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HEDRICK LECTURES

MAA MATHFEST AUGUST 2004

RAMANUJAN GRAPHS AND THE RAMANUJAN CONJECTURES

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Lecture 1 Expander Graphs

- Highly connected but sparse graphs.

$X = (V, E)$ finite

V = set of vertices

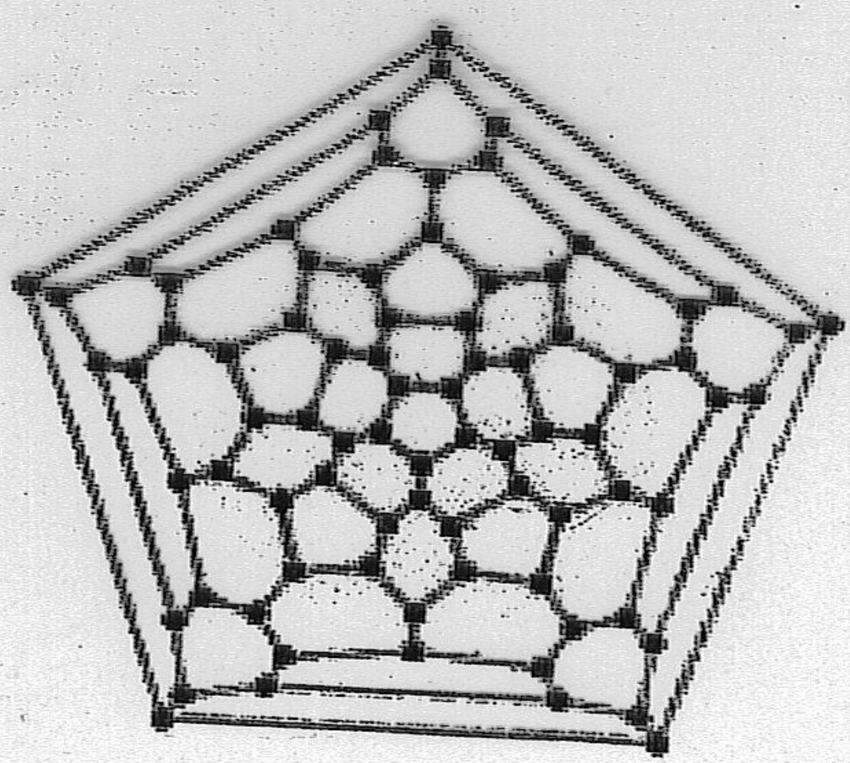
E = set of edges

$|V| = n \qquad |E| = m$

want · $m \approx n$ ie sparse.

· "highly connected"

EXAMPLE



$|V| = 80$

$|E| = 120$

C_{80} "Carbon 80"

planar

girth = 5

3-regular

see GRAVER's MINI-COURSE for fullerine graphs

$\boxed{\text{need} \leq 4 \text{ colors!}}$

Applications to Pure Maths:

- combinatorics
- functional analysis / topology
  Baumes - Connes Conjectures
- measure theory Ruziewicz problem

Theoretical Computer Science:

- Derandomization
- superefficient communication networks
- error correcting codes
- computational group theory
- graphs that are hard to pebble.

⋮

much more!

combinatorics:

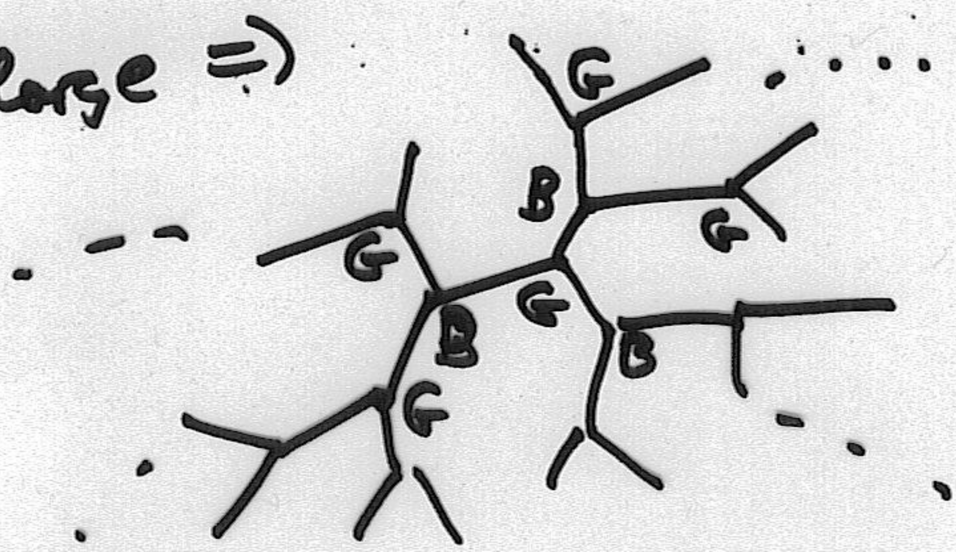
classical problem:—

Given integers g and c find (or at least prove existence) a graph X with $\text{girth}(X) \geq g$ and $\chi(X)$ (chromatic number) $\geq c$.

• girth = length of the shortest closed circuit

• chromatic no = least number of colors to paint V such that no two adjacent v's have same color.

g large $\Rightarrow$



locally 2-colorable

Erdos (1959) such graphs exist!

In fact the random graph of certain shape has this property.

"probabilistic method".

Highly Connected:

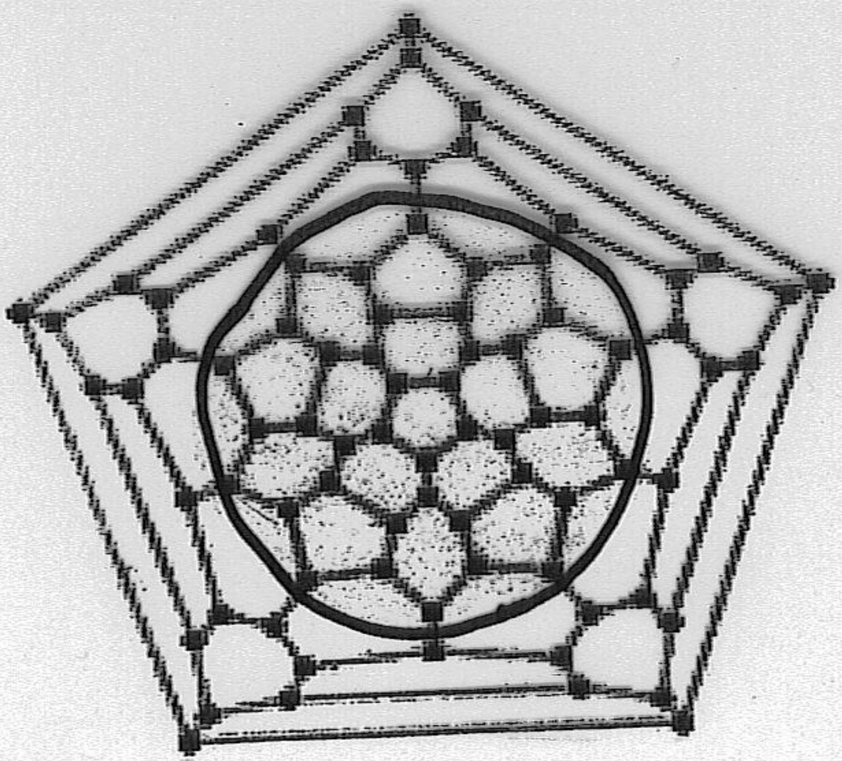
$X = (V, E)$

- Cheeger or expansion coefficient

$$h(X) = \min_{\phi \subsetneq F \subsetneq V} \frac{|\partial F|}{\min(|F|, |F^c|)}$$

F^c = complement of F

∂F is the set of edges running from F to F^c



an optimal decomposition

~~F-vertices in the circle~~

F vertices in the circle $|F|=40$

F^c outside $|F^c|=40$

$h(X)=\frac{1}{4}$ $|\partial F|=10$

(Proof see cover of August 2004 Notices AMS)

