the quotient group of $\coprod_{e \in B_e} M$ divided by the subgroup generated by $e - (-e)$, and the group of coclosed 1-chains $\underline{H}_1(B,M)$ is the quotient of $C_1(B,M)$ by the subgroup generated by $\sum_{e \in e(b)} c(e)$ for each vertex b of B.

(2.3) PROPOSITION. We have isomorphisms

$$H^0(\Omega^2(C_\infty)_s, \mathbf{Z}/n) \xrightarrow{\sim} \mathbf{Z}/n \quad \text{and} \quad H^1(\Omega^2(C_\infty)_s, \mu_n) \xrightarrow{\sim} \underline{H}^1(T, \mathbf{Z}/n)$$

which are compatible with the action of $GL(2,F_\infty)$ and $Gal(F_{\infty,s}/F_\infty)$. Here $\lambda\colon \Omega^2(C_\infty) \to T$ onto the tree where $GL(2,F_\infty)$ acts, and $Gal(F_{\infty,s}/F_\infty)$ acts trivially which induces an action on $\underline{H}^1(T,\mathbf{Z}/n)$.

PROOF. We write $X = \bigcup_{i \in I} X_i$ where I is the set of vertices and edges of T. If i is a vertex v, then X_i is $\lambda^{-1}(B^*(v,1/3))$, and if i is an edge e, then X_i is $\lambda^{-1}(e^*)$ where $e^* = e - \bigcup_v B(v,1/4)$ with the union taken over all vertices of T. The nerve of this covering has vertices I and is $sk(T)$ the first barycentric subdivision of T. Let $I_0 \subset I$ be the subset of $i \in I$ corresponding to the edges, choose isomorphisms $H^1(X_i,\mu_n) \to \mathbf{Z}/n$ by (2.1), and choose orientations for each $i \in I_0$.

Then the composite

$$H^1(X,\mu_n) \longrightarrow \prod_{i \in I} H^1(X_i,\mu_n) \longrightarrow \prod_{i \in I_0} H^1(X_i,\mu_n) \longrightarrow \prod_{i \in I_0} \mathbf{Z}/n$$

is seen, with (1.6), to be an isomorphism $H^1(X,\mu_n) \to \underline{H}^1(T,\mathbf{Z}/n)$. It is injective by (1.6), and the image set corresponds exactly to those functions, extended to be alternating on all ordered edges, satisfying the condition of being coclosed. This proves the proposition.

§3. A SIMPLE TOPOLOGICAL MODEL

To illustrate further the patching technique used in the previous proposition (2.3) and the ideas which go into the calculation of the cohomology of $\Omega^2(C_\infty)/\Gamma$, we consider maps of surfaces onto a graph.

(3.1) DEFINITION. A map $f: X \to B$ from a surface (real oriented 2-manifold) onto a graph is called regular provided f is proper and

(1) $f^{-1}(\text{open 1-simplex}) = S^2 - (2 \text{ disjoint closed discs})$,

(2) $f^{-1}(\text{open star of a vertex}) = S^2 - (\nu(b) \text{ disjoint closed discs})$

where $\nu(b)$ is the number of 1-simplexes of B incident to b .

Observe that the projection f of the boundary X of a tubular neighborhood N of $B \subset \mathbb{R}^3$ is an example of such a regular map $f: X \to B$. The boundary ∂X of X is the disjoint union of $f^{-1}(b)$ where $\nu(b) = 1$.

Now a cross section $s: B \to X$ of $f: X \to B$ always exists and leads to two split short exact sequences

$$0 \longrightarrow \ker H_1(f) \longrightarrow H_1(X) \xrightarrow{H_1(f)} H_1(B) \longrightarrow 0$$

$$0 \longrightarrow H^1(B) \xrightarrow{H^1(f)} H^1(X) \longrightarrow \operatorname{coker} H^1(f) \longrightarrow 0 \quad .$$

The terms $\ker H_1(f) = E^2_{0,1} = H_0(B,\tilde{H}_1(F))$ and $\operatorname{coker} H^1(f) = E_2^{0,1} = H^0(B,\tilde{H}^1(F))$ are part of the spectral sequence of the map $f: X \to B$ and $\tilde{H}_1(F)$ and $\tilde{H}^1(F)$ are systems of coefficients on B whose structure is clear from axioms (1) and (2) .

(3.2) REMARK. Following the argument of the proof of (2.3), we have short exact sequences using the groups $\underline{H}_1(B,\mathbf{Z})$ and $\underline{H}^1(B,\mathbf{Z})$ introduced in (2.2)

$$0 \longrightarrow \underline{H}_1(B,\mathbf{Z}) \longrightarrow H_1(X) \xrightarrow{H_1(f)} H_1(B) \longrightarrow 0$$

and

$$0 \longrightarrow H^1(B) \xrightarrow{H^1(f)} H^1(X) \longrightarrow \underline{H}^1(B,\mathbf{Z}) \longrightarrow 0 \quad .$$

For each edge $e \in B_e$ of B , let $X_e = f^{-1}(\text{open } e)$, and observe that

$$\coprod_{e \in B_e} H_1(X_e) \longrightarrow H_1(X) \xrightarrow{H_1(f)} H_1(B) \longrightarrow 0$$

and

$$0 \longrightarrow H^1(B) \xrightarrow{H^1(f)} H^1(X) \longrightarrow \prod_{e \in B_e} H^1(X_e)$$

are exact sequences. We choose generators of $H_1(X_e)$ and $H^1(X_e)$ such that the generator for $H_1(X_e)$ and for $H_1(X_{-e})$ are negatives of each other in $H_1(X)$.

