CHAPTER 7.

CONTINUOUS SPECTRUM AND THE TRACE FORMULA IN THE CO-FINITE CASE.

In the previous chapter we showed how one may meromorphically continue $E_j(\omega,s)$ to $s=\frac{n}{2}+it$. We also found that $E_j(\omega,\frac{n}{2}+it)$ is actually analytic. From its Fourier expansion in the various cusps it is apparent that for such values of s (i.e. $s=\frac{n}{2}+it$) the Eisenstein series only just miss being in $L^2(\mathcal{H}^{n+1}/\Gamma)$. The situation is similar to harmonic analysis on the real line, where $e^{2\pi i\xi x}$ are eigenfunctions of convolution operators, yet not L^2 eigenfunctions. The functions $e^{2\pi i\xi x}$ furnish continuous spectrum via "wave packets". We will show how $E_j(\omega,\frac{n}{2}+it)$, $j=1,\dots,h$, $t\in\mathbb{R}$ may be used in the same way to construct the continuous spectrum. For simplicity we begin with the case of one cusp. At the end of the chapter we describe the modifications needed to handle the general case. We assume as is customary with one cusp that it is placed at infinity.

Notice that $E(\omega,\frac{n}{2}+it)$ and $E(\omega,\frac{n}{2}-it)$ are multiples of one another. So we need only consider $E(\omega,\frac{n}{2}+it)$ for $t\geq 0$.

Theorem 7.1. Let $f(r)$, $g(r)$ be in $C_0^\infty(0,\infty)$ and let

$$F(\omega)=\int_0^\infty f(r)\,E(\omega,\tfrac{n}{2}+ir)\,dr$$

$$G(\omega)=\int_0^\infty g(r)\,E(\omega,\tfrac{n}{2}+ir)\,dr$$

then $F, G \in L^2(\mathbb{H}^{n+1}/\Gamma)$ and

$$\frac{1}{2\pi} \int_{\mathfrak{F}} F(\omega) \overline{G(\omega)} \, dV(\omega) = \int_0^\infty f(r) \overline{g(r)} \, dr .$$

Also

$$\Delta F(\omega) = \int_0^\infty -\{(\tfrac{n}{2})^2 + r^2\} f(r) E(\omega, \tfrac{n}{2} + ir) \, dr.$$

Proof. As in the previous chapter set

$$E_A(\omega, \tfrac{n}{2}+it) = E(\omega, \tfrac{n}{2}+it) - \delta_A(y)\{y^{n/2+it} + \phi(\tfrac{n}{2}+it) y^{n/2-it}\}$$

where $\delta_A(y) = 1$ if $y \geq A$, $\delta_A(y) = 0$ if $y < A$. We may write

$$F(\omega) = F_1 + F_2 = \delta_A(y) \int_0^\infty (y^{n/2+ir} + \phi(\tfrac{n}{2}+ir) y^{n/2-ir}) \phi(r) \, dr$$

$$+ \int_0^\infty E_A(\omega, n/2 + ir) f(r) \, dr .$$

F_1 is rapidly decreasing in logy as $y \to \infty$ since it is essentially a Mellin transform of f and $|\phi(n/2 + ir)| = 1$. F_2 is in $L^2(\mathfrak{F})$ since the E_A's are. Thus $F \in L^2(\mathbb{H}^{n+1}/\Gamma)$.

Similarly $G = G_1 + G_2 \in L^2$. Now

(*) $$(F,G)_{\mathfrak{F}} = (F_1, G_1) + (F_1, G_2) + (F_2, G_1) + (F_2, G_2) .$$

Two of the terms are of the form

$$\int \delta_A(y)\, f(r)\, \overline{g(r')}\, E_A(\omega, \tfrac{n}{2} + ir)\, \overline{(y^{n/2+ir} + \phi(\tfrac{n}{2}+ir)\, y^{n/2-ir})}\, dr\, dr'\, dV(\omega).$$

These give zero since for $y \geq A$, E_A, has zero Fourier coefficients. One of the terms left is

$$\int_{\mathfrak{F}} F_1(\omega)\, \overline{G_1(\omega)}\, \delta_A(y)\, dV(\omega)$$

which clearly goes to zero as $A \to \infty$. We let $A \to \infty$ since l.h.s. of (*) is independent of A.

We are left with

$$\lim_{A \to \infty} \int f(r)\, \overline{g(r')}\, E_A(\omega, \tfrac{n}{2} + ir)\, \overline{E_A(\omega, \tfrac{n}{2} + ir')}\, dv(\omega)\, dv\, dv' .$$

The inner product formula 1.62 of the last chapter renders the last expression in the form

$$\int_0^\infty f(r)\overline{g(r')} \left\{ \frac{A^{i(r+r')}\, \phi(n/2 - ir') - \phi(n/2 + ir) A^{-i(r+r')}}{i(r+r')} \right.$$

$$\left. + \frac{A^{i(r-r')} - A^{-i(r-r')}}{i(r-r')} + \frac{A^{-i(r-r')}(1-\phi(n/2+ir)\bar{\phi}(n/2+ir'))}{i(r-r')} \right\} dr\, dr'$$

$$= I + II + III.$$

Since $(r+r')$ is bounded below, $I \to 0$ as $A \to \infty$ by the Riemann Lebesgue lemma. Similarly $III \to 0$ for the same reason since $\frac{1-\phi(\ )\bar{\phi}(\ )}{r-r'}$ is bounded. We are left with II which if we do the r integral first give s

$$2 \int_0^\infty \frac{f(r)\sin(\log A(r-r'))}{(r-r')}\, dr \to 2\pi f(r')$$

as is well known since $\frac{\sin x}{x}$ is the Fourier transform of the Dirichlet kernel $\chi_{[-1,1]}(\xi)$.

$$\therefore \qquad II \to 2\pi \int_0^\infty f(r')\, \overline{g(r')}\, dr' .$$

The claim concerning ΔF follows from $\Delta E + s(n-s)E = 0$. □

Corollary 7.2. The map $f \to F$ in 7.1 extends to an isometry of $L^2(0,\infty)$ in $L^2(\mathbb{H}^{n+1}/\Gamma)$.

Auxilary Estimates. Denote by $\mathcal{E}$ the subspace of $L^2(\mathbb{H}^{n+1}/\Gamma)$ spanned by the Eisenstein series, i.e. the image of $L^2(0,\infty)$ under the above map. It is easy to see at this point that $\mathcal{E}$ is an invariant subspace for Δ in view of Theorem 7.1. It is also clear from 7.1, that the spectrum of Δ on $\mathcal{E}$ is absolutely continuous, and is $((\frac{n}{2})^2, \infty)$. The simplicity of this Plancherel formula may be a little deceptive. In fact we will see that the 'density of states' corresponding to this continuous spectrum is essentially $\phi'/\phi\,(\frac{n}{2}+it)$.

The following simple "Bessel inequality" turns out to be very useful in estimating such quantities as $\|E_A(s)\|_{\mathfrak{F}}$ which in term will give us our first control over ϕ'.

7.3 Bessel Inequality.

Let $k(z,\omega)$ be a real point pair invariant of compact support, and let $K(z,\omega) = \sum_{\gamma \in \Gamma} k(z, \gamma\omega)$ be its automorphization.