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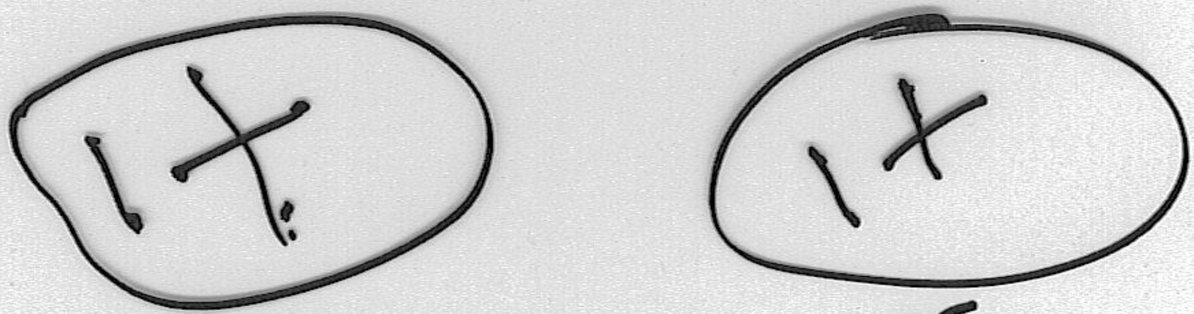
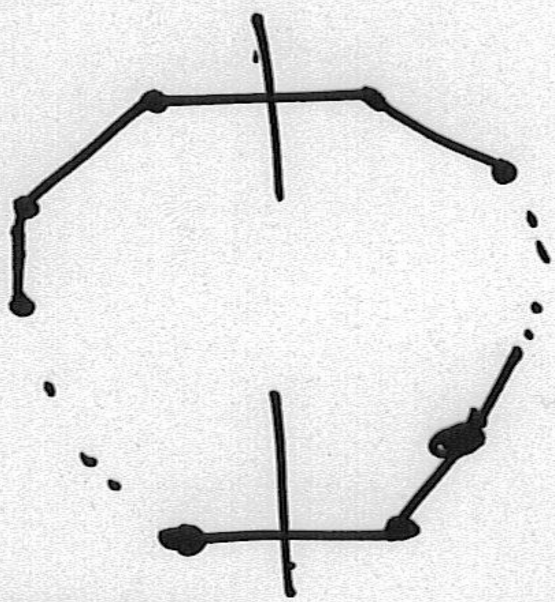
$h(X) \neq 0$ if and only if X is connected.



F_k $\quad$ F^c_k

$\partial(F) = \phi$.

<u>Eg:</u> C_{2n} The $2n$-cycle



F $\quad$ F^c.

$|\partial F| = 2$ $\qquad$ $|F| = n$

$$h(C_{2n}) = \frac{1}{n}$$

$\longrightarrow 0$ as $n \to \infty$

(6) I

Restrict to k-regular graphs

ie every $v \in V$ has k edges coming out of it.

$X_{n,k}$ k-regular with $|V| = n$.

$$|E| = \frac{nk}{2}$$

(sparse)

<u>Definition:</u> An expander is a sequence of $X_{n,k}$'s, k-fixed and $n \to \infty$ s.t. there is $\varepsilon_0 > 0$ and $h(X) \geq \varepsilon_0$ for all X.

(uniform expansion of sets of size $\leq |V|/2$).

"highly connected"

$\boxed{k \geq 3}$

Do they exist?

Yes! Pinsker (1973) by probalistic method.

EXPLICIT?

Spectral Method (Tanner-Alon/Milman)

Recall: spectral theorem for real symmetric matrices.

A real $n \times n$ symmetric

then A is orthogonally diagonalizable

$v_1, v_2, \ldots, v_n$ orthonormal basis of $\mathbb{R}^n$

s.t. $A v_j = \lambda_j v_j$

$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$, eigenvalues.

Variationally:

$$\lambda_1 = \max_{\|v\|=1} v^t A v$$

achieved by v_1

$\langle , \rangle$ standard inner product on $\mathbb{R}^n$.

$$\lambda_2 = \max_{\substack{\|v\|=1 \\ \langle v, v_1 \rangle = 0}} v^t A v$$

etc.

$X_{n,k} = (V, E)$ order V

$\{v_1, v_2, \ldots, v_n\}$

Adjacency Matrix:

$$A = (a_{ij}) \qquad a_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j \\ 0 & \text{otherwise} \end{cases}$$

Note: $A^{\nu} = (a_{ij}^{(\nu)})$

$a_{ij}^{(\nu)} = \#$ of paths from v_i to v_j.

A is real symmetric with row and column sum $= k$.

$$A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

in fact

$$\lambda_0 = k \geq \lambda_1 \geq \lambda_2 \cdots\cdots \geq \lambda_{n-1}.$$

- $X_{n,k}$ is connected iff $\lambda_1 < k$.

Proof: $\Longrightarrow$ $X_{n,k}$ connected
$Af = kf$

$$\Rightarrow \frac{1}{k} A f(v) = \frac{1}{k} \sum_{w \sim v} f(w) = f(v)$$

"f is harmonic"

Let $\max_{v \in V} f(v)$ occur at v_0.

Then $$f(v_0) = \frac{1}{k} \sum_{w \sim v} f(w)$$

= average of nb's of v_0

$\Longrightarrow$ $f(w) = f(v_0)$ for $w \sim v_0$

now repeat for such w to get $f(w') = f(v_0)$ for w' a nbh of w

Since X is connected $f = \text{const.}$

Exercise: $\lambda_{n-1} = -k$ iff $X_{n,k}$ is bipartite. (2 colorable)

$k-\lambda_1$ the spectral gap also measures "high connectedness".

THEOREM: (Tanner; Alon Milman)

$$\frac{k-\lambda_1}{2} \le h(X_{n,k}) \le \sqrt{2k(k-\lambda_1)}$$

- Let $\mu(X) = \max\{|\lambda_j|;\ j \neq 0\}$

X not bipartite and connected $\Rightarrow \mu(X) < k$.

The smaller the μ the more connected.

Properties:

(i). $\chi(X) \geq \dfrac{k}{\mu(X)}$ (Hoffman)

(ii). If $S, T \subset V$

$\#(S,T)$ = number of edges running between S and T

$$\left| \#(S,T) - \frac{|S||T|k}{n} \right| \leq \cancel{\lambda}\sqrt{|S||T|}$$

μ

$\dfrac{|S||T|k}{n}$ = ~~no~~ expected no of edges for a random graph.

("quasi random")

(iii) A random walk on X equidistributes itself quickly if μ is small.

$p_0(v)$ a prob. distn. on V

$p_t(v)$ the distn starting at p_0 and walking t steps

then

$$\| p_t - \text{uniform} \|_1 = \sum_{v \in V} \left| p_t(v) - \frac{1}{n} \right|$$

$$\leq \sqrt{n} \left(\frac{\mu}{k} \right)^t .$$

(iv).

(v)

⋮

see books by F. CHUNG and LINIAL–WIGDERSON.

(references later).

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Proof of (i):

χ colors so $V = F_1 \sqcup F_2 \ldots \sqcup F_\chi$ with F_j having no edges internally.

Say $F = F_{j_0}$ has max size.

$$|F| \geq \frac{n}{\chi}.$$

Set
$$f(v) = \begin{cases} |V-F| & , v \in F \\ -|F| & \text{if } v \notin F. \end{cases}$$

$$\sum f(v) = 0$$

$$\|f\|_2^2 = |F|.|V-F|.|V| \leq |F| n^2$$

also $v \in F$

$$Af(v) = \sum_{w \notin F} \sum_{w \sim v} f(w) = -|F| \sum_{\substack{w \notin F \\ w \sim v}} 1 = -k|F|$$

$$\|Af\|_2^2 \geq k^2 |F|^3$$

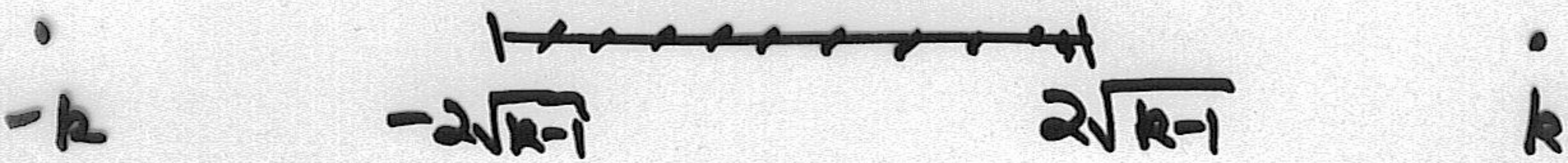
But $\|Af\|_2^2 \leq \mu^2 \|f\|_2^2$ since $\langle f, \mathbb{1} \rangle = 0$

$$\Rightarrow \quad \chi \geq k/\mu.$$

· Limit to how small μ can be:

<u>Proposition</u> (Alon – Boppana)

$$\underline{\lim}_{n\to\infty} \inf_{X_{n,k}} \mu(X) \geq 2\sqrt{k-1}$$



<u>PROOF:</u> $X_{n,k}$ given

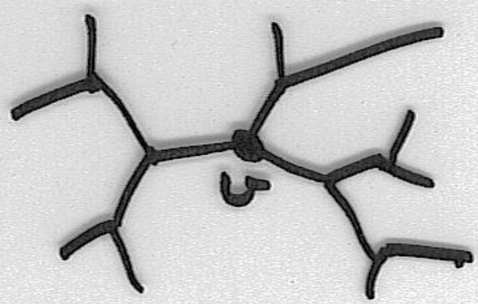
$\nu \geq 1$.

$A^{2\nu}$ has $a^{(2\nu)}(j,j)$

$=$ # of paths from v_j to v_j of length 2ν.