In the case of the previous section B was a tree so that $H_1(B) = 0$ and $H^1(B) = 0$, and thus $\underline{H}_1(B,\mathbf{Z}) \xrightarrow{\sim} H_1(X)$ and $H^1(X) \xrightarrow{\sim} \underline{H}^1(B,\mathbf{Z})$ are isomorphisms.

(3.3) REMARK. For a regular map $f\colon X \longrightarrow B$ of a closed surface onto a finite graph we have $\operatorname{rk} H_1(X) = 2 \cdot \operatorname{rk} H_1(B)$ and the homology group $H_1(B)$ is isomorphic to $\ker H_1(f) \xleftarrow{\sim} \underline{H}_1(B,\mathbf{Z})$. The same statement holds in cohomology.

This assertion follows from the fact that the symplectic homology pairing $x \cdot y$ on $H_1(X)$ is nonsingular, but $x \cdot y$ restricts to zero on either of the direct summands $\ker H_1(f)$ or $H_1(B) = \operatorname{im} H_1(s)$ for a section s of f. These isotropic submodules are then maximal isotropic and thus of the same rank.

(3.4) REMARK. Let $f\colon X \longrightarrow B$ be a regular map of the surface X onto a graph B which has a finite subgraph B_0 such that $B - B_0$ is the disjoint union of m half lines $L_1,\dots,L_m$. Then each $f^{-1}(L_i)$ is a topological punctured disc, and X is a closed surface with m points deleted. We define the cuspidal cohomology $H^1_!$ of B and X by

$$H^1_!(B) = \ker(H^1(B) \longrightarrow H^1(B - B_0))$$

and

$$H^1_!(X) = \ker(H^1(X) \longrightarrow H^1(X - f^{-1}(B_0))) .$$

The cohomology exact sequence in (3.2) becomes

$$0 \longrightarrow H^1_!(B) \longrightarrow H^1_!(X) \longrightarrow \underline{H}^1_!(B,\mathbf{Z}) \longrightarrow 0$$

where $\underline{H}^1_!(B,\mathbf{Z})$ is the subgroup of $\underline{H}^1(B,\mathbf{Z})$ consisting of $c = (c_e)_{e \in B_e}$ with $c_e = 0$ for e in $B - B_0$. Again $\operatorname{rk} H^1_!(B) = \operatorname{rk} \underline{H}^1_!(B,\mathbf{Z})$, and $H^1_!(B)$ and $\underline{H}^1_!(B,\mathbf{Z})$ are isomorphic.

§4. COHOMOLOGY OF THE MODULI SPACE WITH FIXED LEVEL STRUCTURE

With the map $\Omega^2(C_\infty) \to T$ onto the tree, we were able to analyze the cohomology of $\Omega^2(C_\infty)$ in (2.3). The simple topological model is contained in (3.2). Now we consider a subgroup Γ of $GL(2,A)$ and study the cohomology of $\Gamma\backslash\Omega^2(C_\infty)$ using $\Gamma\backslash\Omega^2(C_\infty) \to \Gamma\backslash T$, the mod Γ building map.

The Leray spectral sequence of this map is the covering space spectral sequence whose terms in lowest degree yield the exact sequence with coefficients in $\mathbb{Z}/n$ or μ_n

$$0 \longrightarrow H^1(\Gamma\backslash T) \longrightarrow H^1(\Gamma\backslash\Omega^2(C_\infty)) \longrightarrow H^1(\Omega^2(C_\infty))^\Gamma \longrightarrow H^2(\Gamma\backslash T) \quad .$$

The H^1 groups for the graph $\Gamma\backslash T$ and the rigid analytic space $\Gamma\backslash\Omega^2(C_\infty)$ are defined by coverings and the building map induces coverings.

Under the hypothesis that Γ acts on T with stabilizer subgroups of points only p-groups (p prime to n), the groups $H^*(\Gamma\backslash T) = H^*(\Gamma)$ are just the cohomology groups of Γ with coefficients in μ_n or $\mathbb{Z}/n$. Further, since T is a tree, we have $H^2(\Gamma) = 0$. We obtain a short exact sequence whose last term $H^1(\Omega^2(C_\infty))^\Gamma$ is isomorphic, by (2.3), to $\underline{H}^1(\Gamma\backslash T)^\Gamma$ the group of Γ-invariant coclosed cochains on T with values in μ_n^{-1} or $\mathbb{Z}/n$. This gives the following result.