7.4 Proposition. We have

$$\frac{1}{2\pi} \int_0^\infty |\hat{k}(\tfrac{n}{2}+it)|^2 \, |E(\omega, \tfrac{n}{2}+it)|^2 \, dt \leq \int_{\mathfrak{F}} |k(\omega, \omega')|^2 \, dV(\omega').$$

Proof. In this inequality ω is fixed. As a function of ω', K is of compact support in $\mathfrak{F}$. Thus if

$$h(\omega') = K(\omega,\omega') - \frac{1}{2\pi} \int_0^T \hat{k}(\tfrac{n}{2}+it) E(\omega, \tfrac{n}{2}+it) \cdot \overline{E(\omega', \tfrac{n}{2}+it)} \, dt$$

then

$$\|h\|^2 = \int_{\mathfrak{F}} |K(\omega,\omega')|^2 \, dr(\omega) - \frac{1}{2\pi} \int_0^T |\hat{k}(\tfrac{n}{2}+it) E(\omega, \tfrac{n}{2}+it)|^2 \, dt$$

which follows from the Plancherel formula and $K_o E = s(n-s)E$. The inequality then follows immediately.

<u>Lemma 7.5</u>. Let $k \geq 0$ be a point pair which is supported in $d(z,\omega) \leq \varepsilon$ and with $\int k(z,\omega)\, dv(\omega) = 1$. Then

$$|\hat{k}(s) - 1| < \frac{1}{2} \text{ for } |s| \leq \frac{c}{\varepsilon}$$

for suitable absolute c.

<u>Proof</u>. If $\omega_o = (1,0,\ldots,0)$ then

$$\hat{k}(s) = \int_{d(\omega',z_o) \leq \varepsilon} k(\omega_o,\omega')(y(\omega'))^s \, dv(\omega')$$

and so

$$|\hat{k}(s) - 1| = \int_{d(\omega',\omega_o) \leq \varepsilon} (y(\omega')^s - 1)\, k(z_o,\omega') dV(\omega')$$

and the lemma follows.

We now also assume that k is chosen, as it may be, so that $\max|k| \leq C\, \varepsilon^{-(n+1)}$ for absolute C.

Lemma 7.6. For such k

$$\int_{\mathfrak{F}} |K(z,\omega)|^2 \, dv(\omega) \ll \frac{y^n}{\varepsilon} + \varepsilon^{-(n+1)}$$

where as always $K(\omega_1,\omega_2) = \sum_{\gamma \in \Gamma} k(\omega_1,\gamma\omega_2)$.

<u>Proof</u>. For z in a compact subset of $\mathfrak{F}$ the $\epsilon^{-(n+1)}$ term which is $\max|k|$ makes the statement true, since we are averaging $|k|^2$ over a set of measure $<< \epsilon^{(n+1)}$. So we can think of $y(z)$ being large. Now as in 1.19

$$|K(z,\omega)| << \#\{\ell \in L : |\ell| \leq y\epsilon\}\ \epsilon^{-(n+1)}$$

$$= (O(1) + (y\epsilon)^n)\ \epsilon^{-(n+1)} << \epsilon^{-n+1} + y^n/\epsilon\ .$$

Also

$$(*) \qquad \begin{cases} k(z,\omega) = 0 & \text{if} \quad \omega \in \mathfrak{F} \quad \text{and} \\ y(\omega) \leq ye^{-\epsilon} & \text{or} \quad y(\omega) \geq y\,e^{\epsilon}\ . \end{cases}$$

If $\epsilon \leq c_1/y$ for suitably small c_1 (depending on L only).

$$K(z,\omega) = k(z,\omega)$$

and

$$\int_{\mathfrak{F}} |K(z,\omega)|^2\, dv(\omega) = \int_{d(z,\omega)\leq\epsilon} |K(z,\omega)|^2\, dv(\omega) \leq \epsilon^{-(n+1)}\ .$$

On the other hand if $\epsilon > c_1/y$ then $\epsilon^{-(n+1)} << y^n/\epsilon$

whence

$$|K(z,\omega)| << y^n/\epsilon$$

$\therefore$ by $*$

$$\int_{\mathfrak{F}} |K(z,\omega)|^2\, dv(\omega) << \int_{y\,e^{-\epsilon}}^{ye^{\epsilon}} \left(\frac{y^n}{\epsilon}\right)^2 t^{-(n+1)}\, dt << \frac{y^n}{\epsilon}\ . \qquad \square$$

Corollary 7.7. $\int_0^T |E(\omega, \frac{n}{2} + it)|^2 dt = O(y^n T + T^{n+1})$.

Proof. If we apply the Bessel in equality 7.4, then on the range $[0,T]$, $|\hat{k}(\frac{n}{2} + it)| \geq C_3 > 0$ by 7.5, if $\epsilon = 1/T$. The right hand side of the Bessel inequality is $O(T y^n + T^{n+1})$ by 7.6.

The following factorization of $\phi(s)$ will be used only minimally but is of interest. We have seen in the previous chapter that $\phi(s)$ is bounded in $\mathrm{Re}(s) \geq n/2$ except for possible poles in $(\frac{n}{2}, n]$. Let $n/2 < \sigma_1 < \sigma_2 \ldots < \sigma_N = n$, be these simple poles if any such exist, besides the one at n, which we discussed in Chapter 6.57. Define

$$(7.9) \qquad \phi^*(s) = \prod_{i=1}^{N} \left(\frac{s-\sigma_i}{s-n+\sigma_i} \right) \phi(s).$$

We have introduced the rational factor to render ϕ^* analytic in $\mathrm{Re}(s) \geq n/2$. Clearly this is achieved and the identity $|\phi^*(n/2 + it)| = 1$ still holds. Thus $|\phi^*(s)| \leq 1$ for $\mathrm{Re}(s) \geq n/2$ still holds. Also $\phi^*(s)\, \phi^*(n-s) = 1$.

We may therefore factor out the zeros and poles of ϕ^* (which occur symmetrically about $n/2 + it$) to obtain

$$\phi^*(s) = e^{g(s-n/2)} \prod_{\rho} \left(\frac{s+\bar{\rho}-n}{s-\rho} \right)$$

where $\rho = \beta + i\gamma$ are the poles of ϕ^*.

Here g is some entire function but the bound in $\operatorname{Re}(s) \geq n/2 \Rightarrow e^{g(s-n/2)}$ is bounded, and is of absolute value 1 on $\operatorname{Re}(s) = n/2$. By reflection and $\phi^*(s)\,\phi^*(n-s) = 1$ we see that $g(s-n/2)$ is constant (being bounded and entire), and since $\phi^*(s) \to 1$ on $\operatorname{Re}(s) \to \infty$, ϕ^* being a Dirichlet series we learn that $g \equiv 0$.

$$\therefore \qquad \phi(s) = \prod_{i=1}^{N} \left(\frac{s-\sigma_i}{sn+\sigma_i} \right) \prod_{\rho} \left(\frac{s-n+\bar{\rho}}{s-\rho} \right)$$

we call this the Blashcke factorization.