Let $\rho(2\nu)$ be the # of paths of length 2ν from v to v on a k-regular tree, $\rho(2\nu)$ can be calculated explicitly.

eg $k=3$



clearly $a^{(2\nu)}(j,j) \geq \rho(2\nu)$

$$\mathrm{TRACE}(A^{2\nu}) \geq n\,\rho(2\nu)$$

$$\| \quad k^{2\nu} + \sum_{j=1}^{n-1} \lambda_j^{2\nu}$$

$$\underline{\lim}\ \mu^{2\nu}(x) \geq \rho(2\nu)$$

$$\underline{\lim}\ \mu(x) \geq (\rho(2\nu))^{1/2\nu}$$

but $(\rho(2\nu))^{1/2\nu} \longrightarrow 2\sqrt{k-1}$ (Exercise).

<u>Definition:</u> $X_{n,k}$ is a Ramanujan graph if

$$\mu(X_{n,k}) \leq 2\sqrt{k-1}.$$

- Spectrally optimally highly connected, escpanders.

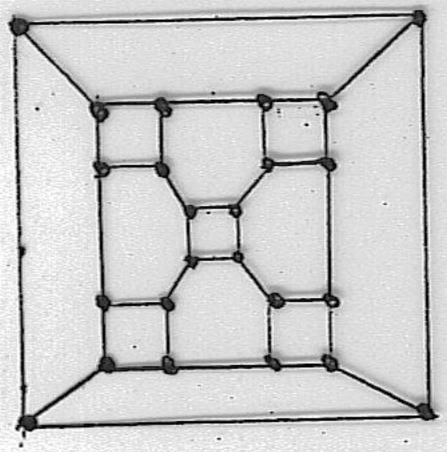
<u>Eg:</u> C_{80}

C_{84} · largest planar Ramanujan graph known.

(there are finitely many planar Ramanujan ~~gra~~ graphs,
Lipton-Tarjan ~~x~~ since $(h \to 0 \quad n \to \infty)$)

SOME PLANAR CUBIC RAMANUJAN GRAPHS

(There are only finitely many such graphs) $2\sqrt{2} = 2.828\ldots$

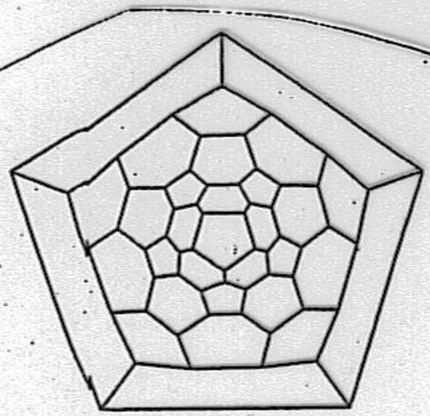


$X(2,3)$ (P. Chiu)

Cayley graph for $PGL(2,\mathbb{F}_3)$

$\lambda_1 = 2.414\ldots$, bipartite

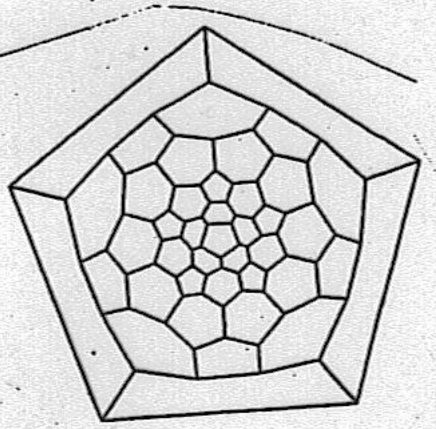
$h = \frac{1}{2}$.



Soccer ball graph (truncated icosahedron C_{60})

Cayley graph for A_5.

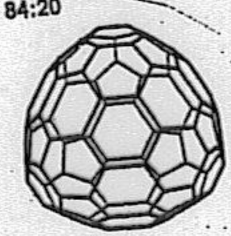
$\lambda_1 = 2.7566\ldots$

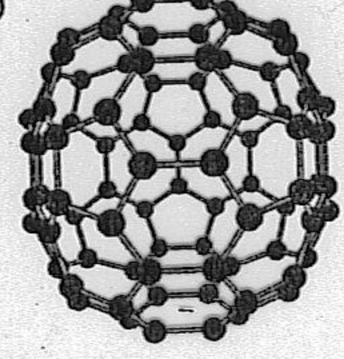


C_{80} with I_h (order 120) symmetry group. Non homogeneous.

$\lambda_1 = 2.818$ (Gamburd).

$h = 1/4$





Apparently these are the largest such graphs known.
(Fowler-Rogers-Fajtlowicz-Hasen-Capoross

$C_{84;20}$, symmetry T_d order 24.

$C_{84,23}$, D_{d2}, $\lambda_1 = 2.8279\ldots$

Ramanujan graphs exist for $k = p^e + 1$. p a prime, $e \geq 1$.

<u>Question</u>: Do they exist for all $k \geq 3$?

<u>Explicit Construction:</u>

$p \equiv 1 \, (4)$ a prime

$$p = x_0^2 + x_1^2 + x_2^2 + x_3^2 \qquad x_j \in \mathbb{Z}$$

$x_0 > 0$ odd

There are $k = p+1$ such (x_0, x_1, x_2, x_3)

eg. $p=5$

$(1, \pm 2, 0, 0)$
$(1, 0, \pm 2, 0)$
$(1, 0, 0, \pm 2)$.

Let $q \equiv 1\,(4)$ q prime

$\mathbb{F}_q = \mathbb{Z}/q\mathbb{Z}$ field q elements

$i = \sqrt{-1}$ $\quad i^2 \equiv -1\,(q)$.

Assume $\left(\frac{p}{q}\right) = 1$ ie

that there is $b \in \mathbb{F}_q$ s.t.

$b^2 \equiv p \pmod{q}$.

$G = SL(2, \mathbb{F}_q)$ $\quad$ 2×2 matrices with det $= 1$

$|G| \sim q^3$ (tend to ∞).

$$S = \left\{ \frac{1}{b} \begin{pmatrix} x_0 + i x_1 & x_2 + i x_3 \\ -x_2 + i x_3 & x_0 - i x_1 \end{pmatrix} \;\middle|\; (x_0, x_1, x_2, x_3) \text{ above} \right\}$$

$|S| = k = p + 1$, $S \subset G$.

$\cdot \quad s \in S$ iff $s^{-1} \in S$.

$$X^{(p,q)} = (G, E)$$

with $V = G = SL(2, \mathbb{F}_q)$

$g, h \in V \quad g \sim h$ if

$g = hs$

with $s \in S$

eg. each g has $|S| = k$ nbh's.

$X^{(p,q)}$ is k-regular.

<u>Theorem:</u> (Lubotzky-Phillips-S, Margulis)

(i) $X^{(p,q)}$ is connected

(ii) girth$(X^{(p,q)}) \geq \frac{2}{3} \log_{k-1} |X|$.

(iii) $X^{(p,q)}$ is Ramanujan.

- $X^{(p,q)}$ are explicit spectrally optimal expanders

- $$\chi\left(X^{(p,q)}\right) \geq \frac{p+1}{\mu(X)} \geq \frac{p+1}{2\sqrt{p}}$$

as if p is large and q is large $X^{(p,q)}$ has large chromatic number and girth.

Recall:

$X = X_{n,k}$ k-regular graph on n vertices.

$k = \lambda_0 \geq \lambda_1 \cdots \geq \lambda_{n-1}$ eigenvalues of adjacency

$$\mu(X) = \max\{|\lambda_j| : j \neq 0\}.$$

. the smaller μ the more connected

X is Ramanujan if

$\mu(X) \leq 2\sqrt{k-1}$. optimal!

They exist if $k = p^e + 1$.

p^e = order of a finite field.

Let $p \equiv 1(4)$ be prime

$x_0^2 + x_1^2 + x_2^2 + x_3^2 = p$ $\quad x_0 > 0$ odd, $x_j \in \mathbb{Z}$

denote these by S

$$|S| = p+1.$$

$q \equiv 1 \, (4)$ another prime

$$\mathbb{F}_q = \mathbb{Z}/q\mathbb{Z} \quad \text{finite field}$$

assume $\left(\frac{p}{q}\right) = 1$ ie $b^2 = p$ in $\mathbb{F}_q$ has a solution

let $i^2 = -1$ in $\mathbb{F}_q$.

$X^{(p,q)}$ is the (CAYLEY) graph with

$$V = G = PSL(2, \mathbb{F}_q)$$

E : $g \sim h$ if $g = hs$ for $s \in S$

$$S = \left\{ \frac{1}{b} \begin{bmatrix} x_0 + i x_1 & x_2 + i x_3 \\ -x_2 + i x_3 & x_0 - i x_1 \end{bmatrix} : x \in S \right\}$$

$|S| = k = p+1$.

$X^{(p,q)}$ is a k-regular graph.