(4.1) PROPOSITION. For a congruence subgroup $\Gamma \subset GL(2,A)$ of the ideal $I \subset A$, which acts on the building T with p-groups for stabilizer subgroups, the mod Γ building map yields the cohomology exact sequence

$$0 \longrightarrow H^1(\Gamma) \longrightarrow H^1(\Gamma\backslash\Omega^2(C_\infty)) \longrightarrow \underline{H}^1(T)^\Gamma \otimes \mu_n^{-1} \longrightarrow 0$$

or with coefficients in $\mathbb{Z}/n$

$$0 \longrightarrow H^1(\Gamma\backslash T,\mathbb{Z}/n) \longrightarrow H^1(M_I^2 \otimes F_{\infty,s},\mathbb{Z}/n) \longrightarrow \underline{H}^1(T,\mathbb{Z}/n)^\Gamma \otimes \mu_n^{-1} \longrightarrow 0 \quad .$$

In order to obtain a useful cohomology, we have to look at a compactification of $\Gamma\backslash\Omega^2(C_\infty)$ or $M_I^2(C_\infty)$, or equivalently a neighborhood of infinity. As in the previous section the cuspidal cohomology $H_!^1(M_I^2 \otimes F_{\infty,s},\mathbb{Z}/n)$ is the subgroup of H^1 consisting of classes equal to zero on some neighborhood of infinity, or equivalently, of classes with image in $\underline{H}_!^1(T,\mathbb{Z}/n)_!^\Gamma \otimes \mu_n^{-1}$, i.e. coclosed Γ-invariant cochains with support compact modulo Γ.

(4.2) REMARK. Let $I_0 \subset \mathrm{Gal}(F_{\infty,s}/F_\infty)$ be the inertia subgroup. In the short exact sequence of (4.1), I_0 acts trivially on the subgroup and quotient group (the associated graded group). Then $(\sigma,x) \mapsto \sigma x - x$ defines a map

$I_0 \times H^1(M_I^2 \otimes F_{\infty,s}, \mathbf{Z}/n) \to H^1(\Gamma\backslash T, \mathbf{Z}/n)$ which is bilinear and $\mathrm{Gal}(F_{\infty,s}/F_\infty)$-equivariant. Identifying μ_n with I_0/I_0^n as $\mathrm{Gal}(F_{\infty,s}/F_\infty)$-modules, we factor this map through

$$\mu_n \otimes \underline{H}^1(T, \mathbf{Z}/n)^\Gamma \otimes \mu_n^{-1} \longrightarrow H^1(\Gamma\backslash T, \mathbf{Z}/n) \quad .$$

(4.3) PROPOSITION. Under the assumptions of (4.1) the short exact sequence of (4.1) restricts to

$$0 \longrightarrow H^1(\Gamma\backslash T, \mathbf{Z}/n) \longrightarrow H^1_!(M_I^2 \otimes F_{\infty,s}, \mathbf{Z}/n) \longrightarrow \underline{H}^1_!(T, \mathbf{Z}/n)^\Gamma \otimes \mu_n^{-1} \longrightarrow 0 \quad .$$

In the limit over $n = \ell^i$ the map of (4.2) induces

$$\underline{H}^1(T, \mathbf{Z}_\ell)^\Gamma_! \longrightarrow H^1(\Gamma\backslash T, \mathbf{Z}_\ell) \quad .$$

This map is an isomorphism after tensoring with $\mathbf{Q}_\ell$.

This proposition is the analogue of (3.4). Finally, we incorporate the isomorphism into the exact sequence to obtain a description of $H^1_!$ of the level I-moduli space as follows with the next definition.

(4.4) DEFINITION. The special representation $\mathrm{sp}_{\mathrm{Gal}}$ is the two-dimensional representation of $\mathrm{Gal}(F_{\infty,s}/F_\infty)$ through $\hat{\mathbf{Z}} \ltimes \mathbf{Z}_\ell(1)$ generated by $(\phi,0)$ and $(0,u)$ with $\phi u \phi^{-1} = u^{q_\infty}$. This representation is given by

$$(\phi,0) \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & q_\infty^{-1} \end{pmatrix} \quad \text{and} \quad (0,u) \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad .$$

This two-dimensional representation has an invariant one-dimensional subspace, and it allows us to give another interpretation of (4.3).

(4.5) THEOREM. Under the assumptions of (4.3) we have an isomorphism of $\mathrm{Gal}(F_{\infty,s}/F_\infty)$-modules

$$H^1_!(M_I^2 \otimes F_{\infty,s}, \mathbf{Q}_\ell) \simeq \underline{H}^1(T, \mathbf{Q}_\ell)^\Gamma_! \otimes \mathrm{sp}_{\mathrm{Gal}} \quad .$$

In the next chapter, we intepret this isomorphism in terms of automorphic forms on the adèle group.

The proofs of the related results on abelian varieties can be found in [SGA 7, p. 20, exposé I].