We introduce an important function

$$W(r) = 1 - \frac{(\phi^*)'}{\phi^*}(n/2+ir) . \tag{7.10}$$

From the Blashcke product

$$-\frac{(\phi^*)'}{\phi^*}(s) = \sum_{\rho} \frac{s-\rho}{s-n+\bar{\rho}} \; \frac{s-\rho-(s-n+\bar{\rho})}{(s-\rho)^2} = \sum_{\rho} \frac{n-\rho-\bar{\rho}}{(s-\rho)(s-n+\bar{\rho})} ,$$

and so

$$-\frac{\phi'^*}{\phi^*}(n/2+it) = \sum_{\rho} \frac{n-2\beta}{(n/2-\beta)^2+(t-\gamma)^2} . \tag{7.10'}$$

It follows that $W(r) \geq 1$, $\forall r$. The function $W(r)$ will appear in the trace formula, and is very important. It counts the amount of continuous spectrum. It also controls various terms related to the continuous spectrum.

<u>Proposition 7.11</u>. $1-|\phi(s)|^2 = O((\sigma - n/2)\, W(t))$ for $n/2 \leq \sigma < \sigma_o < n$.

<u>Proof</u>. Exercise. More importantly

<u>Proposition 7.12</u>.

(i) $$\int_{\mathfrak{F}} \tilde{E}_A(\omega, s)\, \overline{\tilde{E}_A(\omega, s)}\, dV(\omega) = O(W(t)), \qquad n/2 \leq \sigma < \sigma_o < n$$

(ii) $$\int_{\mathfrak{F}} \tilde{E}_A(\omega, n/2 + it)\, \overline{\tilde{E}_A(\omega, n/2 + it)}\, dV(\omega) = W(t) + O(1) = -\frac{\phi'}{\phi}(n/2 + it) + O(1)$$

(the various implied constants depend on A which is fixed here).

<u>Proof</u>. Both follows from the inner product formula. For example in (i) we obtain from that formula

$$\text{l.h.s.} = \frac{A^{2\sigma-n} - A^{n-2\sigma}|\phi(s)|^2}{2\sigma - n} + \frac{\bar{\phi}(s)A^{2it} - \phi(s)A^{-2it}}{2i\,t}$$

$$= \frac{A^{2\sigma-n} - A^{n-2\sigma} + A^{n-2\sigma}(1 - |\phi(s)|^2)}{2\sigma - n} + O(1) = O(W(r))$$

by 7.11. (ii) follows from the inner product formula after putting

$s_1 = s_2 = \sigma + it$ and taking limit as $\sigma \to n/2$ which gives the ϕ'/ϕ term. Next we show that W gives a pointwise bound for E.

Proposition 7.13.

For a fixed $\omega \in \mathfrak{F}$

$$E(\omega, s) = O(W(t)|t|^{n+1}) \quad \text{for } n/2 \leq \sigma \leq \sigma_o .$$

Proof. We choose A much larger than $y(\omega)$ then

$$\int_{\{\omega' : y(\omega') \leq A\} \cap \mathfrak{F}} |E(\omega', s)|^2 AV(\omega')$$

$$\leq \int_{\mathfrak{F}} |\tilde{E}_A(\omega', s)|^2 \, dV(\omega') = O(W(t)) \quad \text{where } s = \sigma + it.$$

Choose a point pair invariant $k(\omega_1, \omega_2)$ with sufficiently small support so that

$$K(z,\omega) = k(z,\omega) \quad \text{for} \quad y(z) < A .$$

Then $\int_{\mathfrak{F}} K(\omega, z) E(z,s) \, dV(z) = h(s) E(\omega, s)$

$$\therefore \quad h(s) E(\omega, s) = \int_{\mathfrak{F}} k(\omega, z) \tilde{E}_A(z,s) \, dV(z)$$

$$E(\omega, s) = O\left(\frac{V(t)}{|h(s)|} \left(\int |k(\omega, z)|^2 \, dV(z) \right)^{1/2} \right) .$$

As before if support of k is in a ball radius $<< 1/a_t$, then $|h(s)-1| < 1/2$ and then $\max|k| < t^{n+1}$.

i.e. $$E(\omega, s) = O(W(t)t^{n+1}) .$$

These last few propositions give us control over the size of E via W. Actually we can use the Bessel inequality to obtain an important estimate for W.

Theorem 7.14.

$$\int_0^R W(t)dt = O(R^{n+1}) .$$

For a general group Γ this is the best order of magnitude bound that we know for W. Bounding W further, turns out to be one of the most important problems in the theory and we will return to this point after the trace formula. We will need the following lemma

Lemma 7.15.

$$\int_{\mathfrak{F}} |E_A(\omega, n/2+it)|^2 dV(\omega)$$

$$\leq C \int_{\mathfrak{F}\cap\{\omega | y(\omega) \leq Ct+A\}} |E(\omega, n/2+it)|^2 dV(\omega)$$

for suitable C.

Proof. It is a matter of estimating

$$(*) \qquad \int_{\mathfrak{F}_L} \int_{Ct+A} |E_A(\omega, n/2+it)|^2 dV(\omega)$$

since in the rest of the region $E_A = E$. In (*) E_A has no zero Fourier coefficients. The non-zero Fourier coefficients from Section 6.3 satisfy

$$c''_e(y) + \left[\frac{t^2 + 1/4}{y^2} - 4\pi^2|e|\right] c_e(y) = 0 \tag{7.16}$$

where

$$a_e(y) = y^{n/2-1/2} c_e(y)$$

$a_e(y)$, $e \in L^*$ being the Fourier coefficients.

From 7.16 it is clear that for $y \geq y_o >> t$

$$c''_e(y) \geq \pi^2 |e|^2 c_e(y) . \tag{7.17}$$

Since we know our solutions $a_e(y)$ are rapidly decreasing at infinity we see that

$$c_e(y_o)\ c'_e(y_o) < 0 \qquad \text{(and hence } c_e(y) \text{ do not change sign).}$$

Since in fact any solution of 7.16 with

$$f(y_o)\, f'(y_o) \geq 0 \qquad \text{(not both zero)}$$

must clearly by 7.17 go to infinity at least as fast as $\exp(\pi|e|y)$ as must its derivative.

Let $x(y)$ be the solution of 7.16 with

$$x(y_o) = c_e(y_o)$$
$$x'(y_o) = -c'_e(y_o) \qquad \text{(say } c_e(y_o) > 0)$$

then $x(y)$, $x'(y) \to \infty$ like $e^{\pi y}$ at least.

However

$$x'(y)\,\psi(y) - x(y)\psi'(y) = \text{cont.} \Rightarrow 0 \le c_e(y) \le e^{-\pi y} c_e(y_o)$$

$c_e'(y_o)e^{-\pi y} \le c_e'(y) \le 0.$

From this it is easy to see that

$$\int_{y_o}^{\infty} |a_e(y)|^2 \frac{dy}{y^2} \le C \int_{y_o}^{y_o+1} |a_e(y)|^2 \frac{dy}{y^2}$$

and from this the lemma follows by adding over the non zero coefficients and using Parseval.