- $X^{(p,q)}$ is Ramanujan

$$\mu\left(X^{(p,q)}\right) \le 2\sqrt{k-1} = 2\sqrt{p}\,.$$

More general cases (Chiu, Morgenstern, Pizer, Livne/Jordan ...)

<u>Random $X_{n,k}$'s?</u>

- They are expanders

Are they Ramanujan?
Is random best?

<u>THEOREM</u> (J. FRIEDMAN 2003)
$k \ge 3$, $\varepsilon > 0$ fixed then

$$\mathrm{Prob}\left\{\mu(X_{n,k}) \le 2\sqrt{k-1} + \varepsilon\right\} \longrightarrow 1$$

as $n \to \infty$.

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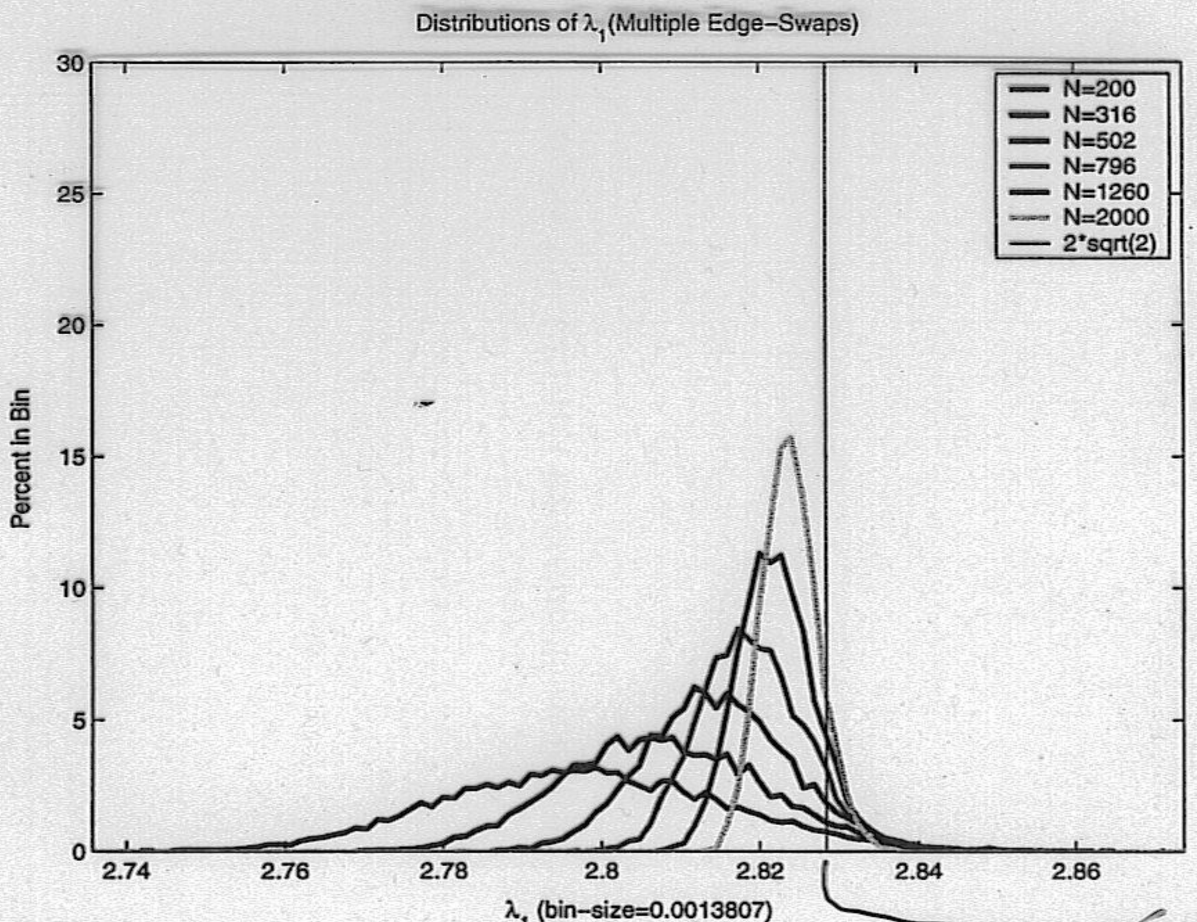
T. NOVIKOFF II

COURANT UNDERGRAD HONOURS LAB.



$2\sqrt{2}$

Figure 2: Shows the distributions of $\lambda_1(G)$ for a few different graphs sizes, collected using the appropriate number of edge-swaps at each step. Note that the distribution gets thinner as the graph size increases, and also that the distribution is slowly shifting to the right.

The movement ~~to~~ of the mean to $2\sqrt{2}$ is faster than the standard deviation. Renormalizing:

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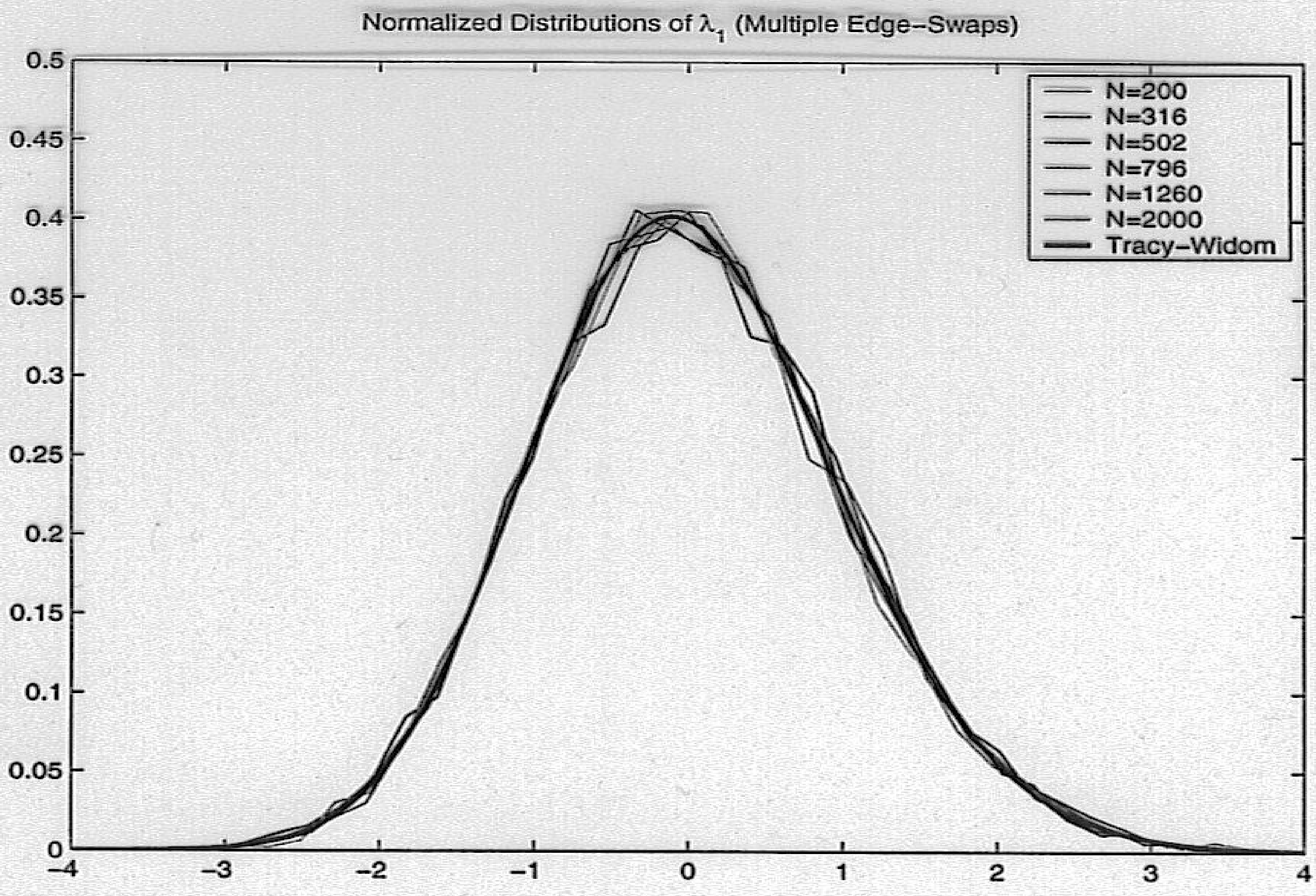

Figure 3: Shows the distributions of Figure 3, normalized to have mean=0, variance=1, and area=1. Plotted as well is the Tracy-Widom distribution for beta=1, but the distribution for beta=2 may as well have been plotted, for as one can see in Figure 4, the two are practically indistinguishable once normalized. The distributions of λ_1 are clearly not symmetric about their mean, but rather possess the same sort of asymmetry as the Tracy-Widom distributions.