CHAPTER 5. APPLICATIONS TO RELATIONS BETWEEN AUTOMORPHIC FORMS AND GALOIS REPRESENTATIONS

In this chapter we calculate the limit cohomology $\underline{H} = \varinjlim_H H^1(M_H^2, \bar{\mathbb{Q}}_\ell)$ as a representation of $GL(2,\mathbb{A}_F) \times Gal(\bar{F}/F)$. This representation decomposes into a direct sum of $\pi \otimes \sigma(\pi)$ where $\pi = \pi' \otimes sp_{GL(2,F_\infty)}$ and $\sigma(\pi)$ is a two-dimensional Galois representation with $\sigma(\pi)|D_\infty = sp_{Gal}$.

This defines a certain map $\pi \longmapsto \sigma(\pi)$ which is given in §2 and studied further in §3 with the congruence formula for the Frobenius action on the two-dimensional Galois representation. In the last section we sketch the proof of the local Langlands' conjecture in characteristic p for $GL(2)$. This is a proof of a local result using a global theorem.

§1. COCLOSED 1-COCHAINS AND THE SPECIAL REPRESENTATION

For any $X \in \mathbb{P}_1(F_\infty)$ we denote the quotient linear map by $\gamma_X: F_\infty^2 \to F_\infty^2/X$, and observe that if L is a lattice in F_∞^2, then $\gamma_X(L)$ is a lattice in F_∞^2/X. For an ordered 1-simplex $\vec{a} = (L_0 > L_1)$ in $I(F_\infty^2) = T$, the opposite simplex $-\vec{a}$ is represented by $(L_1 > \pi L_0)$, and $\pi\gamma_X(L_0) = \gamma_X(\pi L_0) \subset \gamma_X(L_1) \subset \gamma_X(L_0)$ are lattices in the 1-dimensional space F_∞^2/X so that either $\gamma_X(\pi L_0) = \gamma_X(L_1)$ or $\gamma_X(L_1) = \gamma_X(L_0)$.

(1.1) NOTATIONS. For an ordered 1-simplex $\vec{a}$ of the tree $T = I(F_\infty^2)$, let $P(\vec{a})$ denote the subset of $X \in \mathbb{P}_1(F_\infty)$ with $\gamma_X(L_1) = \gamma_X(L_0)$ where $\vec{a} = (L_0 > L_1)$.

From the above remark we see that $\mathbb{P}_1(F_\infty) = P(\vec{a}) \cup P(-\vec{a})$ is a partition of the projective line. If $\vec{a}_0, \ldots, \vec{a}_s$ are all the ordered 1-simplexes issuing from a vertex $\{L_0\}$, then $\mathbb{P}_1(F_\infty) = P(\vec{a}_0) \sqcup \cdots \sqcup P(\vec{a}_s)$ is also a partition of the projective line. Thus $P(-\vec{a}_0) = P(\vec{a}_1) \sqcup \cdots \sqcup P(\vec{a}_s)$ is a partition of any $P(\vec{a})$ which leads to the assertion that the $P(\vec{a})$ generate the Boolean algebra of open compact subsets of $\mathbb{P}_1(F_\infty)$.

(1.2) REMARK. The end or boundary points of the tree $T = I(F_\infty^2)$ are given by half infinite simplicial paths. Fixing a vertex L_0 of T, we assign to each point $X \in \mathbb{P}_1(F_\infty)$ a sequence of lattices

$L_0 > L_1 > \cdots > L_m > L_{m+1} > \cdots$ with $L_m > L_{m+1} > \pi L_m$ and $L_m \supset \pi^{m-1} L_0$ satisfying $\gamma_X(L_i) = \gamma_X(L_{i+1})$ for $i \geq 0$. All half-infinite simplicial paths from L_0 arise this way and each one determines an end of T.

(1.3) NOTATIONS. For a group M we map the coclosed cochains of T into the M-valued measures on $\mathbb{P}_1(F_\infty)$ by $c \mapsto \mu_c$ where $\mu_c(P(\vec{a})) = c(\vec{a})$. The map is defined $\underline{H}^1(T,M) \to \mathrm{Meas}(\mathbb{P}_1(F_\infty),M)$.

For the ordered 1-simplexes $\vec{a}_0,\ldots,\vec{a}_s$ issuing from a vertex, we have

$$\mu_c(P(-\vec{a}_0)) = \mu_c(P(\vec{a}_1)) + \cdots + \mu_c(P(\vec{a}_s))$$

from the coclosed condition $c(-\vec{a}_0) = c(\vec{a}_1) + \cdots + c(\vec{a}_s)$. This relation is sufficient to show that μ_c is a finitely additive set function on the family of compact open subsets of $\mathbb{P}_1(F_\infty)$. From the partition $\mathbb{P}_1(F_\infty) = P(\vec{a}) \sqcup P(-\vec{a})$ we obtain $\mu_c(\mathbb{P}_1(F_\infty)) = 0$ so that μ_c has total mass equal to zero.

(1.4) PROPOSITION. The function $c \mapsto \mu_c$ is an isomorphism $\underline{H}^1(T,M) \to \mathrm{Meas}(\mathbb{P}_1(F_\infty),M)_0$ of the M-valued coclosed cochains of $\dot{T}$ onto the M-valued measures of total mass zero.