Corollary 7.16.

$$\int_0^T \|E_A(\cdot\, , \tfrac{n}{2} + it)\|_{\mathfrak{F}}^2 \, dt = O(T^{n+1}) .$$

Proof.

$$\int_0^T |E(\omega, \tfrac{n}{2} + it)|^2 \, dt = O(y^n T + T^{n+1})$$

by integrating this in $\mathfrak{F} \cap \{ y \le C'T\}$ gives

$$\int_0^T \int_{\mathfrak{F}\cap\{y \le c'T\}} |E(\omega, \frac{n}{2}+it)|^2 \, dt \, dV(\omega) = O(T^{n+1}).$$

Theorem 7.14 now follows from 7.12(ii).

As we have already pointed out this bound on $W(t)$ is already the best general bound on W that we know!

Remark. In the general case of h cusps all of the above estimates may be proved for $E_j(\omega, \frac{n}{2}+it)$, $E_j(\omega, s)$ etc. in a similar fasion by use of the inner product formula, and Bessel inequality. We do not stop at this point to describe this any further - but rather leave it to the end when we describe what happens in the general case in the trace formula.

The Trace Formula. We have now developed enough preliminaries to derive the non compact trace formula. As usual let k be a C^∞ compact support point pair and let $K = \sum_{\gamma=\Gamma} k(z, \gamma\omega)$. We have seen that K does not give rise (unless it is trivial) to a Hilbert Schmidt kernel, where $\mathfrak{F}$ is finite volume but not compact. However by use of the Eisenstein series which have been the subject of the past number of chapters, we may isolate exactly the non-compact part of K. After removing this contribution we will have a trace class operator. Earlier we already introduced the space $\mathcal{E}$ spanned by the Eisenstein series.

$$\mathcal{E} = \{ F \in L^2(H^{n+1}/\Gamma) : F(z) = \int_0^\infty f(t)\, E(z, \tfrac{n}{2} + it)\, dt \quad \text{where} \quad f \in L^2(0, \infty) \} .$$

The action of K on $\mathcal{E}$ is given by

$$\int_{\mathfrak{F}} K(z,\omega)\Big(\int_0^\infty f(t)\, E(\omega, \tfrac{n}{2} + it)\, dt\Big) dV(\omega) = \int_0^\infty h(t) f(t)\, E(\omega, \tfrac{n}{2} + it) dt$$

where $k \leftrightarrow h$ is as usual $h(t) = \hat{k}(\frac{n}{2} + it)$.

Thus clearly $\mathcal{E}$ is invariant under our operator K as is $\mathcal{E}^\perp$ (since K is self-adjoint). We now show that K restricted to $\mathcal{E}^\perp$ is a compact operator. The easiest way of seeing this and one which allows an easy computation of the trace is to introduce $H(z,\omega)$ a kernel which corresponds to the action of K on $\mathcal{E}$. Let

$$H(z,\omega) = \frac{1}{4\pi} \int_{-\infty}^\infty \hat{k}(\tfrac{n}{2} + it)\, E(z, \tfrac{n}{2} + it)\, \overline{E(\omega, \tfrac{n}{2} + it)}\, dt$$

$$= \frac{1}{2\pi} \int_0^\infty \hat{k}(\tfrac{n}{2} + it)\, E(z, \tfrac{n}{2} + it)\, \overline{E(\omega, \tfrac{n}{2} + it)}\, dt .$$

Firstly there is a question of convergence. Hence k is smooth and of compact support, $\hat{k}$ and hence h is rapidly decreasing, and the key estimates 7.13 and 7.14 imply that the integral defining $H(z,\omega)$ converges. In fact by the Plancherel formula, and Bessel inequality

$$\int_{\mathfrak{F}} |H(z,\omega)|^2 \, dV(\omega) = \int_0^\infty h(t)|E(z, \tfrac{n}{2}+it)|^2 \, dt$$

$$\leq 2\pi \int_{\mathfrak{F}} |K(z,\omega)|^2 \, dV(\omega) ,$$

one can also see convergence.

The action of $H(z,\omega)$ is given by

Proposition 7.17. For $\phi \in C_0^\infty(0, \infty)$ and

$$g(\omega) = \int_0^\infty \phi(t) \, E(\omega, \tfrac{n}{2}+it)dt \quad \text{then}$$

$$\int_{\mathfrak{F}} H(z,\omega) \, g(\omega)dV(\omega) = \int_0^\infty \phi(t) \, h(t) \, E(z, \tfrac{n}{2}+it) \, dt, \text{ and if } u \in \mathcal{E}^{\perp}$$

then $\int H(z,\omega)u(\omega)dV(\omega) = 0$.

Proof. Follows directly from the Planchevrel formula.

It follows that the action of H on $\mathcal{E}$ is identical with that of K on $\mathcal{E}$, while on $\mathcal{E}^{\perp}$, H is zero. Thus the trace of K on $\mathcal{E}^{\perp}$ is the same as that of $K - H$ on the whole space, and we therefore are led to the study of $K - H$.

As is usual in this business we define the cut off functions

$$H_A^{(1)}(z,\omega) = \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\,E_A(z,\tfrac{n}{2}+it)\,\overline{E_A(\omega,\tfrac{n}{2}+it)}\,dt\,.$$

<u>Lemma 7.18</u>. H_A' is a compact operator (actually Hilbert Schmidt)

<u>Proof</u>.

$$\Big(\int_{\mathfrak{F}} |H_A^{(1)}(z,\omega)|^2\,dV(z)dV(\omega)\Big)^{1/2} \le \frac{1}{4\pi}\int_{-\infty}^{\infty} |h(t)|\,\|E_A(\cdot,\tfrac{n}{2}+it)\|^2\,dt$$

We have seen that $\|E_A(\cdot\,,\frac{n}{2}+it)\|^2 = O(W(t))$ and the estimate 7.14 on W, proves 7.18.

We now decompose H into four parts; write $z=(y,x)$, $\underset{\rho=\sigma+it}{\omega} = (\eta,\xi)$ and define

$$H(z,\omega) = H_A^{(1)}(z,\omega) + \frac{1}{4\pi}\int \delta_A(z)\,\delta_A(\omega)[\,y^s+\phi(s)\,y^{n-s}]\cdot[\overline{\eta^s+\phi(s)\eta^{n-s}}]h(t)dt$$

$$+\,H_A^{(3)}(z,\omega) = H_A^{(1)} + H_A^{(2)} + H_A^{(3)}\,.$$

<u>Lemma 7.19</u>.

$$H_A^{(2)}(z,\omega) = \delta_A(z)\,\delta_A(\omega)\,K_o(y,\eta) + H_A^{(4)}(z,\omega)$$

where $H^{(4)}$ is compact and $\int_{\mathfrak{F}} H^{(4)}(z,z)\,dV(z) \to 0$ as $A\to\infty$, and $K_o(y,\eta) = \int_{\mathbb{R}^n} k(\omega_1,\,\omega_2+t)\,dt$ as in 1.26.

Lemma 7.20. $H_A^{(3)}(z,\omega)$ is compact and $\int_{\mathfrak{F}} H_A^{(3)}(z,z)dV(z) \to 0$ as $A \to \infty$.

Proof of 7.19. $H_A^{(2)}$ has four factors when multiplied out. Two of these give

$$(y\eta)^{n/2} \frac{1}{4\pi} \int_{-\infty}^{\infty} h(t)[(y/\eta)^{it} + (\eta/y)^{it}]dt$$

$$= (y\eta)^{n/2} \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t)(y/\eta)^{it}\, dt$$

since h is even.