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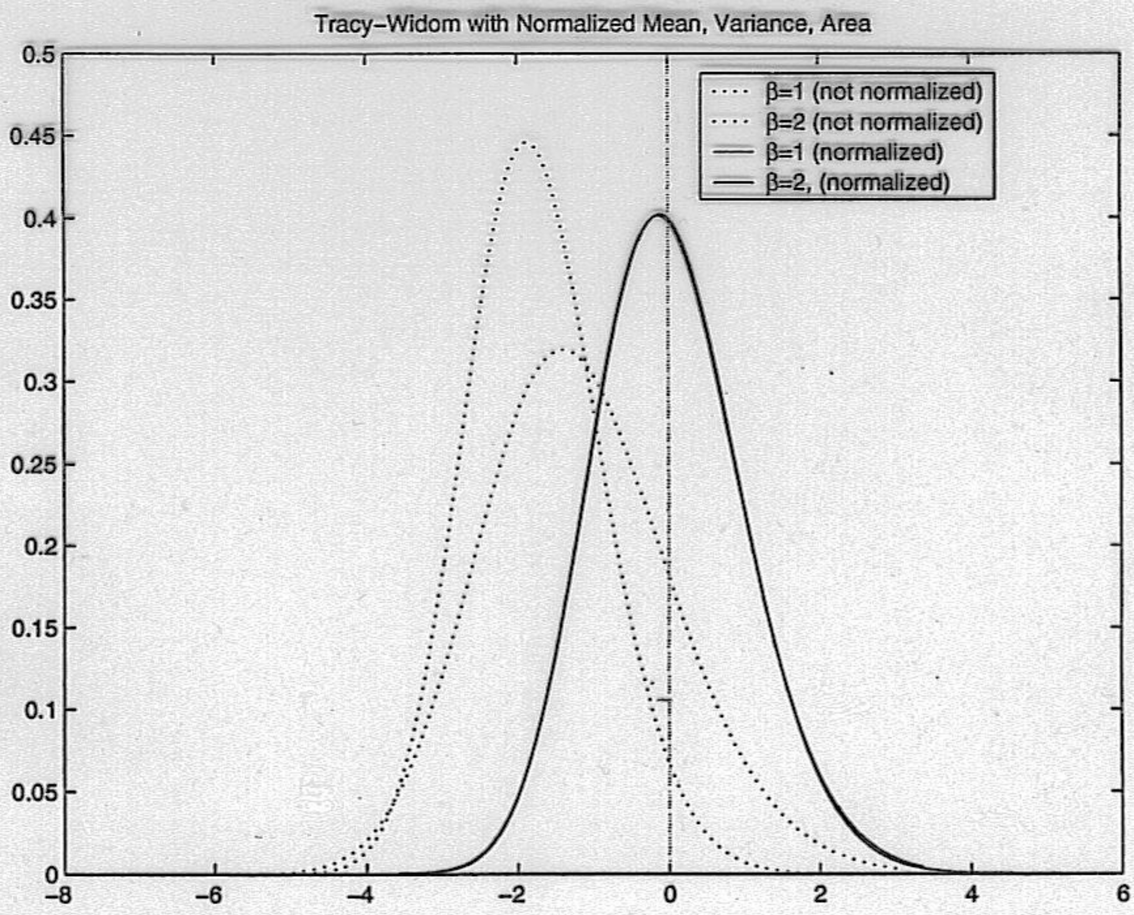

Figure 4: The dotted curves are the Tracy-Widom distributions for "$\beta = 1$" (GOE) and "$\beta = 2$" (GUE), as they were presented in the original paper by Tracy et al, and the solid curves are the same curves normalized to have mean=0, variance=1, and area=1.

these suggest~~s~~ That

$\text{Prob}(X_{n,k} \text{ is Ramanujan}) = 0.52\ldots$!

$\boxed{\text{bet}}$

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Modular forms (classical)

periodic functions (or forms or better still eigenfunctions) on homogeneous spaces.

$\mathbb{H}$ upper half plane
$= \{ z = x+iy \mid y > 0 \}$.

z a complex variable

$G = SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ ad-bc = 1 \right\}$

G is a (Lie) group with the usual topology from $\mathbb{R}^4$.

G acts projectively on $\mathbb{H}$

$$gz = \frac{az+b}{cz+d}, \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$z \to gz$ preserves $\dfrac{dx\,dy}{y^2}$.

G acts transitively on $\mathbb{H}$

ie if $z_0 \in \mathbb{H}$ is fixed and $z \in \mathbb{H}$ there is $g \in G$ such that $g z_0 = z$

Stabilizer of $i = \sqrt{-1} \in \mathbb{H}$

$$= \{ g i = i \} = \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} : 0 \le \theta < 2\pi \right\}.$$

$:= K$

compact.

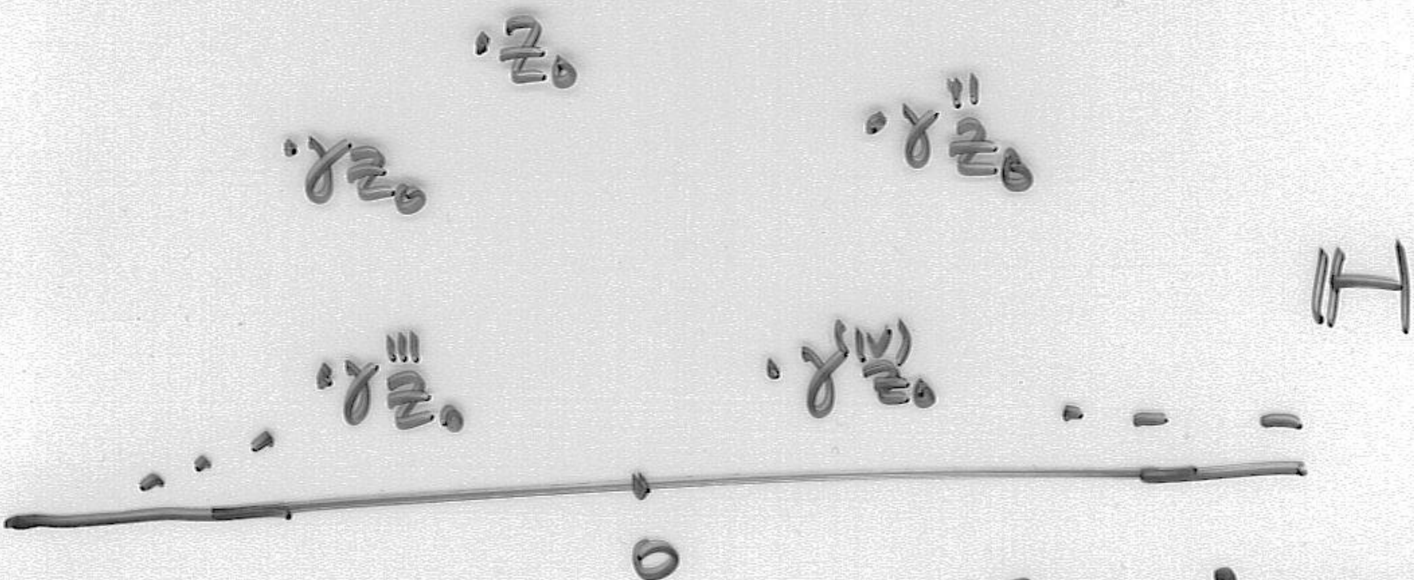
The cosets.

$$G/K \cong \mathbb{H}$$ (in fact topologically)

Number theory (or discrete math) enters via

$$\Gamma = SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{l} ad - bc = 1 \\ a, b, c, d \in \mathbb{Z} \end{array} \right\}.$$

<u>discrete</u> subgroup of $SL(2, \mathbb{R})$.



Fix z_0, the images $\{\gamma z_0\}$, $\gamma \in \Gamma$ have no accumulation point in $\mathbb{H}$ (Γ discrete in G, K compact).

Form $X = \Gamma \backslash \mathbb{H}$

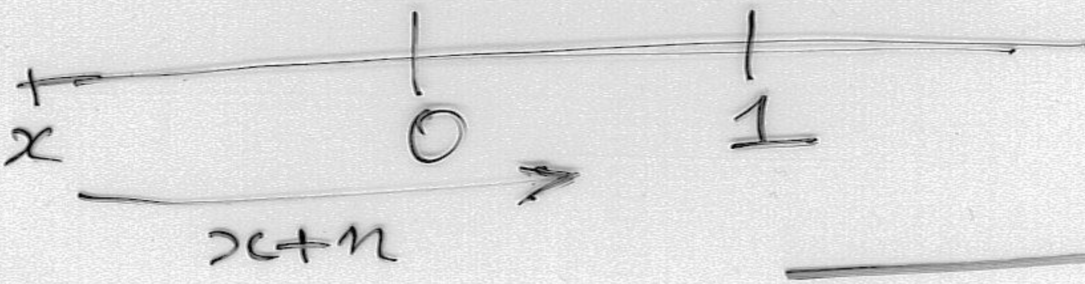
the modular surface.