PROOF. The inverse of $c \mapsto \mu_c$ is given by $\mu \mapsto c_\mu$ where $c_\mu(\vec{a}) = \mu(P(\vec{a}))$. The coclosed condition for c_μ follows from the finite additivity and total mass zero by the relations made explicit in (1.3). This proves the proposition.

Now the measures on $\mathbb{P}_1(F_\infty)$ are linear functionals on the space C^∞ of locally constant functions on $\mathbb{P}_1(F_\infty)$. This space $C^\infty(\mathbb{P}_1(F_\infty))$ is an important representation space under translation by $GL(2,F_\infty)$ on $\mathbb{P}_1(F_\infty)$. This representation is related to the special representation of the group $GL(2,F_\infty)$. It is the key link between cohomology as described by coclosed cochains and representation theory.

(1.5) DEFINITION. Let D be a ring of scalars. Then the special representation sp (or sp(D)) of $GL(2,F_\infty)$ (with values in D) is defined on the module $V_{sp} = C_0^\infty(\mathbb{P}_1(F_\infty),D)/D$ where D also denotes the subspace of constant functions and the action of $GL(2,F_\infty)$ is given by translation of functions $(sf)(x) = f(s^{-1}x)$ for $f \in C_0^\infty(\mathbb{P}_1(F_\infty),D)$, $x \in \mathbb{P}_1(F_\infty)$, and $s \in GL(2,F_\infty)$.

(1.6) PROPOSITION. The function $c \mapsto \mu_c$ of (1.5) is an isomorphism $\underline{H}^1(T,D) \to \mathrm{Hom}_D(V_{sp},D)$, the algebraic dual.

PROOF. This is immediate from (1.5) and the fact that $\mathrm{Meas}(\mathbb{P}_1(F_\infty),D)$ is the algebraic dual of $C^\infty(\mathbb{P}_1(F_\infty),D)$.

Now we use the group action on V_{sp} to obtain still another version of the previous two propositions.

(1.7) PROPOSITION. The function $\psi(f,s) \mapsto \psi(f,e) = \phi(f)$ defines an isomorphism

$$\mathrm{Hom}_{GL(2,F_\infty)}(V_{sp},C^\infty(GL(2,F_\infty))) \longrightarrow \mathrm{Hom}_D(V_{sp},D)$$

with inverse $\phi(f) \mapsto \psi(f,s) = \phi(sf)$.

PROOF. The $GL(2,F_\infty)$-morphism condition on $\psi(f,s)$ for $s,t \in GL(2,F_\infty)$ takes the form $\psi(tf,s) = \psi(f,st)$. Setting $s = 1$, we obtain $\phi(tf) = \psi(f,t)$ and the two maps are inverse to each other.

We use the notation $L(GL(2,F_\infty))$ for $C^\infty(GL(2,F_\infty))$. Let Γ be a subgroup of $GL(2,F_\infty)$. It acts on the tree T , the space $\mathbb{P}_1(F_\infty)$, the representation V_{sp} and its dual V'_{sp} , and also on $GL(2,F_\infty)$. Hence the isomorphisms of (1.4), (1.6) and (1.7) restrict to isomorphisms

$$\underline{H}^1(T,D)^\Gamma \xrightarrow{\sim} \mathrm{Hom}_D(V_{sp},D)^\Gamma \xleftarrow{\sim} \mathrm{Hom}_{GL(2,F_\infty)}(V_{sp},L(GL(2,F_\infty)/\Gamma)) \quad .$$

Now we assume that Γ is a subgroup of $GL(2,A)$ of finite index. For each parabolic P over the global field F of $GL(2)$ with unipotent radical U , we form $f_P(x) = \int_{u\in U/\Gamma\cap U} f(xu)du$. Note that U is conjugate to $\left|\begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}\right|$ by an element of $GL(2,F)$.

(1.8) DEFINITION. A function f on $GL(2,F_\infty)/\Gamma$ is cuspidal provided $f_P(x) = 0$ for all parabolic P of $GL(2)$ over F . Let $L_0(GL(2,F_\infty)/\Gamma)$ denote the subspace of cuspidal $f \in L(GL(2,F_\infty)/\Gamma)$.

For $f \in L(GL(2,F_\infty)/\Gamma)$ which is the image of an element of V_{sp} by a homomorphism in $\mathrm{Hom}_{GL(2,F_\infty)}(V_{sp},L(GL(2,F_\infty)/\Gamma))$, the function f is cuspidal if it has compact support modulo Γ and the center of $GL(2)$. This gives the next proposition.

(1.9) PROPOSITION. The above isomorphism

$$\underline{H}^1(T,D)^{\Gamma} \xrightarrow{\sim} \mathrm{Hom}_{GL(2,F_\infty)}(V_{sp}, L(GL(2,F_\infty)/\Gamma))$$

restricts to an isomorphism

$$\underline{H}^1(T,D)^{\Gamma}_{!} \xrightarrow{\sim} \mathrm{Hom}_{GL(2,F_\infty)}(V_{sp}, L_0(GL(2,F_\infty)/\Gamma)) \quad ,$$

where recall $\underline{H}^1(T,D)^{\Gamma}_{!}$ denotes the coclosed cochains which are Γ-invariant and have compact support modulo Γ.