On the other hand

$$K_o(y,\eta) = K_o(\omega_1,\omega_2) = \int k(\omega_1, \omega_2 + t)\, dt$$

$$= \int k\left(\frac{(y-\eta)^2+\xi^2}{y\eta}\right)d\xi = (y\eta)^{n/2} g(\log y/\eta)$$

$$= (y\eta)^{n/2} \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t)(y/\eta)^{it}\, dt .$$

Thus these two terms give rise to the $\delta_A(z)\ \delta_A(\omega)\ K_o(y,\eta)$ term.

The other terms are similar to each other, we consider for example

$$\frac{1}{4\pi} \int_{-\infty}^{\infty} h(t)\, \phi(\tfrac{n}{2}+it)(y\eta)^{n/2-it}\, dt = \frac{1}{i} \int_{Re(s)=n/2} \hat{k}(s)\phi(s)(y\eta)^{n-s}\, ds .$$

Now in view of the rapid decay in t of $\hat{k}(\sigma+it)$ we may shift this contour integral a little to the right and still not pick up any poles of ϕ. Thus for some $\sigma_1 > \frac{n}{2}$ we will have the last term

$$O(\delta_A(\eta)\,\delta_A(y)(y\eta)^{n-\sigma_1}) .$$

From this everything in 7.19 follows.

<u>Proof of 7.20</u>. As in 7.19 $H_A^{(3)}$ is a sum of various terms all of which are essentially the same. Consider for example

$$\delta_A(y)\int_{-\infty}^{\infty} H(t)\,y^{n/2-it}\,\phi(\tfrac{n}{2}+it)\,\overline{E_A(\omega,\tfrac{n}{2}+it)}\,dt .$$

The usual estimate on $W(t)$ gives convergence, and allows us to shift the integration line of

$$\frac{1}{i}\,\delta_A(y)\int_{Re(s)=\frac{n}{2}} \hat{k}(s)y^{n-s}\,\phi(s)\,E_A(\omega,n-s)\,ds$$

over to $Re(s)=n+1$ say. In doing so we will go over the finitely many poles of E. Now since $\phi(s)\,E(\omega,\,n-s) = E(\omega,s)$ it follows that $\phi(s)E_A(\omega,\,n-s) = E_A(\omega,s)$. Thus the above becomes

$$G(z,\omega) + \delta_A\frac{(y)}{i}\int_{Re(s)=n+1} \hat{k}(s)y^{n-s}\,E_A(\omega,s)\,ds$$

where $G(z,\omega)$ is the sum of the residues. Now $E_A(\omega,s)=O(1)$ on $Re(s)=n+1$ and so the integral in the last term is $O(1/y)$. Thus this term is a compact kernel and its trace $\to 0$ as $A \to \infty$. Now the poles of $E_A(\omega, s)$ are L^2 eigenfunctions for $s \in (\frac{n}{2}, n)$, therefore the $G(z,\omega)$ term is a sum of terms of the form

$$\delta_A(y) y^{n-\sigma_i} g(\omega) .$$

These are also Hilbert Schmidt kernels, and have trace $\to 0$ with $A \to \infty$.

We saw in 1.20 that

$$K(z,\omega) = \delta_A(z)\, \delta_A(\omega)\, K_o(y, \eta) + K_A(z,\omega)$$

where K_A is Hilbert Schmidt. We therefore have

$$(7.21) \qquad K - H = K_A - [H_A^{(1)} + H_A^{(3)} + H_A^{(4)}] .$$

We have shown that all of the kernels on the right of the (7.21) are Hilbert Schmidt and so

Theorem 7.22. $K - H$ is Hilbert Schmidt.

It follows as in the compact case, it follows that there is a sequence $0 = \lambda_o < \lambda_1 \leq \lambda_2 \ldots$ (we do not claim there are infinitely many λ's - see later) and orthonormal $u_j \in L^2(H^{n+1}/\Gamma)$ such that

$$\Delta u_j + \lambda_j u_j = 0, \quad \|u_j\|_2 = 1 , \quad \text{and} \quad \mathcal{E}^\perp = \text{span}\{u_j\} .$$

Now assuming further that K-H is trace class, which it will be under our assumptions on k (see our discussion in the compact case) it follows that

$$(7.22)\qquad \begin{cases} \Sigma\ \hat{k}(s_n) = \int_{\mathfrak{F}} \{K(z,z) - H(z,z)\}\, dV(z) \\[2ex] = \lim_{A\to\infty} \int_{\mathfrak{F}} \{K_A(z,z) - H_A^{(1)}(z,z)\}\, dV(z) \end{cases}$$

in view of Lemmas 7.19, 7.20.

The same on the left of 7.22, is our usual sum as in the compact quotient case, thus to complete the calculation we must evaluate the right hand side of 7.22.

We begin with the H term

$$\int_{\mathfrak{F}} H_A^{(1)}(z,z)dV(z) = \frac{1}{4\pi} \int_{-\infty}^{\infty}\int_{\mathfrak{F}} \hat{K}(\tfrac{n}{2}+it)|E_A(\omega,\tfrac{n}{2}+it)|^2\, dt\, dV(z).$$

By the inner product formula this gives

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\{\, 2\log A - \frac{\phi'}{\phi}(\tfrac{n}{2}+it) + \frac{A^{2it}\overline{\phi(\tfrac{n}{2}+it)} - A^{-2it}\phi(\tfrac{n}{2}+it)}{2it}\}\, dt$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} \{\, A^{2it}[\frac{\phi(\tfrac{n}{2}-it) - \phi(\tfrac{n}{2}+int)}{2it}]h(t) + \phi(\tfrac{n}{2}+it)[\frac{\sin(t\log A)}{2t}]\}\, dt$$

$$+ \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\ 2\log A - \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\ \frac{\phi'}{\phi}(\tfrac{n}{2}+it)\, dt\ .$$

Now as $A \to \infty$, the first term in the first integrand goes away by the Riemann Lebesque lemma. The second term in the first integral is an approximate identity, and since $\int_{-\infty}^{\infty} \frac{\sin x}{x} = \pi$ it leads to

$$\frac{1}{4}\,\phi(\tfrac{n}{2})\,h(0) \quad \text{as} \quad A \to \infty .$$

We therefore have

$$(7.23) \quad \int_{\mathfrak{F}} H_A^{(1)}(z,z)dV(z) = g(0)\ \log A - \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\frac{\phi'}{\phi}(\tfrac{n}{2}+it)\,dt$$

$$+ \frac{1}{4}\,\phi(\tfrac{n}{2})h(0) + o(1) \quad \text{as} \quad A \to \infty .$$

Now we look at

$$\int_{\mathfrak{F}} K_A(z,z)dV(z),$$

we have seen a number of times that $K_A - K_o$ is rapidly decreasing into the cusp, so

$$\int_{\mathfrak{F}} K_A(z,z)dV(z) = \int_{S_A} K(z,z)dV(z) + \int_{y \geq A} (K(z,z) - K_o(z,z))dV(z)$$

where

$$S_A = \{z \in \mathfrak{F} : y(z) \leq A\}.$$

This leaves us with $\int_{S_A} K(z,z)\,dV(z)$.

