Added by Derek Robinson, November 2017 Chapter II of Langlands thesis established the density of the analytic elements in any continuous representation T of a Lie group on a Banach space X. In this note we explain how the arguments and estimates used to establish the density provided the foundation for a much deeper understanding of the differential structure of the group representations and eventually a productive approach to the analysis of diffusion processes on Lie groups.

Let B and $U(t)$ denote the operator and holomorphic semigroup of Theorem 8. One may normalize B, by the addition of a multiple of the identity, to ensure that $U(t)$ is uniformly bounded, i.e. $\|U(t)\| \leq M$ for all $t > 0$. Then the fractional powers B^γ, $\gamma \in \langle 0, 1\rangle$, are well defined. Let $X_a(B)$ denote the subspace of analytic elements of B, the $x \in W_\infty$ such that $\|B^n x\| \leq b^n n!$ for all positive integers n. Similarly let $X_a(B^{1/m})$ denote the analytic elements of $B^{1/m}$. These can be identified as the $x \in W_\infty$ such that $\|B^n x\| \leq c^n (nm)!$. Finally let $X_a(T)$ denote the analytic elements of the representation T, i.e. the $x \in W_\infty$ such that $\|x\|_n \leq a^n n!$. Then the final characterization of the analytic elements is given by

$$X_a(T) = X_a(B^{1/m}) \qquad (*)$$

for all continuous representations of the group and all the strongly elliptic m-th order operators. Since $X_a(B^{1/m})$ is dense in X, by general semigroup theory, this result incorporates the density of $X_a(T)$. Now we outline a path to establishing $(*)$.

If $x \in W_\infty$ then $x \in D(B^n)$ and $\|B^n x\| \leq c^n \|x\|_{mn}$ for all $n \geq 1$ with $c > 0$. Therefore if $x \in X_a(T)$ then $\|B^n x\| \leq (ac)^n (mn)!$ and $x \in X_a(B^{1/m})$. Hence $X_a(T) \subseteq X_a(B^{1/m})$. But the converse inclusion is much more difficult to establish. First, Langlands parametrix estimates establish that the resolvent of B exists and $\|(\lambda - B)^{-1} x\| \leq N \lambda^{-1} \|x\|$ for all large $\lambda > 0$. A slight elaboration of these arguments gives bounds $\|A_\alpha (\lambda - B)^{-1} x\| \leq N' \lambda^{-1+|\alpha|/m} \|x\|$ for all large $\lambda > 0$ whenever $|\alpha| \leq m - 1$. But since the semigroup U is holomorphic one has $\|BU(t)\| \leq b\, t^{-1}$ for all $t \in \langle 0, 1]$. Combining these estimates gives $\|A_\alpha U(t)\| \leq a(b+1) t^{-|\alpha|/m}$ for all $t \in \langle 0, 1]$ and $|\alpha| \leq m - 1$. Then by an induction argument one concludes that $\|U(t)x\|_k \leq a\, b^k k! t^{-k/m} \|x\|$ for all $x \in X$, $t \in \langle 0, 1]$ and $k \geq 1$. In particular one has a straightforward verification of Langlands conclusion that $U(t)X \subseteq X_a(T)$. Since $X_a(B) = \bigcup_{t>0} U(t)X$ one then deduces that $X_a(B) \subseteq X_a(T)$.

The possibility of the stronger inclusion $X_a(B^{1/m}) \subseteq X_a(T)$ was suggested by Roe Goodman (J. Funct. Anal. **3** (1969) 246–264). He established this result with B a Laplacian, i.e. a sum of squares, in any unitary representation. Ed Nelson then observed that the result extended to a large class of representations satisfying a general regularity property. Nelson's argument was given in an appendix to Goodman's paper. It is a modification of his earlier result on analytic domination (Ann. Math. **3** (1959) 572–615).

First $D(B) \subseteq (\lambda - B)^{-1} X \subseteq W_{m-1}$ for all large λ. Therefore $W_m \subseteq D(B) \subseteq W_{m-1}$. But it is not always true that $D(B) = W_m$. This is the case for unitary representations and the Goodman–Nelson arguments extend to all Banach space representations for which it is valid, specifically for all representations such that $\|x\|_m \leq a\,(\|Bx\| + \|x\|)$ for all $x \in W_\infty$. The key observation is that $\|(\operatorname{ad} A)_\alpha (B) x\| \leq b^{|\alpha|} \|x\|_m \leq a\, b^{|\alpha|} (\|Bx\| + \|x\|)$ for all α since $(\operatorname{ad} A_i)(B) = A_i B - B A_i$ is m-th order, i.e. the adjoint action does not change the order. Then it follows by a modification of the analytic domination result of Nelson that if $\|B^n x\| \leq c^n (nm)!$ for all $n \geq 1$ then $x \in X_a(T)$, i.e. $X_a(B^{1/m}) \subseteq X_a(T)$.

Secondly, if $D(B) \neq W_m$ then the foregoing reasoning does not apply. Nevertheless $(*)$ follows by interpolation arguments. First the spaces W_k are invariant under the group representation T and all the previous considerations apply to W_k and the representation $T|_{W_k}$, which we also denote by T. Then it follows readily that $X_a(T) = X_{k,a}(T)$ where $X_{k,a}$ denotes the analytic elements of T on W_k. Hence if W_γ, $\gamma \in \langle 0,1\rangle$, are the real interpolation spaces between X and W_1 then each W_γ is T-invariant and $W_1 \subseteq W_\gamma \subseteq X$. It follows that $X_a(T) = X_{\gamma,a}(T)$ with $X_{\gamma,a}$ the analytic elements of T on X_γ. But one has bounds $c\,\|B^{1/m}x\| \leq \|B^{1/m}x\|_\gamma \leq C\,\|B^{1/m}x\|_1$ where B now is the common notation for the elliptic operator on X, W_γ and W_1. Moreover, $\|x\|_1 \leq a\,(\|Bx\| + \|x\|)$ by the estimates on the resolvent of B. Therefore $c\,\|B^{1/m}x\| \leq \|B^{1/m}x\|_\gamma \leq a\,C\,(\|B^{(m+1)/m}x\| + \|B^{1/m}x\|)$. Hence $X_a(B^{1/m}) = X_{\gamma,a}(B^{1/m})$. Combining these conclusions one observes that $(*)$ is valid on X if and only if it is valid on X_γ. Therefore the problem of verifying $(*)$ is replaced by the problem of establishing that $D(B) = W_{\gamma,m}$. But this follows by an extension of the standard theory of interpolation spaces. The argument, which is quite circuitous, exploits various equivalent identifications of the spaces in terms of the group representation and the semigroup. Full details of all these arguments can be found in Chapters I–II of Elliptic Operators and Lie Groups, Oxford Univ. Press, 1991.

Finally Langlands proved that the action of U is given by a semigroup kernel K which is integrable with respect to Haar measure. This result was central to his proof of the density of $X_a(T)$ in contrast to the arguments outlined above which are independent of the kernel. Further analysis of the kernel, notably with a Lie group version of the Nash inequalities, establish that K is of 'Gaussian' type but that is another story; a tortuous tale.