§2. LIMIT COHOMOLOGY AND AUTOMORPHIC FORMS

Recall in 2(5.7) we have an adelic and a local description of the C_∞-valued points $M^r_H(C_\infty)$ of the moduli scheme M^r_H for open compact subgroups $H \subset GL(r,\hat{A})$. Our aim in this section is to relate $\bar{\mathbb{Q}}_\ell$-valued automorphic forms to the limit cohomology $\varinjlim_H H^1_!(M^2_H(C_\infty),\bar{\mathbb{Q}}_\ell)$ using the isomorphism in (1.9) and the cohomology calculation (4.5). For this we make use of the special representations sp_{Gal}, see Ch. 4(4.5), and $sp_{GL(2)}$, see (1.5), with values in $\bar{\mathbb{Q}}_\ell$, the algebraic closure of the ℓ-adic numbers.

(2.1) DEFINITION. The space $L_0(GL(2,F)\backslash GL(2,\mathbb{A}_F))$ of cuspidal automorphic forms with values in $\bar{\mathbb{Q}}_\ell$ consists of functions $f\colon GL(2,F)\backslash GL(2,\mathbb{A}_F) \longrightarrow \bar{\mathbb{Q}}_\ell$ such that

(a) f is invariant by an open compact subgroup,

(b) the $GL(2,F_\infty)$-transforms of f generate a finite direct sum of irreducible representations, and

(c) f is cuspidal, i.e. $\int_{U(F)\backslash U(\mathbb{A}_F)} f(ux)du = 0$ for all x where U consists of matrices

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \quad .$$

For a function f satisfying (a) and (b) in (2.1), we have that f is in L_0 if and only if it has support compact modulo the center of $GL(2,F)$.

Now we are prepared to relate cohomology and automorphic forms. In (1.9) for a group D of scalars, we studied an isomorphism

$$\theta\colon \underline{H}^1(T,D) \longrightarrow \mathrm{Hom}_{GL(2,F_\infty)}(V_{sp}, C^\infty(GL(2,F_\infty),D))$$

which preserved the action of $GL(2,F_\infty)$ and restricted to certain submodules as an isomorphism. We apply this now to the case $D = C^{\infty}(GL(2,\mathbb{A}_F^f),\bar{\mathbb{Q}}_\ell)$ and obtain an isomorphism

$$\theta\colon \underline{H}^1(T,C^{\infty}(GL(2,\mathbb{A}_F^f),\bar{\mathbb{Q}}_\ell)) \xrightarrow{\sim} \mathrm{Hom}_{GL(2,F_\infty)}(V_{sp},C^{\infty}(GL(2,\mathbb{A}_F^f) \times GL(2,F_\infty),\bar{\mathbb{Q}}_\ell)) \quad .$$

Using the two coset space descriptions of M_H^r, see 2(5.7), and the cohomology calculation 4(4.5), we have a restriction of this θ to θ_H where θ_H is canonical on the associated graded group and defined

$$\theta_H\colon H_!^1(M_H,\bar{\mathbb{Q}}_\ell) \xrightarrow{\sim} \mathrm{Hom}_{GL(2,F_\infty)}(V_{sp},L_0(GL(2,F)\backslash GL(2,\mathbb{A}_F^f) \times GL(2,F_\infty)/H)) \otimes sp_{Ga}$$

as (Centralizer (H) in $GL(2,\mathbb{A}_F)$) $\times$ $Gal(F_{\infty,s}/F_\infty)$ -representations. For this we use the calculation

$$H_!^1(M_H,\bar{\mathbb{Q}}_\ell) = \coprod_{xH \in GL(2,\mathbb{A}_F^f)/H} \left\{\begin{matrix}\text{coclosed cochains of } T\\ \text{invariants by } xHx^{-1} \cap GL(2,F)\\ \text{with compact support}\end{matrix}\right\} \otimes sp_{Gal}$$

$$= \left\{\begin{matrix}\text{coclosed cochains on } Gal(2,\mathbb{A}^f) \times T\\ \text{invariant by } H \text{ and } GL(2,F) \text{ with}\\ \text{compact support}\end{matrix}\right\} \otimes sp_{Gal}$$

which maps by the restriction θ_H of θ. Observe that invariance by H implies that the function is locally constant on $GL(2,\mathbb{A}_F^f)$.

Now we assemble all the isomorphisms θ_H together with the transfer morphisms $M_H \longrightarrow M_{xHx^{-1}}$ to define an isomorphism in the limit. This limit isomorphism is one of the main results of the theory.