Fundamental domain:

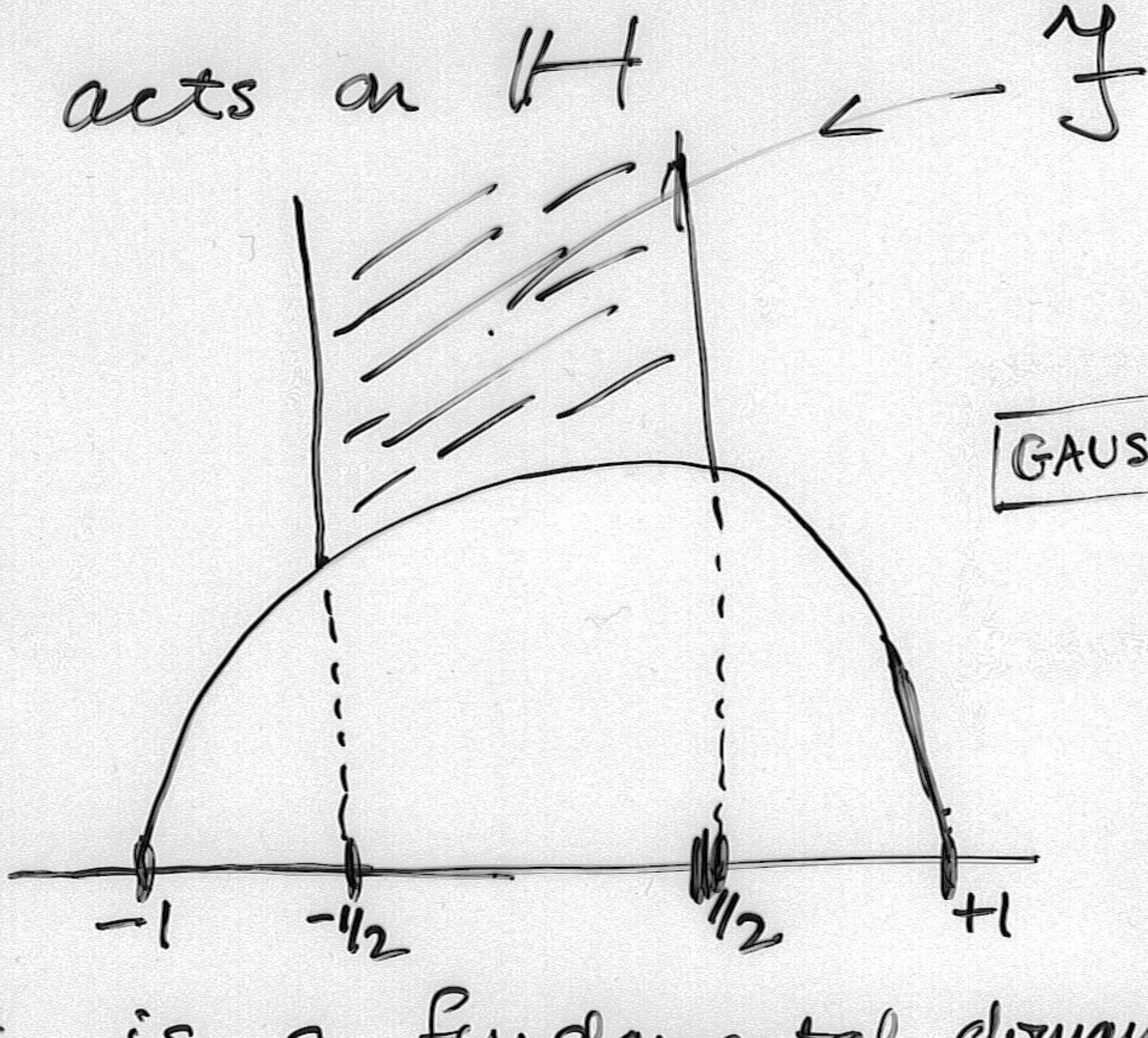
(11) II

aside: $(\mathbb{R},+)$, $\mathbb{Z} \subset \mathbb{R}$ discrete subgroup

$$\mathbb{R}/\mathbb{Z} \cong [0,1) = \bigcirc \text{ circle}$$



Γ acts on $\mathbb{H}$



GAUSS.

$\mathcal{F}$ is a fundamental domain.

PROOF:

$$\begin{bmatrix}1&1\\0&1\end{bmatrix}=T\in\Gamma \quad , T^n=\begin{bmatrix}1&n\\0&1\end{bmatrix}$$

$$T^n z = z+n$$

$$S=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\in\Gamma \quad (S, T \text{ generate})$$

$z\in\mathbb{H}$ find $\gamma\in\Gamma$ s.t. $\gamma z\in\mathcal{F}$

if $z\in\mathcal{F}$ — done.

otherwise choose n s.t. $z' = T^n z$

$$-\tfrac{1}{2} < \mathrm{Re}(z') \leq \tfrac{1}{2} \quad , \mathrm{Im}(z') = \mathrm{Im}(z)$$

if $z'\in\mathcal{F}$ — done.

else $\mathrm{Im}(z') \leq 1$ in fact $|z'|<1$

set $z'' = Sz'$, $\mathrm{Im}(z'') = \dfrac{\mathrm{Im}(z')}{|z'|^2} \geq \mathrm{Im}(z')$

now repeat.

process must terminate — or else the sequence with

$$-\tfrac{1}{2} \le \mathrm{Re}(z^j) \le \tfrac{1}{2}, \quad \mathrm{Im}(z) \le \mathrm{Im}(z^j) \le 1.$$

has accumulation point (Heine-Borel!)

Let $k \geq 2$ an even integer.

A holomorphic function f on $\mathbb{H}$ is a modular form for $\Gamma = SL(2,\mathbb{Z})$ of weight k if

(i) $$f(\gamma z) = (cz+d)^k f(z), \quad \gamma \in \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma.$$

(ii). since $f(Tz) = f(z+1) = f(z)$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z}$$

we assume $a_n = 0$ for $n < 0$.

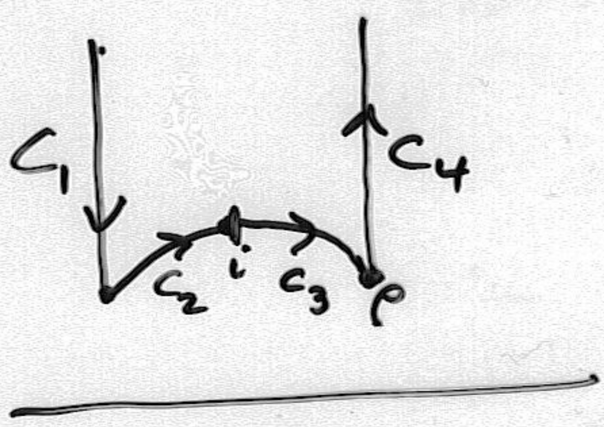
The space of such forms is a vector space denoted

$$\mathcal{M}_k(\Gamma).$$

Key fact it is finite dimensional.

$$\dim \mathcal{M}_k(\Gamma) = \begin{cases} \left[\frac{k}{12}\right] & \text{if } \frac{k}{2} \equiv 1 \ (6) \\ \left[\frac{k}{12}\right]+1 & \text{otherwise.} \end{cases}$$

Proof: $f \not\equiv 0$, $f \in \mathcal{M}_k$



$\rho = e^{2\pi i/6}$

$C = C_1 \cup C_2 \cup C_3 \cup C_4$

$$\frac{1}{2\pi i} \int_C \frac{f'}{f}(z)\,dz = \# \text{ of zeroes of } f$$

(Cauchy).

I

$T C_1 = -C_4$ ~~$T_2 C_2$~~ $S C_2 = -C_3$

$$f(Tz) = f(z), \quad f(Sz) = z^k f(z)$$

$$\int_{C_1} \frac{f'}{f} + \int_{C_4} \frac{f'}{f} = 0$$

$$\int_{C_2} \frac{f'}{f} + \int_{C_3} \frac{f'}{f} = 2\pi i \, \frac{\nu}{12}.$$

so the no' of zeroes $\leq \frac{\nu}{12}.$

$\Rightarrow$ finite dimensional.

19-th century modular forms — generate elements in $\mathcal{M}(\Gamma)$ by various means, finite dimensionality $\Rightarrow$ relations among coeff.

<u>EG</u> Jacobi

$$r_4(n) = \#\{x_1^2 + x_2^2 + x_3^2 + x_4^2 = n\}$$

if n is odd

$$r_4(n) = 8 \sum_{d \mid n} d$$

explicates Lagrange

Let

$$S_k(\Gamma) = \{ f \in \mathcal{M}_k(\Gamma) : a_f(0) = 0 \}$$

a codimension 1 subspace of $\mathcal{M}_k(\Gamma)$.

called the space of CUSP forms.