Now

$$K(z,z) = \sum_{\sigma \in \Gamma} k(z,\sigma z).$$

However for y large $k(z,\sigma z) = 0$ unless σ is the identity or is an element of Γ_∞, since otherwise $d(z,\sigma z) \to \infty$, as y gets large and k as always has compact support. Thus

$$\sum_{\sigma} \int_{S_A} K(z,\sigma z)dV(z) = \sum_{\substack{\sigma \notin \Gamma_\infty \\ \text{or a conjugate} \\ \text{of } \Gamma_\infty}} \int_{\mathfrak{F}} k(z,\sigma z)dV(z) + \sum_{\substack{\sigma \in \Gamma_\infty \\ \text{or conjugate} \\ \text{of } \Gamma_\infty}} \int_{S_A} k(z,\sigma z)\, dV(z) .$$

The first term in the last equation is handled in exactly the same way as the compact quotient.

The identity term of the second sum is also handled as before. The new term is whats left over in the second term. The centralizer of $\ell \in \Gamma_\infty$ is Γ_∞ and so the second sum leads to

$$\sum_{\ell \in L}{}' \sum_{\sigma \in \Gamma/\Gamma_\infty} \int_{\sigma S_A} k(z, z+\ell)\, dV(z)$$

where ' denotes, omit $\ell = 0$.

We have seen that as $\sigma \in \Gamma/\Gamma_\infty$, $\sigma\mathfrak{F}$ fills up the strip $\{(y,x) : y > 0,\ x \in \mathfrak{F}_L\}$ and therefore since $k(z,z+\ell) = 0$ if y is small enough, we are led to

$$\int_{y < A} \int_{x \in \mathfrak{F}_L} \sum_{\ell \in L}{}' k\left(\frac{|\ell|^2}{y^2}\right) \frac{dy\, dx}{y^{n+1}} = V(\mathfrak{F}_L) \int_{y < A} \sum_{\ell \in L}{}' k\left(\frac{|\ell|^2}{y^2}\right) \frac{dy}{y^{n+1}} .$$

Write $v = |\ell|^2/y^2$ or $y = |\ell|/v$ gives

$$V(\mathfrak{F}_L) \sum_{\ell \in L}{}' \int_{\frac{|\ell|^2}{A^2}}^{\infty} k(v) \frac{v^{n/2-1}}{|\ell|^2} \, dv.$$

Let

$$B_L(x) = \sum_{|\ell| \leq x}{}' \frac{1}{|\ell|^n} .$$

It is an easy computation using lattice point count (or any other method) that

$$B_L(x) = \frac{2\pi^{n/2}(\log x + \gamma_L)}{\Gamma(\frac{n}{2})} + O(x^{-\delta})$$

for some $\delta > 0$, and where γ_L is a constant like the Euler constant depending on the lattice.

It follows that

7.24
$$\left\{ \begin{aligned} \int_{S_A} k(z,z)\, dV(z) &= \frac{1}{2} \int_0^{\infty} k(t) t^{n/2-1} B_L(A\sqrt{t})\, dt \\ &\quad + II \qquad = I + II \end{aligned} \right.$$

where II are terms like those in the compact quotient.

Now

$$I = \frac{1}{2}\int_0^\infty k(t)\, t^{n/2-1}\left[\frac{2\pi^{n/2}}{\Gamma(n/2)}(\log(A\sqrt{t}) + \gamma_L) + O((\sqrt{t})^{-\delta})\right] dt$$

$$= \left(\frac{1}{2}\int_0^\infty k(t)\, t^{n/2-1}\, dt\right)\left[\frac{2\pi^{n/2}}{\Gamma(n/2)}(\log A + \gamma_L)\right]$$

$$+ \frac{1}{2}\,\frac{\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty k(t)\, t^{n/2-1}\log t\, dt + O(A^{-\delta}) .$$

To express everything in terms of g, recall that $\pi^{n/2}\, \mathcal{J}_{\frac{n}{2}}\, k(t) = Q(t)$ where

$$\mathcal{J}_\alpha f = \frac{1}{\Gamma(\alpha)}\int_0^\infty f(x+t)\, t^{\alpha-1}\, dt$$

and where $Q(e^u + e^{-u} - 2) = g(u)$. Thus

$$I = (\log A + \gamma_L)\, g(0) + \frac{1}{2}\,\frac{\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty k(t)\, t^{n/2-1}\log t\, dt .$$

Now

$$k(x) = \begin{cases} \pi^{-n/2}(-1)^{n/2}\, Q^{(n/2)}(x) & \text{if } n \text{ is even} \\ (-1)^{\frac{n+1}{2}}\, \mathcal{J}_{1/2}\, Q^{\frac{n+1}{2}}(x) & \text{if } n \text{ is odd} . \end{cases}$$

Say n is even we have

$$I = (\log A + \gamma_L)\, g(0) + \frac{1}{2}\,\frac{(-1)^{n/2}}{\Gamma(n/2)}\int_0^\infty Q^{(n/2)}(t)[t^{n/2-1}\log t\,]dt$$

integrating by parts $\frac{n}{2} - 1$ times gives

$$(\log A + \gamma_L) g(0) + \frac{1}{2} \int_0^\infty Q'(t)[\log t + c_n]\, dt .$$

Similarly one can show a similar such expression with n odd for appropriate c'_n. Thus

$$I = (\log A + \gamma_L) g(0) + c_n\, g(0) + \frac{1}{2} \int_0^\infty Q'(t) \log t\, dt$$

$$(7.25) \qquad = c_{n,\Gamma}\, g(0) + \frac{1}{2} \int_0^\infty \{u + 2 \log(1 - e^n)\}\, dg(u) + g(0) \log A$$

$$= c_{n,\Gamma}\, g(0) + \frac{1}{4} h(0) - \frac{1}{2\pi} \int_{-\infty}^\infty h(t) \frac{\Gamma'(1+it)}{\Gamma(1+it)}\, dt .$$

Putting these together we obtain the following trace formula for the case of one cusp.

For g of compact support:

$$(7.26) \quad \left\{ \begin{aligned} & \sum_j h(r_j) - \frac{1}{2\pi} \int_{-\infty}^\infty h(t) \frac{\phi'}{\phi}\left(\frac{n}{2} + it\right) dt \\ & \qquad = \overbrace{\text{ident.} + \text{hyp.} + \text{elliptic}}^{\text{as fefore}} + c'_{n,\Gamma}\, g(0) \\ & \qquad + \frac{1 - \phi(n/2)}{2} h(0) - \frac{1}{\pi} \int_{-\infty}^\infty h(t) \frac{\Gamma'(1+it)}{\Gamma(1+it)}\, dt . \end{aligned} \right.$$

The r_j is related to the s_j as in the compact case, via

$$r_j^2 + \frac{n}{2} = s_j .$$

Also the sum $\Sigma\, h(r_j)$ is a sum over $\pm\, r_j$'s which is why we multiplied all coefficients by 2 .