(2.2) THEOREM. With the above notations the limit of the θ_H defines an isomorphism of $GL(2,\mathbb{A}_F) \times Gal(F_{\infty,s}/F_\infty)$ -representations

$$\boxed{\underline{H} = \varinjlim_H H_!^1(M_H,\bar{\mathbb{Q}}_\ell) \longrightarrow \mathrm{Hom}_{GL(2,F_\infty)}(V_{sp},L_0(GL(2,F)\ GL(2,\mathbb{A}_F))) \otimes sp_{Gal}} \quad .$$

A basic result in the theory of automorphic forms for $GL(2)$, see [J-L, prop. 11.1.1], is that the representation of $GL(2,\mathbb{A}_F)$ on L_0 decomposes with multiplicity one

$$L_0(GL(2,F)\backslash GL(2,A_F)) = \coprod_{\pi\in\Pi} \pi$$

where Π is a set of irreducible admissible representations of the adèlic group $GL(2,\mathbb{A}_F)$. Now each $\pi \in \Pi$ is of the form $\pi = \otimes_v \pi_v$ where π_v is an irreducible admissible representation with a $GL(2,\mathcal{O}_v)$-invariant vector for almost all v . From the theorem we have the next corollary.

(2.3) COROLLARY. As $GL(2,\mathbb{A}_F^f)$, $Gal(F_{\infty,s}/F_\infty)$ modules, we have an isomorphism

$$\underline{H} \xrightarrow{\sim} \bigoplus_{\pi \in \Pi,\, \pi_\infty = sp} \left[\left(\bigotimes_{v \neq \infty} \pi_v\right) \otimes sp_{Gal}\right]$$

and as $GL(2,\mathbb{A}_F^f)$, $Gal(F_s/F)$ modules we have a mapping $\pi \mapsto \sigma(\pi)$ of two-dimensional $Gal(F_s/F)$-modules such that $\sigma(\pi)|Gal(F_{\infty,s}/F_\infty) = sp_{Gal}$ and an isomorphism

$$\underline{H} \xrightarrow{\sim} \bigoplus_{\pi \in \Pi,\, \pi_\infty = sp} \left[\left(\bigotimes_{v \neq \infty} \pi_v\right) \otimes \sigma(\pi)\right] \quad .$$

The mapping $\pi \mapsto \sigma(\pi)$ is a form of the reciprocity mapping of class field theory between automorphic representations equal to the special representation at ∞ and Galois representations of dimension two equal to the special representation at ∞ .

§3. PROPERTIES OF THE CORRESPONDENCE $\pi \mapsto \sigma(\pi)$

Let Π_∞ denote the set of representations π in Π with $\pi_\infty = sp_{GL(2)}$, and let Σ denote the set of compatible families of ℓ-adic representations σ of $Gal(\bar{F}/F)$ which are two dimensional and irreducible for all $\ell \neq p$, and let Σ_∞ denote the $\sigma \in \Sigma$ with $\sigma_\infty = sp_{Gal}$.

From (2.2) and (2.3) we have a function still denoted $\pi \mapsto \sigma(\pi)$ defined $\Pi_\infty \longrightarrow \Sigma_\infty$. For an irreducible representation π of $GL(2,\mathbb{A}_F)$, let ω_π denote the scalar action defined by π restricted to the center of $GL(2,\mathbb{A}_F)$. Also we use the reciprocity map from abelian class field theory

$$Gal(\bar{F}/F) \longrightarrow GL(1,F)\backslash GL(1,\mathbb{A}_F)$$

so that for a 1-dimensional character χ of the ideal class group we have a character χ of $Gal(\bar{F}/F)$ by composing with the reciprocity map.

(3.1) PROPOSITION. For $\pi \mapsto \sigma(\pi)$ we have

(1) $\sigma(\pi \otimes \chi) = \sigma(\pi) \otimes \chi^{-1}$ for χ a 1-dimensional character.

(2) $\det \sigma(\pi) = \omega_\pi^{-1}(-1)$ for the central character ω_π.

PROOF. (1) Observe that $\pi \otimes \sigma(\pi) = (\pi \otimes \chi) \otimes (\sigma(\pi) \otimes \chi^{-1})$ is a subrepresentation of $\underline{H}$ from which the first assertion follows.

(2) This relation is rather involved to work out completely. For this one uses the cup product in cohomology and the alternating bilinear form that it defines on $\underline{H}$. In a group GL(2) define $s^\vee = s/\det$ where this is the contragredient relative to the alternating form so that

$$sx \wedge [\det(s)]^{-1} sy = s \wedge y \quad .$$

For π a representation on V, $s \mapsto \pi(s^\vee)$ is isomorphic to the dual representation, see [Deligne 1973, pp. 102-3] for references. For a form ψ with $\psi(\pi(s)x, \pi(s^\vee)y) = \psi(x,y)$ we have

$$\psi(\pi(s)x, \pi(s)y) = \omega_\pi(s)\psi(x,y) \quad ,$$

and ψ is unique up to a constant factor. One has $\psi(x,y) = \omega_\pi(-1)\psi(y,x)$. These considerations are coupled with properties of the cup product form to yield the stated relation.

Now we are in a position to study the relation between π_v and $\sigma(\pi)_v | D_v$ for all v such that $GL(2, \mathcal{O}_v)$ leaves a 1-dimensional space of the representation space of π_v invariant. Then $\pi_v = \mathrm{Ind}_B^G(\chi_1, \chi_2)$ (unitary), and it is classified by a quadratic polynomial $T^2 - a_v T + b_v$ with roots $\chi_1(\pi_v)$ and $\chi_2(\pi_v)$, called the Hecke polynomial. Such a place v is called unramified.