Eg $\dim \mathcal{M}_{12}(\Gamma) = 2$

and $\dim S_{12}(\Gamma) = 1$

The unique member in $S_{12}(\Gamma)$ is $\Delta(z)$ the discriminant of the genus 1 curve $\mathbb{C}/\langle 1, z\rangle$ from elliptic functions

$$\Delta(z) = e^{2\pi i z} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right)^{24}$$

$$:= \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}$$

$\tau(1) = 1$, $\tau(2) = -24$, $\tau(3) = 252$
$\tau(4) = -1472$, $\tau(5) = 4830$,
$\tau(6) = -6048$, $\tau(7) = -16744, \ldots$

RAMANUJAN CONJECTURES (20 th CENTURY).

(i). $\tau(m)\tau(n) = \tau(mn)$ if $(m,n) = 1$

in fact

$$\tau(m)\tau(n) = \sum_{d \mid (m,n)} d^{11} \tau\left(\frac{mn}{d^2}\right)$$

(ii) $|\tau(p)| \leq 2p^{11/2}$.

(i) proved by Mordell (1920 ?) + Hecke

(ii). Deligne (+ Weil Conjectures) 1974.

$S_k(\Gamma)$ f.d. vector space

$n \geq 1$ integer

Define the linear transformation

$T_n : S_k(\Gamma) \longrightarrow S_k(\Gamma)$ by

$$T_n f(z) = n^{k-1} \sum_{\substack{a \geq 1 \\ ad = n \\ 0 \leq b < d}} d^{-k} f\left(\frac{az+b}{d}\right)$$

[comes from orbits $\Gamma \backslash \Delta(n)$
$\Delta =$ integer 2×2 matrices of det n]

The T_n's commute in fact

$$T_n T_m = \sum_{d \mid (n,m)} d^{k-1} T_{\frac{nm}{d^2}}$$

T_n's are self adjoint w.r.t.

$$\langle f, g \rangle = \int_{\mathcal{Y}} f(z)\overline{g(z)}\, y^{k} \frac{dx\,dy}{y^2}$$

$$\langle T_n f, g \rangle = \langle f, T_n g \rangle \quad \text{(Petersson)}$$

Spectral theorem + T_n's commute simultaneously diagonalize them.

(i) of Ramanujan follows from

$\dim S_k = 1$ so $T_n \Delta = \lambda_n \Delta$

and λ_n's satisfy the T_n relations

and ~~ea~~

$$\tau(n) = \lambda(n).$$

II

Ramanujan-Petersson Conjecture (proved by Deligne) states that the eigenvalues $\lambda_f(p)$ of T_p on $S_k(\Gamma)$ satisfy

$$|\lambda_f(p)| \leq 2p^{\frac{k-1}{2}}.$$

Everything generalizes to the subgroups $\Gamma(N)$ of $\Gamma = SL(2,\mathbb{Z})$

$N \geq 1$ an integer

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (N) \right\}$$

$\Gamma(N)$ is a normal subgroup of Γ of index $\approx N^3$.

$\mathcal{M}_k(\Gamma(N))$ are the modular forms:

$$f(\gamma z) = (cz+d)^k f(z)$$

for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(N)$.

$z \in \mathbb{H}$ upper half plane.

Classical constructions of modular forms:

Eisenstein (periodize)

If $k > 2$ an even integer

$$E_k(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k}$$

converges absolutely. By substituting and resumming

$$E_k(\gamma z) = (cz+d)^k E_k(z).$$

$E_k \in \mathcal{M}_k(\Gamma)$, $\Gamma = SL(2,\mathbb{Z})$

(similar construction for $\Gamma(N)$).

$E_k(z)$ has an explicit Fourier expansion:

$$E_k(z) = c(k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, e^{2\pi i n z}$$

$c(k)$ a nonzero constant

where $\sigma_\nu(n) = \sum_{d|n} d^\nu$ (explicit elementary)

- E_k is never a cusp form.

<u>Jacobi's construction:</u>

Let $F(x_1, \ldots, x_n) = x^t A x$.

be an integral positive definite quadratic form in an even number of variables (A integral)

E G $x_1^2 + x_2^2 + x_3^2 + x_4^2$

for $z \in \mathbb{H}$ set

$$\Theta_F(z) = \sum_{m \in \mathbb{Z}^n} e^{2\pi i F(m) z}$$

(converges rapidly).

$$= \sum_{\nu=0}^{\infty} r_F(\nu) e^{2\pi i \nu z}$$

Where $r_F(\nu) = \#\{m \in \mathbb{Z}^n : F(m) = \nu\}$.

Now . $\Theta_F(z+1) = \Theta_F(z)$ periodicity of $\exp(2\pi i)$

(deeper) . $\Theta_F(-1/z)$ is almost equal to $z^k \Theta(z)$, $k = \frac{n}{2}$.

define an equivalence relation on Λ'

$$\alpha \sim \beta \quad \text{if}$$

$$p^m \alpha = \pm p^n \beta \quad \text{for some } m, n.$$

Using Jacobi's formula for $r_4(p^\nu)$ and some special arithmetic properties of integral quaternions (see references) $\Longrightarrow$

(1) $\Lambda = \Lambda'/\sim$ is a free group

(2) S generates Λ and

Cayley-graph (Λ, S) is a $p+1$ regular tree.

Cayley(G, S) $\quad V = G$

$g_1 \sim g_2$ if $g_1 = g_2 s$, $s \in S$ &

(3) $\Lambda(2q) = \{ \alpha = x_0 + x_1 i + x_2 j + x_3 k :$
$q \mid x_j ,\ j = 1, 2, 3 \}$

$\Lambda(2q)$ is a normal subgroup

Cayley graph $(\Lambda/\Lambda(2q), S)$

$k = p+1$

$$\equiv X^{(p,q)} .$$

(4). Spectrum$(X^{(p,q)}) \setminus \{k\}$

$\subset$ spectrum $(T_p$ acting on $S_2(\Gamma(q)))$.

---

(4) is proven by counting ~~paths~~ points in $\Lambda(2q)$ once using the spectrum of $(\Lambda/\Lambda(2q), S)$ and once using Jacobi's formula and representation by forms in 4 variables.

$\Rightarrow$ $k=2$

Ramanujan Conjecture $\Rightarrow$

$$|\lambda_j(T_p)| \leq 2p^{1/2}$$

$\Rightarrow$ $X^{(p,q)}$ is Ramanujan.

---

§ general F

- $F(x_1, x_2, x_3, x_4)$ quaternary form

~~[illegible]~~ $\Theta_F(z) = \mathcal{E}is(z) + \mathrm{Cusp}(z)$

so

$$r_F(n) = \mathcal{E}_F(n) + C_F(n)$$

- $$C_F(n) = O_\varepsilon(n^{\frac{1}{2}+\varepsilon})$$

(Raman Conj)

$\mathcal{E}_F(n)$ = # of representations of n ~~by~~ in terms of solving $F(x) = n$ in congruences (SINGULAR SERIES)

if not 0, $\approx n$. For n roughly correct.

References:

- J. P. Serre
  "A course in arithmetic"
  Springer

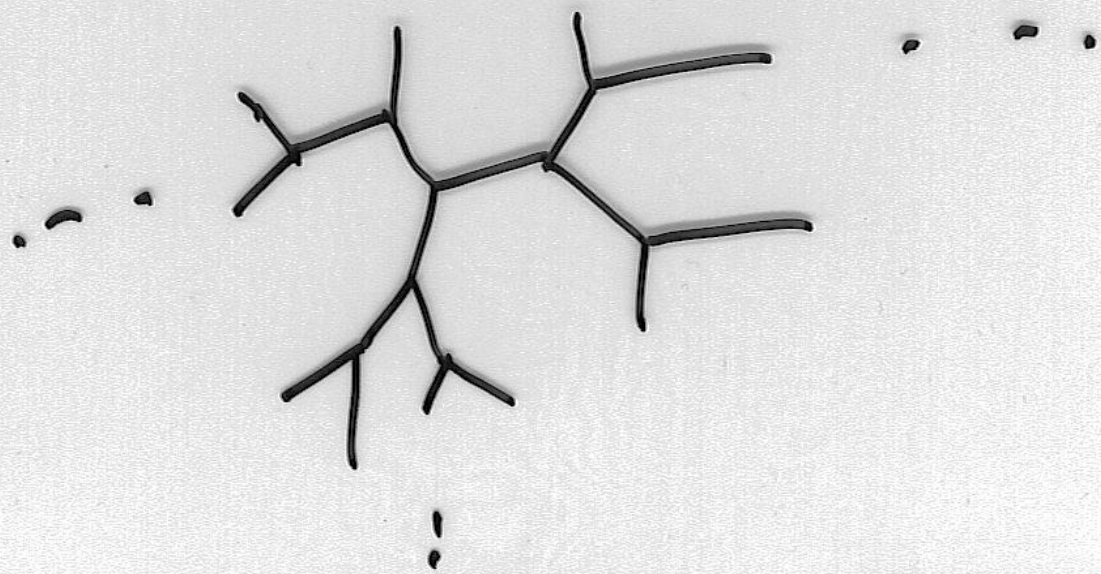
- P. SARNAK "Some Applications of modular forms"
  ~~C~~ Cambridge tracts Vol 99.