7.27 The General Finite Volume Case.

We have discussed the Eisenstein series in the general case of h_1 cusps, and proven the basic facts concerning these just as with the case of one cusp. One then proceeds in directly the same fasion, forming $K(z,\omega)$ and

$$H(z,\omega) = \frac{1}{2\pi} \sum_{j=1}^{h_1} \int_{-\infty}^{\infty} h(t)\, E_j\, (z, \tfrac{n}{2}+it)\, \overline{E_j(\omega, \tfrac{n}{2}+it)}\, dt$$

Under the usual condition on k, one shows that K-H is compact. Everything proceeds as before, the only thing to point out is in using the Maass - Selberg formula 1.62, along $s = \frac{n}{2} + it$, on taking a trace we are left only with $\phi'/\phi(\frac{n}{2}+it)$ where $\phi = \det.(\Phi_{ij})$. In fact it is the determinant ϕ which controles essentially everything, including the matrix Φ_{ij} .

Remark. If $\phi(s)$ is regular at some point s, with $\mathrm{Re}(s) < n/2$, $t \neq 0$, then so is $\Phi(s)$ (and hence also $E_i(z,s)$).

To see this, for $\mathrm{Re}(s) > \frac{n}{2}$, $t \neq 0$ we know everything is holomorphic. Now

$$\Phi(s)\,\Phi(n-s) = I .$$

Thus $\Phi(s)$, if $\mathrm{Re}(s) < \frac{n}{2}$ is regular if $\Phi(n-s)$ is invertable which depends only on whether $\phi(n-s) \neq 0$.

We leave the actual computation of the trace in this general case to the reader. The trace formula becomes:

Theorem 7.28. For g even of compact support and $h = \hat{g}$, $h_1 = \#$ of cusps

$$\sum_j h(r_j) - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t)\, \frac{\phi'}{\phi}\left(\frac{n}{2}+it\right)dt$$

$$= \text{ident.} + \text{elliptic} + \text{hyp.} +$$

$$C_{n,p,h}\, g(0) + \frac{1}{2}\left(h_1 - \operatorname{trace}\left(\Phi\left(\tfrac{n}{2}\right)\right)\right) h(0)$$

$$- \frac{h_1}{\pi} \int_{-\infty}^{\infty} h(t)\, \frac{\Gamma'}{\Gamma}(1+it)\, dt .$$

This is still not the most general case since we have assumed that the parabolic subgroups Γ_j corresponding to the cusp K_j is a rank n lattice. Actually in view of in Chapter , we can assume that our given general cofinite Γ has a subgroup Γ' of finite index, which has no elements of finite order. Γ' will then fall within the theory we have just developed.

As far as the analysis goes, there are no new difficulties since if $E_j(z,s)$ are the Eisenstein series for Γ' , then defining

$$F_j = \sum_{\gamma \in \Gamma/\Gamma'} E_j(\gamma z, s)$$

yield Γ invariant functions. It is easy to see that on

$$[\text{span}\,\{F_j(z, \tfrac{n}{2} + it)\}\,]^{\perp}$$

$K(z,\omega)$ is compact. The rest is then as before.

7.29 Spectral Resolution.

In view of 7.22 we have the following spectral expansion in the L^2 sense of an arbitrary $f \in L^2(\mathcal{H}^{n+1}/\Gamma)$.

$$(7.30) \qquad f = \sum_j \langle f, u_j \rangle u_j(z) + \sum_{j=1}^{h} \int_{-\infty}^{\infty} \langle f, E_j(\cdot, \tfrac{n}{2} + it)\rangle E_j(z, \tfrac{n}{2} + it)dt$$

where $\langle\ ,\ \rangle$ is L^2 inner product over $\mathfrak{F}$.

With this comes also the Parseval identity (this is the correct version of the preliminary Bessel inequality used earlier).

$$(7.31) \qquad \|f\|_2^2 = \sum_{j=1}^{h} \int_{-\infty}^{\infty} |\langle f, E_j(\cdot, \tfrac{n}{2} + it)\rangle|^2 dt + \sum_j |\langle f, u_j \rangle|^2 .$$

The action of any function of Δ on an $f \in L^2$ is then easily calculated from the expansion 7.30

$$K_o f = \sum_j \langle f, u_j \rangle \hat{K}(\lambda_j) u_j(z) + \sum_{j=1}^{h} \int_{-\infty}^{\infty} h(t) \langle f, E_j \rangle E_j(z, \tfrac{n}{2} + it) dt.$$

We turn to immediate applications of the trace formula, just as in the case of a compact quotient.

7.32 Weyl's Law.

The important new term in the trace formula 7.28 (different to the compact case) is the second term on the left hand side - i.e. $\int h \frac{\phi'}{\phi} dt$ term. This term comes from the Eisenstein series and hence the continuous spectrum. Except for finitely many poles ϕ has it's poles to the left of $\mathrm{Re}(s) = \frac{n}{2}$ (the poles and zero are symmetric via $\phi(s)\phi(n-s) = 1$) and so $\int_{-T}^{T} \phi'/\phi(\frac{n}{2} + it) dt$ measures the winding due to the poles. This is even more clear from the partial fraction expansion 7.10 for $(\phi^*)'/(\phi)^*$ which differs from ϕ'/ϕ by only finitely many factors and hence asymptotically is not significantly different. If we now apply the exact same argument as we did in the compact case to obtain the asymptotics of the eigenvalues, we obtain the following analogue of Weyl's law:

Theorem 7.33.

$$\sum_{|r_j| \leq T} 1 - \frac{1}{2\pi} \int_{-T}^{T} \frac{\phi'}{\phi}\left(\frac{n}{2}+it\right)dt = CT^{n+1} - \frac{h_1}{\pi} T \log T + C_1 T + O\left(\frac{T^n}{\log T}\right)$$

where $C =$.

Notice that only in dimension $n = 1$ the terms $T \log T$ and T not swollowed by the big O term. The new terms, $T \log T$ and T come from the $g(0)$ and $\int \frac{\Gamma'}{\Gamma}$ terms in 7.28.

The term $-\int_{-T}^{T} \frac{\phi'}{\phi}$ is a winding number, counts the continuous spectrum, and for large T is of course positive. Thus Weyl's law is correct when we count both the discrete and continuous spectrum in this way. One is led naturally to the question as to which (if any) of the two terms on the left of 7.33 is the dominant term. In fact there is no obvious reason why $\sum_{|r_j| \leq T} 1$ term should even be unbounded. Notice 7.33 gives the bound $\int_{-T}^{T} \phi'/\phi = O(T^{n+1})$ which is our key estimate 7.14! We are led to one of the most important unsolved problems of the theory.

Conjecture 7.34. (Roelcke - Selberg).

(a) Weakest form: $N(T) = \sum_{|r_j| \leq T} 1$ is unbounded as a function of T. Put another way this conjecture says that there are infinitely many L^2 eigenfunctions for Δ, on any cofinite quotient of hyperbolic space.

(b) Stronger form: The $N(T)$ term in 7.33 is the dominant term, i.e.

$$N(T) \sim C\,T^{n+1} \quad \text{as} \quad T \to \infty .$$

(c) Strongest form: $\int_{-T}^{T} \frac{\phi'}{\phi}(\frac{n}{2}+it)dt = O(T^{1+\varepsilon}) \qquad \forall\, \varepsilon > 0.$

One reason for conjecturing part (c) is that this term corresponds to the continuous spectrum, which in turn comes from the Eisenstein series which correspond to automorphizations of functions of y alone - so one may expect the asymptotics to be of a one dimensional problem.