Further, if $\sigma(\pi)_v$ is unramified, then it is characterized by the characteristic polynomial of Frobenius Fr_v. This is also a quadratic polynomial $T^2 - a'_v T + b'_v$. The basic result, which we sketch, is the following theorem.

(3.2) THEOREM. For a place v such that π_v and $\sigma(\pi)_v$ are unramified, the Hecke polynomial of π_v twisted by $|\det(s)|^{1/2}$ equals the characteristic polynomial of Frobenius.

PROOF. By an Eichler-Shimura type congruence formula

$$(Fr_v)^2 - a_v(Fr_v) + b_v = 0 \quad .$$

This congruence formula is proved, as in the case of elliptic curves, by considering Drinfel'd modules mod v and the action of Frobenius.

Next, from (3.1) and the considerations related to alternating forms, we deduce that $b_v = b'_v$ since $b'_v = \det \sigma(\pi)$. Hence for roots α and β of the Hecke polynomial, we see that α,β or α,α or β,β are the characteristic roots of Fr_v. The condition on the constant terms of the polynomials implies that the roots of the characteristic polynomial of Fr_v are α and β.

§4. LOCAL LANGLANDS' CONJECTURE IN CHARACTERISTIC p

Now we sketch the steps in the proof of the local Langlands' conjecture in characteristic ρ for GL(2) which was outlined in a letter from Deligne to Drinfel'd dated 1/21/1975.

(4.1) The map $\rho \mapsto \pi(\rho)$. Let ρ be a two-dimensional representation of $W(\bar{K}/K)$ and the local Weil group of the local field K in characteristic ρ. We can take $K = F_v$ for a function field F. Since the Artin conjecture is true for Artin L-functions in the function field case, see A. Weil [1948], we can apply [J-L, proposition 12.6, p. 408] to obtain $\pi(\rho)$ from a global automorphic representation.

(4.2) Injectivity of $\rho \mapsto \pi(\rho)$. This is corollary (1.7) of proposition (1.6) in Gelbart and Jacquet, "A Relation Between Automorphic Representations of GL(2) and GL(3)" [1978]. It is an argument on $GL(2) \times GL(2)$ which uses corollary 19.16 of [J].

(4.3) Surjectivity of $\rho \mapsto \pi(\rho)$. As with the injectivity, this depends on the local result that $\varepsilon(\pi \otimes \chi)$ depends only on ω_π for χ very ramified, see [J-L, proposition 3.8(iii), p. 116] and $\varepsilon(V \otimes \chi)$ depends only on $\det V$ for χ very ramified, see [Deligne, 1973, proof of 4.16, p. 546]. Now consider global π and $\sigma(\pi) = \sigma$ given by (3.2). Here π and σ have the same global L and ε factors and the same local L and ε factors at all, $v \notin S$, where S is finite. We wish to prove that L_v and ε_v are equal for all v. The global functional equations and product formulas have the form

$$\prod_v L_v(\pi \otimes \chi) = \prod_v \varepsilon_v(\pi \otimes \chi) \prod_v L_v(\check{\pi} \otimes \chi^{-1} \otimes \omega_1)$$

and

$$\prod_v L_v(\sigma \otimes \chi) = \prod_v \varepsilon_v(\sigma \otimes \chi) \prod_v L_v(\check{\sigma} \otimes \chi^{-1} \otimes \omega_1) \quad .$$

Now divide the one expression by the other using the equality for $v \notin S$ to obtain

$$\prod_{v\in S} \varepsilon_v'(\pi \otimes \chi) = \prod_{v\in S} \varepsilon_v'(\sigma \otimes \chi)$$

where $\varepsilon'(\tau) = \varepsilon(\tau)(L(\tau^v \times \omega_1)/L(\tau))$. Now choose a global χ which equals 1 at a fixed $v_0 \in S$ and which is very ramified at $w \in S - \{v_0\}$. Then $\varepsilon_w' = \varepsilon_w$ depends only on the central character and determinant. Hence both sides are equal at $w \in S - \{v_0\}$, which implies $\varepsilon_{v_0}'(\pi) = \varepsilon_{v_0}'(\sigma)$. From this we recover $\varepsilon_v(\pi) = \varepsilon_v(\sigma(\pi))$ and $L_v(\pi) = L_v(\sigma(\pi))$ for all v .

For a finite set $S = \{v_0,\ldots,v_m\}$ of places and local discrete series representations π_i at v_i , one can prove by looking at a quaternion algebra ramified at each place of S that there is a global representation π with $\pi_{v_0} = \pi_0$ and with π_{v_i} and π_i differing by an unramified twisting for $i = 1,\ldots,m$. Using (3.2), we form $\sigma = \sigma(\pi)$ and define $\rho = \sigma_{v_0}$. Then $\rho \mapsto \pi_0 = \pi(\rho)$ has the desired property giving surjectivity.

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