TREES:

Λ

k regular tree

3



$$Af(v) = \sum_{w \sim v} f(w) \qquad f : V \to \mathbb{C}$$

$$\ell^2(\Lambda) = \{ f : \sum_{v \in \Lambda} |f(v)|^2 < \infty \}.$$

$A : \ell^2(\Lambda) \to \ell^2(\Lambda)$ is bounded self adjoint.

H. KESTEN (thesis)

$$\text{spectrum}(A) = [-2\sqrt{k-1}, 2\sqrt{k-1}].$$

<u>Alon-Boppana:</u>

spect $X_{n,k}$ as $n \to \infty$ is at least as large as that of the universal cover Λ.

Set $\Delta = A - k$ = "Laplacian"

$$\Delta f(v) = \sum_{w \sim v} f(w) - k f(v).$$

---

a related conjecture for $\Gamma(N) \backslash \mathbb{H}$

hyperbolic geometry instead of complex analysis (free ourselves from holomorphic!)

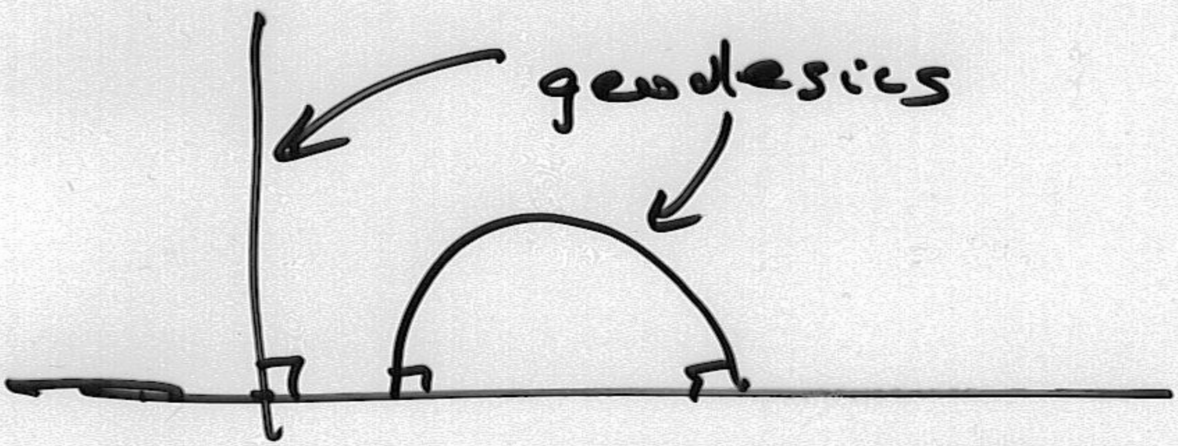
$\mathbb{H}$ model of non Euclidian geometry

(Lobachevski & Bolyai)

$$ds = \frac{|dz|}{y} \qquad dA = \frac{dx\,dy}{y^2}$$

Δ_H = Laplacian for ds

$$= y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$



$G = SL(2, \mathbb{R})$ acts on $\mathbb{H}$

$z \to \dfrac{az+b}{cz+d}$ preserves hyp. distances

$$d_H(gz, gw) = d_H(z, w)$$

As with Kesten, computes

$$\text{Spectrum}\,(\Delta \text{ on } L^2(\mathbb{H})) = \left[\frac{1}{4}, \infty\right).$$

ie variationally

$$\inf_{\substack{f \text{ compact} \\ \text{support smooth} \\ \text{on } \mathbb{H}}} \frac{\int_{\mathbb{H}} |\nabla_H f|^2 \frac{dx\,dy}{y^2}}{\int_{\mathbb{H}} |f|^2 \frac{dx\,dy}{y^2}} = \frac{1}{4}$$

Spectrum of Δ on $L^2(\Gamma(N)\backslash\mathbb{H})$

$$\Delta\phi + \lambda\phi = 0$$

$$\phi(\gamma z) = \phi(z) \qquad \gamma \in \Gamma(N)$$

$$\int_{\mathcal{F}_{\Gamma(N)}} |\phi(z)|^2 \frac{dx\,dy}{y^2} = 1$$

$f_0 = \text{constant} \qquad \lambda_0 = 0.$

λ_1 ? next largest

Variationally:

$$\lambda_1(\Gamma(N)) = \inf_{\substack{\int_{\mathcal{F}_{\Gamma(N)}} f(z) \frac{dx\,dy}{y^2} = 0}} \frac{\int_{\mathcal{F}_{\Gamma(N)}} |\nabla_H f(z)|^2 \frac{dx\,dy}{y^2}}{\int_{\mathcal{F}_{\Gamma(N)}} |f(z)|^2 \frac{dx\,dy}{y^2}}$$

<u>Selberg Conjecture (1965):</u>

$$\lambda_1(\Gamma(N)\backslash \mathbb{H}) \geq \frac{1}{4}.$$

(ie no worse than universal cover $\mathbb{H}$)

NOT PROVEN!

strong approximations exist

- Kim-S (JAMS 2003) $\lambda_1(\Gamma(N)\backslash\mathbb{H}) \geq \frac{975}{4096} = 0.238\ldots$

21 century problem:

General Ramanujan Conjectures:

concern such spectral problems

for

$$\Gamma \backslash SL(n, \mathbb{R}) \quad , n \geq 2$$

for $\Gamma = \Gamma(N) = \{ A \in SL(n, \mathbb{Z}) : A \equiv I \pmod{N} \}$.

(i) For Hecke like operators

(ii) For the invariant differential operators

The assertion is that for cuspidal functions the joint spectra are contained in as small as possible set — that is the ones corresponding to the L^2-spectra of the universal covering spaces;

One gets higher dimensional Ramanujan complexes called "Ramanujan Buildings" (who ordered those?)

see recent papers by

W. Li (GAFA) 2004, Lubotzky-Samuels-Vishne preprint 2004

References:

a "Notes on the generalized Ramanujan Conjectures"

P. SARNAK

www.math.princeton.edu

→ sarnak preprints.

Reason: Poisson summation

$f(u) = f(u_1, \ldots u_n)$

be rapidly decreasing as $u \to \infty$ and smooth

then

$$\sum_{m \in \mathbb{Z}^n} f(m) = \sum_{m \in \mathbb{Z}^n} \hat{f}(m)$$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$$

is the Fourier transform

apply this to

$$f(u) = e^{-2\pi i z u^t A u}$$

f is almost its own Fourier transform. $\perp$

- $\Theta_F(z)$ is a modular form of weight $k = \frac{n}{2}$ for $\Gamma(4 \cdot \det A)$.

Example

$$F_1(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

so

Θ_{F_1} is weight 2 on $\Gamma(4)$.

This space is small enough that Θ_F = Eisenstein series wt. 2

$\Longrightarrow$ equating coefficients

$$r_4(n) = 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d .$$

A more recent construction of weight 2 forms:

E an elliptic curve $a, b \in \mathbb{Z}$

$$y^2 = 4x^3 - ax - b \quad \text{—— } (*)$$

$$N = \text{discr.} = a^3 - 27b^2$$

For p a prime $p \nmid N$ set

$$a(p) = p - \#\{(x,y) \in \mathbb{F}_p \times \mathbb{F}_p \text{ satisfying } *\}$$

Extend to all integers:

- For $p \mid N$ a more careful defn is needed
- For $(m,n) = 1$ set $a(mn) = a(m)\,a(n)$
- $a(p^{n+1}) = a(p^n)\,a(p) - p\,a(p^{n-1})$ $n \geq 1$.

Wiles modularity (20^th century)

$$f(z) = \sum_{m=1}^{\infty} a(m)\, e^{2\pi i m z}$$

is a modular cusp form of weight 2 for $\Gamma(N)$.

Sums of 4 squares and multiplication:

H be the Hamilton quaternions

$$\alpha = x_0 + x_1 i + x_2 j + x_3 k$$

$x_j \in \mathbb{R}$, $i^2 = j^2 = k^2 = -1$

$\alpha\beta$ $\qquad ij = -ji = k \ \ldots$

$$\bar{\alpha} = x_0 - x_1 i - x_2 j - x_3 k.$$

$$N(\alpha) = \alpha\bar{\alpha} = x_0^2 + x_1^2 + x_2^2 + x_3^2 .$$

$$N(\alpha\beta) = N(\alpha)\, N(\beta).$$

$H(\mathbb{Z})$ quaternions with $x_j \in \mathbb{Z}$.

Fix $p \equiv 1 (4)$ a prime.

$\Lambda' = \{\alpha \in H(\mathbb{Z}) : \alpha \equiv 1 (2)$ and $N(\alpha) = p^m, m \geq 0\}$.

Λ' is closed under multiplication.

$S = \{\alpha \in \Lambda' : N(\alpha) = p\}$

S generates Λ'.