Evidence of the possible truth of 7.34 comes from arithmetic groups where one can calculate $\phi(s)$ explicitly and one finds that the conjecture is true in it's strongest form. In fact we saw in (7.10)''' that

$$\int_{-T}^{T} \frac{\phi'}{\phi}(\frac{n}{2}+it)dt = O(T) + 2\pi \sum_{|\gamma| \leq 2T} 1$$

where $\beta + i\gamma$ is a pole of $\phi(s)$. Thus part (c) follows if

$$\phi(s) = \frac{a(s)}{b(s)}$$

with $a(s)$ entire and $b(s)$ entire of order 1. This happens to be the case in all the examples (arithmetic) of Chapters 6 that we calculated, since $b(s)$ is typically a zeta function of a number field.

In particular we have

Theorem 7.35. In the case of the modular group $SL_2(\mathbb{Z})$ the discrete spectrum satisfies Weyl's law. Thus most of the spectrum is discrete.

Theorem 7.35 is a triumph of the trace formula, it gives the existence of L^2 spectrum, which as we shall see in the next section gives us cusp forms.

These mysterious objects will be a major concern for the rest of the book!

7.36 Asymptotics of Geodesics.

The other immediate application of the trace formula is to the asymptotics of the lengths of geodesics on hyperbolic manifolds. Everything proceeds exactly as in the compact case. Indeed if we look at the proof for the compact quotient case we observe that the $\int \frac{\phi'}{\phi}$ and $\int \frac{\Gamma'}{\Gamma}$ terms as well as the $h(0)$ terms may all be incorporated in Lemma 1 of that section. The rest of the proof follows mutatus mutandis.

Theorem 7.37. Let $M = H^{n+1}/\Gamma$ be a finite volume hyperbolic manifold and let $\Pi(x)$ be as before, the number of closed geodesics on M whose length is less than x. Then

$$\Pi(x) = Li(e^{nx}) + Li(e^{(\frac{n}{2}+t_1)x}) \ldots + Li(e^{(\frac{n}{2}+t_k)x}) + O(e^{(n-\frac{n}{n+2})x})$$

where $t_j = ir_j$ corresponds to the discrete eigenvalues of Δ on $L^2(M)$, in $[0, (\frac{n}{2})^2)$.

The Selberg zeta function $z(s)$ may be formed out of the hyperbolic transformations in a fasion exactly the same as before. Its analytic continuation, functional equation follow as before, the only difference being the poles of $\phi(s)$ which appear as poles of $z(s)$ as well.

We end this chapter by defining cusp forms:

We have already introduced the space $\mathcal{E}$ of functions spanned by the Eisenstein series $E_j(z, \frac{n}{2} + it)$ along the line $s = \frac{n}{2}$. Consider this space together with the finite dimensional space of residues of the E_j's at the poles of E in $(\frac{n}{2}, n)$, which we call R. We have seen each member of R is in $L^2(\mathcal{H}^{n+1}/\Gamma)$.

If u is C^∞ of compact support in $\mathfrak{F}$ then for $\mathrm{Re}(s) > n$ we have

$$F_j(s) = \int_{\mathfrak{F}} u(z) E_j(z,s)\, dV(z)$$

$$= \int_0^\infty \int_{\mathfrak{F}_j} u(z^{(j)})\, (y^j)^s \frac{dx^{(j)}\, dy^{(j)}}{(y^j)^{n+1}} = \int_0^\infty \hat{u}_j(y^{(j)}, 0)(y^j)^{s-n} \frac{dy^{(j)}}{(y^j)} .$$

Where $\hat{u}_j(y^{(j)}, 0)$ is the zero'th coefficient of u in the j^{th} cusp, i.e. $F_j(s)$ is the Mollin transform of $\hat{u}_j(y)$. Thus

$$y^{-n} \hat{u}_j(y,0) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_j(s)\, y^{-s}\, ds .$$

Shifting the integral across to $c = \frac{n}{2}$ gives

$$\hat{u}_j(y,0) = \frac{y^{n/2}}{2\pi i} \int_{-\infty}^{\infty} F_j(\tfrac{n}{2}+it)y^{-it}\, dt + \sum_{\sigma_K} y^{n-\sigma_K} \operatorname{Res}(F\, E, S = \sigma_K)$$

residue of F_j on $(\frac{n}{2}, n)$.

The residues of F correspond to the poles of E in $(\frac{n}{2}, n]$, and clearly give

$$\hat{u}_j(y,0) = \frac{y^{n/2}}{2\pi i} \int_{-\infty}^{\infty} \langle u(\cdot),\ E_j(.,\tfrac{n}{2}+it)\rangle\, y^{-it}\, dt$$

$$+ \sum_{\sigma_K} \langle u(.),\ \operatorname{Res}\ (E_j,\ s=\sigma_K)\rangle\, y^{n-\sigma_K} .$$

The last equation makes sense (and so entends) for all $u \in L^2(\mathbb{H}^{n+1}/\Gamma)$, since, by the spectral resolution $\langle u(.),\ E_j(.\,,\ \frac{n}{2}+it)\rangle \in L^2(\mathbb{R})$, and $\operatorname{Res}(E_j\,,\ s=\sigma_K) \in L^2(\mathbb{H}^{n+1}/\Gamma)$. It follows that if $u \in (\mathcal{E} \oplus R)^{\perp}$ then $\hat{u}_j(y,0) \equiv 0$. Conversely if $\hat{u}_j(y,0) \equiv 0$ it is clear by growth considerations that $\langle u(.),\ \operatorname{Res}(E_j,\ s=\sigma_K)\rangle = 0$, and $\langle u(.), E_j(.\,,\frac{n}{2}+it)\rangle = 0$. If this is so for every j then $u \in (\mathcal{E} \oplus R)^{\perp}$.

<u>Definition 7.38</u>. We let C be the space of <u>cuspidal</u> functions, i.e. the subspace of $L^2(\mathbb{H}^{n+1}/\Gamma)$ consisting of all $u \in L^2$ for which

$$\int_{\mathfrak{F}_j} u(z^j)\, dx^j = 0 \qquad \forall\, y^{(j)},\ j=1,\dots,h_1 ,$$

i.e. all L^2 functions which have zero Fourier coefficient in every cusp. It is easy to see that this definition makes sense on L^2 functions and defines a closed subspace of L^2.

We have shown in the remarks above 7.38 that

$$C = (\mathcal{E} \oplus R)^{\perp} .$$

Thus $L^2 = \mathcal{E} \oplus R \oplus C$, each of the subspaces is invariant under function of Δ. Also we have seen that $K(z,\omega)$ as usual is compact on $R \oplus C$, and in particular on C. Thus C has an orthonormal basis of eigenfunctions

$$v_1, v_2, \ldots, \qquad \Delta v_j + \lambda_j v_j = 0 .$$

<u>Definition 7.39</u>. A cusp <u>form</u> is an L^2 eigenfunction of Δ which has zero Fourier coefficient in each cusp i.e. an L^2 eigenfunction of Δ which is in C.

We observe that since R is finite dimensional the conjecture 7.34 in its weakest form is equivalent to C being infinite dimensional, or that there are infinitely many cusp forms.

This definition of cusp form is modeled on the classical definition of an analytic automorphic form being one which has zero coefficient in its Fourier expansion, equal to zero, see [ ].